Math 313/513, Spring 2021 **Martin Bies**

Homework 5

Due: Thursday, March 4 – 10:00 am EST

Problem 1: Base change [10 Points]

In this exercise, you will find that in some basis it is easy to understand a linear transformation. We exemplify this for the linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\mathcal{B}_1 = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad A_{\mathcal{B}_1\mathcal{B}_1} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{1}$$

1. Here I am interested in your intuition: What do you expect this linear transformation φ to do? Is it a rotation? Or maybe a reflection? Something else?
2. Consider the basis $\mathcal{B}_2 = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$:
   - Find the transformation matrix $T_{\mathcal{B}_2\mathcal{B}_1}$ for the base change from $\mathcal{B}_1$ to $\mathcal{B}_2$.
   - Compute $T_{\mathcal{B}_1\mathcal{B}_2}$ and $A_{\mathcal{B}_1\mathcal{B}_2} = A_{\mathcal{B}_1\mathcal{B}_1} T_{\mathcal{B}_1\mathcal{B}_2}$.
   - From your intuition and $A_{\mathcal{B}_1\mathcal{B}_2}$: Formulate a hypothesis for how φ acts.
3. Compute $A_{\mathcal{B}_2\mathcal{B}_2} = T_{\mathcal{B}_2\mathcal{B}_1} A_{\mathcal{B}_1\mathcal{B}_1} T_{\mathcal{B}_1\mathcal{B}_2}$ and verify your hypothesis.

Problem 2: Transformation matrix in `Python` [10 Points]

1. Write a `Python` function:
   Input: Two basis $\mathcal{B}_1$, $\mathcal{B}_2$ of $\mathbb{R}^n$. Output: $T_{\mathcal{B}_2\mathcal{B}_1}$.
2. Write a `Python` function:
   Input: Two basis $\mathcal{B}_1$, $\mathcal{B}_2$ of $\mathbb{R}^n$ and $A_{\mathcal{B}_1\mathcal{B}_1}$ Output: $A_{\mathcal{B}_2\mathcal{B}_2}$.
3. We will identify in `Python` basis $\mathcal{B}'$ such that $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ is easy:

$$A_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 3/2 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \tag{2}$$

   - Rotate the standard basis $\mathcal{B}$ by $\alpha \in I = \{0°, 1°, \ldots, 720°\}$ about the z-axis:

$$\mathcal{B}' = \{R_z\vec{e_1}, R_z\vec{e_2}, R_z\vec{e_3}\}, \quad R_z = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

     For all $\alpha \in I$, verify that $\mathcal{B}'$ is a basis (use your function from the midterm).
   - Compute $A_{\mathcal{B}'\mathcal{B}'}$ with 5 digit precision for all angles $\alpha \in I$.
   - For which $\alpha \in I$ is $A_{\mathcal{B}'\mathcal{B}'}$, as computed in `Python`, approximately diagonal?
4. **Math 513** We supplement these observations by a theoretical analysis. By hand, verify $R_z^{-1} = R_z^T$ and compute $A_{\mathcal{B}'\mathcal{B}'}$. Show that $A_{\mathcal{B}'\mathcal{B}'} = R_z^T A_{\mathcal{B}\mathcal{B}} R_z$ is diagonal if and only if $\alpha \in \left\{ -\frac{\pi}{4} + \frac{\pi}{2} \cdot k \,|\, k \in \mathbb{Z} \right\}$.

Problem 3: Orthogonal vector spaces and decomposition [10 Points]

1. For the standard inner product $\langle \cdot, \cdot \rangle_{\text{Std}}$ in $\mathbb{R}^n$, verify all axioms of inner products.
2. Name another inner product $\langle \cdot, \cdot \rangle_2$ in $\mathbb{R}^n$ and verify that it satisfies all axioms.
3. Find two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ such that $\langle \vec{x}, \vec{y} \rangle_2 = 0$ and $\langle \vec{x}, \vec{y} \rangle_{\text{Std}} \neq 0$.
4. Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R})$.
   - Verify that $N(A)$ and $R(A)$ are orthogonal in $\langle \cdot, \cdot \rangle_{\text{Std}}$.
   - For every $\vec{x} \in \mathbb{R}^3$, compute $\vec{x}_N \in N(A)$ and $\vec{x}_R \in R(A)$ with $\vec{x} = \vec{x}_N + \vec{x}_R$.

Problem 4: Inner products in exotic vector spaces [10 Points]

1. Prove that the following is an inner product in Pol_n:

$$\langle \cdot, \cdot \rangle_1 : \text{Pol}_n \times \text{Pol}_n \to \mathbb{R}\,,\ (p, q) \mapsto \int_0^1 p(x) \cdot q(x) dx\,. \tag{4}$$

2. Compute the length of the vectors 1, x, x^2, x^3, x^4 of Pol_4 with $\langle \cdot, \cdot \rangle_1$.
3. **Bonus** Find a basis $\mathcal{B} = \{P_1, P_2, P_3\}$ of Pol_2 such that $\langle P_i, P_j \rangle_1 = 0$ whenever $i \neq j$ and such that $\langle P_i, P_i \rangle_1 = 1$ for all $P_i \in \mathcal{B}$.
4. Argue that also the following is an inner product in Pol_n:

$$\langle \cdot, \cdot \rangle_2 : \text{Pol}_n \times \text{Pol}_n \to \mathbb{R}\,,\ \Big(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{n} b_j x^j\Big) \mapsto \sum_{i=0}^{n} a_i b_i\,. \tag{5}$$

   Find a basis $\mathcal{B} = \{P_1, P_2, P_3\}$ of Pol_2 with $\langle P_i, P_i \rangle_2 = 1$ and $\langle P_i, P_j \rangle_2 = 0$, $i \neq j$.
5. **Math 513:** Find an inner product in $\mathbb{M}(m \times n, \mathbb{R})$. Justify your answer.