(Last) Homework 11

Due: Thursday, April 22 – 10:00 EST

Problem 1: Matrix exponentials [10 Points]

In this question we investigate matrix exponentials of $X, Y \in \mathbb{M}(n \times n, \mathbb{R})$. We define the commutator $[X, Y] := XY - YX$ and recall $e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$.

1. Prove that $\left(e^X\right)^T = e^{\left(X^T\right)}$.
2. Show that in general $e^X e^Y \neq e^{X+Y}$.
3. Show the following:
   - If $XY = YX$, then $e^X e^Y = e^{X+Y}$.
   - $\left(e^X\right)^{-1} = e^{-X}$ (even when X is not invertible).
4. Use these results to proof the following instance of the *Campbell-Baker-Hausdorff* formula:
   If $[X, [X, Y]] = [Y, [X, Y]] = 0$, then $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$.
   Hint: Consider $f(\lambda) = e^{\lambda X} e^{\lambda Y} e^{-\lambda(X+Y)}$ and establish $f'(\lambda) = \lambda [X, Y] \cdot f(\lambda)$.
5. **Math 513:** Can you find $X \in \mathbb{M}(2 \times 2, \mathbb{R})$ such that $e^X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$?

Problem 2: $\mathfrak{so}(3)$ and $\mathrm{SO}(3)$ [10 Points]

In this question we employ linear algebra to investigate the relationship between the Lie algebra $\mathfrak{so}(3)$ and the Lie group $\mathrm{SO}(3)$. In general, Lie algebras can be understood as the tangent space of Lie groups at the identity. In this sense, they linearize a Lie group. An important application is to study properties of Lie groups from their Lie algebra. Of particular interest to quantum mechanics and quantum field theory are representations of Lie groups, which in essence are the mathematical counterpart of elementary particles, such as quarks, electrons, neutrinos and even the Higgs itself.

1. A matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ is skew-symmetric iff $A^T = -A$. Verify that the following is a basis of the $\mathbb{R}$-vector space of the skew-symmetric 3×3-matrices:

$$L_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

2. Verify the *commutation relations* ($[\cdot, \cdot]$ is the above-defined commutator):

$$[L_x, L_y] = L_z, \qquad [L_z, L_x] = L_y, \qquad [L_y, L_z] = L_x. \tag{2}$$

3. The vector-space of skew-symmetric 3×3-matrices with the commutators eq. (2) defines the Lie algebra $\mathfrak{so}(3)$. Show the *Jacobi identity*

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0\,,\qquad A,B,C\in\mathfrak{so}(3)\,. \tag{3}$$

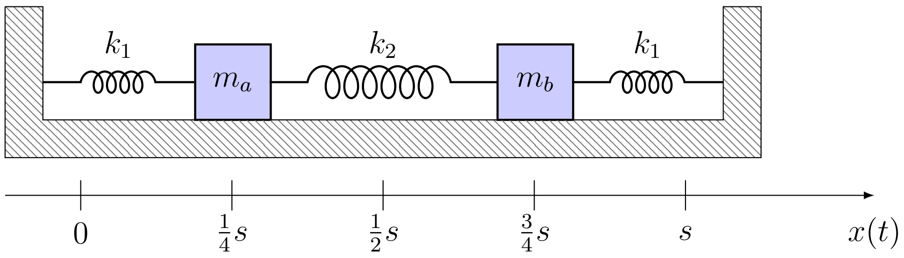
4. Show that for every $X \in \mathrm{so}(3)$, $e^X \in \mathrm{O}(3) = \{M \in \mathbb{M}(3\times 3,\mathbb{R})|M \text{ orthogonal}\}$.

5. Compute $e^{\alpha L_x}$, $e^{\beta L_y}$, $e^{\gamma L_z}$. What is the geometric meaning of these matrices?

One can show that the matrix exponentials of matrices in $\mathfrak{so}(3)$ give matrices in SO(3), i.e have determinant equal to one. An even less trivial result is that $\exp\colon \mathfrak{so}(3) \to \mathrm{SO}(3)$ is surjective. In general, the exponential map from a Lie algebra to its Lie group is not surjective. However, $\exp\colon \mathfrak{so}(3) \to \mathrm{SO}(3)$ is surjective because SO(3) is compact and connected.

Problem 3: A coupled spring-mass-system [10 Points]

In this exercise we investigate the following spring-mass system:



The dynamics of this system is described by a coupled system of ordinary differential equations (ODEs). We will formulate and solve this system and finally interpret the solutions. In the following $m_a, m_b, k_1, k_2 > 0$.

