Math 313: Computational linear algebra

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Contents

1 Preface

Generalities The aim of this course is to give an overview over linear algebra with an emphasize on its arithmetics and applications. For this reason, we restrict ourselves to linear algebra over $\mathbb{R}$ and $\mathbb{C}$. Deeper insights into the theory of linear algebra over arbitrary fields (and eventually algebra over arbitrary rings) are taught in the advanced courses, such as Math 314.

It has become tradition, that this course follows the textbook *Introduction to linear algebra* by *Gilbert Strang*. This course is no exception. For convenience, these notes aim to collect the discussed material.

What is this course about? Very broadly speaking, we will study lines. A little more specifically, we will study the geometry of linear systems of equations.

While linear systems of equations look deceptively simple, they underlie a vast amount of (applied) mathematics. Ideas that you will encounter here have some surprising and far-reaching applications. To appreciate this fact, we must understand the mathematics behind linear equations.

An example of such applications derives from an attempt to answer the following question: *Why should we care about lines?* Recall from calculus the idea that, in order to understand a complicated curve, we can attempt to zoom in and see what these objects look *locally*. This leads to the study of the tangent line, which is an example of a linear structure and vector space! So, in extrapolating to higher-dimensional objects, we may conclude that while the world around us may possess complicated geometries, if we look locally, we see linear structures.

Of course, we lose information when we zoom in. The resulting linear structure is in general not identical to the original geometric structure. Still, this linear "approximation" retains a vast amount of information. This is why we can try to understand such complicated structures from studying their linear approximations. At times, it is then possible to extract information about the original geometric structure from these linear structures. For example, (for equidimensional) curves, surfaces etc. the dimension of the tangent space matches the dimension of the original geometric object.

Acknowledgements These lecture notes are based on the course that Vasu Tewari taught in 2019. Vasu, thank you very much indeed for sharing your notes! Also, I would like to thank Peter McGrath for useful conversations on the 2020 course.

Typos, mistakes and feedback Please send messages regarding typos, mistakes and general feedback to mbies@sas.upenn.edu. Thank you!

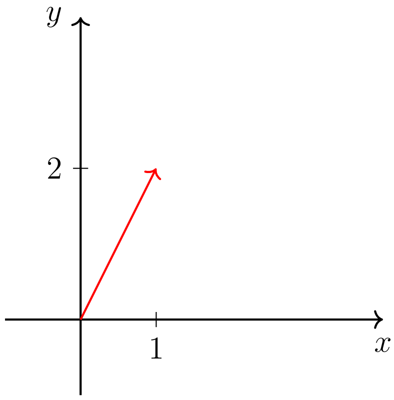
2 Solving Linear Equations

2.1 Revision: Vectors

Note:
You should have encountered vectors before, e.g. in Math 240 or Math 260. Here is a quick revision.

Example 2.1.1 (A vector in 2 dimensions)**:**
$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a vector in 2 dimensions. The entries 1 and 2 are the components of $\vec{v}$.

Remark:
We will write our vectors as column vectors. We can picture the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as follows:



(2.1)

Example 2.1.2 (A vector in 3 dimensions)**:**
Similarly, we can work in three dimensions. For instance, the following is a vector in 3-dimensions:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}. \tag{2.2}$$

Exercise:
Draw an image of this vector $\vec{v}$ in three dimensions.

Note:
These images get no easier in higher dimension. However, we may still abstractly think of vectors by merely listing their components. In a sense, this is the first place where we see the benefits of abstraction.

Definition 2.1.1:
A vector $\vec{v} \in \mathbb{R}^n$ is a column with n real components:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \qquad v_i \in \mathbb{R}. \tag{2.3}$$

Definition 2.1.2 (Addition)**:**
For $\vec{v}, \vec{u} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we define addition and scalar multiplication:

$$\vec{v} + \vec{u} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} := \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}, \tag{2.4}$$

$$c \cdot \vec{v} = c \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} := \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \\ \vdots \\ c \cdot v_n \end{bmatrix}. \tag{2.5}$$

Note:
A 1-dimensional vector is essentially a real number. The algebraic operations for vectors are lifted from addition and multiplication of real numbers.

Exercise (Addition and scalar multiplication pictorially)**:**
Consider $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $c = 2$. Draw images of $\vec{v} + \vec{u}$ and $c \cdot (\vec{v} + \vec{u})$.

Note:
Later in the course, we will see that addition and scalar multiplication allow us to general lines, planes and, more generally speaking, any linear object in any dimension.

