Math 313/513, Spring 2021 **Martin Bies**

Homework 2

Due: Thursday, February 4 – 10:00 am EST

Problem 1: Solving linear systems [10 Points]

1. Solve $A\vec{x} = \vec{b}$ with the method of elimination and back substitution:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 4 & 6 & 1 & 0 \\ -2 & 2 & 0 & 4 \\ 1 & 3 & 5 & 7 \end{bmatrix} , \qquad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix} . \tag{1}$$

2. The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 6)$, $(2, 3)$ and $(-1, 0)$. Find and solve a matrix equation for the unknowns (a, b, c).

3. Find A^{-1} and B^{-1} (if they exist) by Gauss-Jordan elimination:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} , \qquad B = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} . \tag{2}$$

Problem 2: "Order" of elimination with back substitution [10 Points]

Consider $A \in \mathbb{M}(n \times n, \mathbb{R})$ and $\vec{b} \in \mathbb{R}^n$. We apply the method of elimination with back substitution (MEB) to $[A|\vec{b}]$. It is assumed that no permutations of the rows of A are required and that $A\vec{x} = \vec{b}$ is non-singular. Hence, we perform the transformation

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \to \left[\begin{array}{cccc|c} u_{11} & u_{12} & \cdots & u_{1n} & c_1 \\ & u_{22} & \cdots & u_{2n} & c_2 \\ & & \ddots & \vdots & \\ & & & u_{nn} & c_n \end{array}\right] , \tag{3}$$

followed by back substitution. The first step is to turn the second row into the form

$$\left[\begin{array}{cccc|c} 0 & a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12} & \ldots & a_{2n} - \frac{a_{21}}{a_{11}} \cdot a_{1n} & b_2 - \frac{a_{21}}{a_{11}} \cdot b_1 \end{array}\right] . \tag{4}$$

For this we first compute $\frac{a_{21}}{a_{11}}$ and then use this result to perform the remaining additions and multiplications. At most, this requires n additions and $n+1$ multiplications. The first back substitution $x_n = \frac{c_n}{u_{nn}}$ requires (at most) one multiplication.

1. Consider $A \in \mathbb{M}(2 \times 2, \mathbb{R})$, $\vec{b} \in \mathbb{R}^2$:
   - Show that MEB requires at most 6 multiplications and 3 additions.
   - Find A, $\vec{b}$ for which MEB requires stricly less than 9 operations.

2. Repeat for $A \in \mathbb{M}(3 \times 3, \mathbb{R})$, $\vec{b} \in \mathbb{R}^3$.
3. MEB for $A \in \mathbb{M}(n \times n, \mathbb{R})$, $\vec{b} \in \mathbb{R}^n$ requires (at most) $\frac{n \cdot (n^2+3n-1)}{3}$ multiplications and $\frac{(n-1) \cdot n \cdot (2n+5)}{6}$ additions.
   - Find the total number of operations.
   - Use the computer science $\mathcal{O}$-notation to classify MEB.
4. **For Math 513:**
   Show that for $A \in \mathbb{M}(n \times n, \mathbb{R})$, $\vec{b} \in \mathbb{R}^n$, MEB requires (at most) $\frac{n \cdot (n^2+3n-1)}{3}$ multiplications and $\frac{(n-1) \cdot n \cdot (2n+5)}{6}$ additions.

Problem 3: Speed of LU factorizations in `Python` [10 Points]

Perform the following steps in a `Jupyter` notebook:

1. Create a matrix $A \in \mathbb{M}(3 \times 3, \mathbb{R})$ with random entries.
2. Compute the PLU-factorization $A = PLU$. Print P, L, U.
3. Verify that $P \cdot P^T = I$. What type of matrix is P?
4. Write a function `LUtime` which accepts an integer n as input, creates a matrix $M \in \mathbb{M}(n \times n, \mathbb{R})$ with random entries and returns the time that it takes to compute the PLU factorization of this matrix M.
5. Execute this function for $n \in \{1, 2, \ldots, 2000\}$ and plot `LUtime`(n).
6. Fit a cubic to this data and interpret the result (cf. problem 2).

Problem 4: Column picture [10 points]

Consider the following vectors in $\mathbb{R}^3$:

$$\vec{u} = \begin{bmatrix} -3 \\ -2 \\ 4 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \qquad \vec{w} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}. \tag{5}$$

1. Verify or falsify if these vectors are contained in a plane in $\mathbb{R}^3$.
2. Consider the matrix

$$A = \begin{bmatrix} -3 & 1 & 2 \\ -2 & 2 & 0 \\ 4 & -3 & -1 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}. \tag{6}$$

   - Use the result of 1. to verify or falsify that $A\vec{x} = \vec{0}$ has a solution $\vec{x} \neq \vec{0}$.
   - Suppose that $A\vec{x} = \vec{b}$ has a solution. Explain if it has other solutions.