1. A mass on a spring:
   The motion $x\colon \mathbb{R}\to\mathbb{R}\,,\ t\mapsto x(t)$ of a mass m attached to a spring with Hook constant k is governed by the ODE (the so-called *equation of motion*)

   $$m\cdot x''(t) = -k\cdot x(t)\,. \tag{4}$$

   - Show that $x(t) = A\cdot\cos(\omega t) + B\cdot\sin(\omega t)$ with $\omega = \sqrt{\frac{k}{m}}$ solves this ODE.
   - Relate A and B to $x_0 = x(0)$ and $v_0 = x'(0)$.

2. Coupled equal masses on springs:
   At time t, the mass m_a is at position $\frac{s}{4} + x_a(t)$ and m_b at position $\frac{3s}{4} + x_b(t)$. The displacements $x_a(t)$, $x_b(t)$ are governed by the system of ODEs:

   $$\begin{aligned} m_a\cdot x_a''(t) &= -k_1 x_a(t) + k_2\left(x_b(t)-x_a(t)\right)\,,\\ m_b\cdot x_b''(t) &= -k_1 x_b(t) + k_2\left(x_a(t)-x_b(t)\right)\,. \end{aligned} \tag{5}$$

   We define $\vec{x}(t) = \begin{bmatrix} x_a(t) & x_b(t)\end{bmatrix}^T$ and assume $m_a = m_b$.

- Find $A \in \mathbb{M}(2 \times 2, \mathbb{R})$ such that eq. (5) is equivalent to $\vec{x}''(t) = A\vec{x}(t)$.
- Find $S, \Lambda \in \mathbb{M}(2 \times 2, \mathbb{R})$, S invertible and Λ diagonal, with $A = S\Lambda S^{-1}$.
- Solve eq. (5) in terms of $\vec{y}(t) = \begin{bmatrix} y_a(t) & y_b(t) \end{bmatrix}^T = S^{-1}\vec{x}(t)$.
- Use $\vec{y}(t) = S^{-1}\vec{x}(t)$ to solve eq. (5) for $\vec{x}(t)$. Express all constants by

$$x_a = x_a(0)\,, \qquad v_a = x_a'(0)\,, \qquad x_b = x_b(0)\,, \qquad v_b = x_b'(0)\,. \qquad (6)$$

Problem 4: A coupled spring-mass-system in `Python` [10 Points]

We will now repeat exercise 3-2 in `Python` to study how rich the dynamics of this simple system is.

1. Write a `Python` function:
   - Input: Initial values x_a, v_a, x_b, v_b, positive spring constants k_1, k_2, positive masses m_a, m_b, the positive box length s and times $t_{\min}, t_{\max} \in \mathbb{R}$.
   - Processing 1: Compute $\vec{y}(t)$ and $\vec{x}(t)$ as discussed in problem 3-2.
   - Processing 2: Construct the set $T := \{t_{\min}, t_{\min} + 0.1, t_{\min} + 0.2, \dots, t_{\max}]$.
   - Output 1: Plot $\frac{s}{4} + y_a(t)$ and $\frac{3 \cdot s}{4} + y_b(t)$ in one diagram for $t \in T$.
   - Output 2: Plot $\frac{s}{4} + x_a(t)$ and $\frac{3 \cdot s}{4} + x_b(t)$ in a second diagram for $t \in T$.

   Test your function for

   $$k_1 = k_2 = 1\,, \qquad m_a = m_b = 1 \qquad x_a = x_b = 1\,, \qquad v_a = v_b = 1\,, \qquad (7)$$

   $t_{\min} = 0$, $t_{\max} = 100$ and $s = 16$. Does the plot fit with your expectation?

2. Let us now study the limit $m_b \to \infty$. To this end, set $m_b = 10$ and describe how the plot changes. Qualitatively, explain the changed behavior.

3. Without using your function, explain what behavior to expect for $m_b \to 0$.

4. Let us plot the system for asymmetric initial conditions. To this end, we consider

   $$k_1 = k_2 = 1\,, \qquad m_a = m_b = 1 \qquad x_a = v_a = 1\,, \qquad x_b = v_b = 2\,, \qquad (8)$$

   and $t_{\min} = 0$, $t_{\max} = 100$, $s = 16$. Qualitatively, describe how the diagrams change relative to the symmetric case in 4-2. Does this fit your expectation?

5. Finally, let us consider the plots for a choice with different spring constants:

   $$k_1 = 10, \qquad k_2 = 1\,, \qquad m_a = m_b = 1 \qquad x_a = v_a = 1\,, \qquad x_b = v_b = 2\,, \; (9)$$

   and $t_{\min} = 0$, $t_{\max} = 100$, $s = 24$. Qualitatively, describe the plots.