Math 313/513, Spring 2021 **Martin Bies**

Homework 4

Due: Thursday, February 18 – 10:00 am EST

Problem 1: Solutions to linear systems [10 Points]

1. Consider the following linear systems:
   (a) homogeneous system $A\vec{x} = \vec{0}$ of 2 equations in 2 unknowns,
   (b) inhomogeneous system $A\vec{x} = \vec{b}$ of 7 equations in 7 unknowns,
   (c) inhomogeneous system $A\vec{x} = \vec{b}$ of 4 equations in 7 unknowns,
   (d) homogeneous system $A\vec{x} = \vec{0}$ of 4 equations in 7 unknowns,
   (e) inhomogeneous system $A\vec{x} = \vec{b}$ of 5 equations in 2 unknowns,
   (f) homogeneous system $A\vec{x} = \vec{0}$ of 5 equations in 2 unknowns.

   For each of those situations, indicate all the possibilites that can occure:

   (I) no solution, (II) unique solution, (III) infinitely many solutions.

   No justification is required.

2. For $a_1, a_2 \in \mathbb{R}$ we consider the linear system $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 1 & a_1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & a_2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{1}$$

   Find all solutions to $A\vec{x} = \vec{b}$ as function of $a_1, a_2 \in \mathbb{R}$.

Problem 2: Bases of vector spaces [10 Points]

1. Show that the following five vectors in $\mathbb{R}^3$ are linearly dependent:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}. \tag{2}$$

2. Find a basis of $\mathbb{R}^3$ among the five vectors above. Justify your answer.

3. Find 4 linearly independent vectors in Pol_4 and $\mathbb{M}(3 \times 3, \mathbb{R})$.

4. **For Math 513**: Derive $\dim_{\mathbb{R}}(\mathrm{Pol}_n)$ and $\dim_{\mathbb{R}}(\mathbb{M}(m \times n, \mathbb{R}))$.

Problem 3: Inequalities for ranks [10 Points]

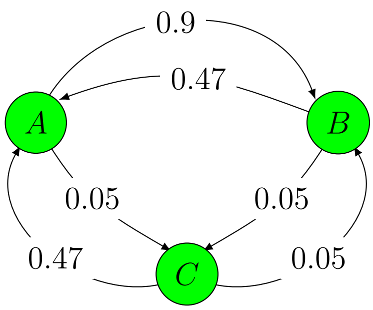
Consider $A \in \mathbb{M}(m \times n, \mathbb{R})$ and $B \in \mathbb{M}(n \times l, \mathbb{R})$.

1. Argue that $\mathrm{rk}(A) \leq n$ and $\mathrm{rk}(A) \leq m$.
2. Consider the map $\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$. For all of the following cases, compare $\mathrm{rk}(A)$ with n and m:

   (a) φ_A is injective, (b) φ_A is surjective, (c) φ_A is bijective.
3. **For Math 513:** Is it always true that $\mathrm{rk}(B) \leq \mathrm{rk}(AB)$?

Problem 4: A Markov process [10 Points]

We consider a network of three vessels A, B, C, which each contain 0.5 litre of water. Every hour, 90% of the water contained in A is being pumped into B. Similar transitions happen among all vessels. The transition rates are as follows:



(3)

1. Find $M \in \mathbb{M}(3 \times 3, \mathbb{R})$, $\vec{x} \in \mathbb{R}^3$ s.t. $M\vec{x}$ lists the water levels after one hour.
2. Assume there was a non-trivial vector $\vec{x}$ with $M\vec{x} = \vec{0}$. Is this consistent with the above system? Compare with a `Python`-computation of $N(M)$.
3. After n-hours, the water levels are $\vec{x}^{(n)} := M^n\vec{x}$. Write a `Python`-function to compute powers of a matrix. Use it to compute $\vec{x}^{(n)}$ for $n \in I = \{0, 1, \ldots, 10\}$.
4. For each $n \in I$, compute the sum $\Sigma^{(n)}$ of the components of $\vec{x}^{(n)}$:

$$\Sigma^{(n)} := x_1^{(n)} + x_2^{(n)} + x_3^{(n)} \,. \tag{4}$$

5. Draw the three components of $\vec{x}_n$ and the sum $\Sigma^{(n)}$ against $n \in I$. What do we learn about this vessel-system? In particular, interpret $\Sigma^{(n)}$.