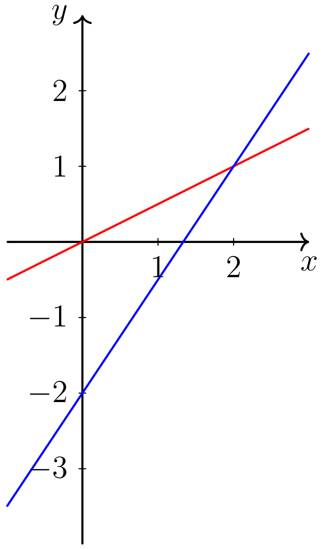
2.2 Approaches to systems of linear equations

For the next few classes, we will work with n equations in n unknown, i.e. as many equations as variables.

Note (Row picture)**:**
Let us consider the following set of equations:

$$\begin{aligned} x - 2y &= 0\,, \\ 3x - 2y &= 4\,. \end{aligned} \tag{2.6}$$

The equations $x - 2y = 0$ and $3x - 2y = 4$ define 2 lines in the plane:



(2.7)

We can therefore interpret this set of equations as the task to find the intersection of those two lines. We term this perspective the *row picture*.

Note (Column picture)**:**
Let us again consider the system of equations

$$\begin{aligned} x - 2y &= 0\,, \\ 3x - 2y &= 4\,. \end{aligned} \tag{2.8}$$

It makes sense to consider the coefficients of x and y in both equations simultaneously. We may thus rewrite this system as

$$x \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}. \tag{2.9}$$

We are thus trying to find scalar multiples of the vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$, such that their sum matches $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$. We term this perspective the “column picture”.

Exercise:
How can we geometrically determine the right scalars x and y?

Note (Matrix picture)**:**
Yet another way to view the equations

$$\begin{aligned} x - 2y &= 0\,, \\ 3x - 2y &= 4\,. \end{aligned} \tag{2.10}$$

is the matrix picture. Here, we use the coefficients to construct the 2×2 *coefficient matrix*

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix} . \tag{2.11}$$

We record the RHS of eq. (2.10) and collect the unknowns in the vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then we can rewrite eq. (2.10) as

$$\begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} . \tag{2.12}$$

Definition 2.2.1 (Multiplication of matrix and vector)**:**

$$\begin{bmatrix} \\ n \times n \\ \text{matrix } A \\ \\ \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \begin{bmatrix} \rotatebox{90}{\text{Column 1 of } A} \end{bmatrix} + x_2 \cdot \begin{bmatrix} \rotatebox{90}{\text{Column 2 of } A} \end{bmatrix} + \cdots + x_n \cdot \begin{bmatrix} \rotatebox{90}{\text{Column } n \text{ of } A} \end{bmatrix} . \tag{2.13}$$

Consequence:
The following approaches to systems of linear equations are equivalent:

- linear systems,
- equations involving column vectors,
- matrix equations $A\vec{x} = \vec{b}$.

Remark:
For a system of three equations in three variables, the row picture corresponds to finding the intersection of three planes. On the other hand, the column picture concern combinations of 3-dimensional vectors. As a general theme through the course, we will find that the column picture is more revealing!

2.3 The method of elimination with back substitution

Example 2.3.1:
Let us return once again to the system of linear equations

$$\begin{aligned} x - 2y &= 0 \,, \\ 3x - 2y &= 4 \,. \end{aligned} \tag{2.14}$$

By multiplying the first equation by 3 and subtracting it from equation 2, we obtain a new system of equations:

$$\begin{aligned} x - 2y &= 0\,, \\ 4y &= 4\,. \end{aligned} \tag{2.15}$$

At this point, we can solve for y and obtain $y = 1$. Once we know that, we can plug this value into the first equation and solve for x. We obtain $x = 2$.

Note:
This procedure is a special instance of the *method of elimination with back substitution.* We aspire to do this in general, i.e. given a system

$$A\vec{x} = \vec{b}\,, \tag{2.16}$$

we desire to transform this system into the form

$$U\vec{x} = \vec{c}\,, \tag{2.17}$$

where U is upper trangular and, as a consequence, $U\vec{x} = \vec{c}$ is readily solvable.

Example 2.3.2:
To see how this can be achieved, let us consider another example:

$$\begin{aligned} x + 2y + z &= 2\,, \\ 3x + 8y + z &= 12\,, \\ 4y + z &= 2\,. \end{aligned} \tag{2.18}$$

First, notice that the variable names do not play any role. Therefore, we may ditch them. The coefficient matrix A and the column vetor $\vec{b}$ are the only pieces of information that are relevant. We put them in the so-called *augmented matrix*:

$$\left[\begin{array}{ccc|c} \color{blue}{1} & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]. \tag{2.19}$$

Now, let us aim for a triangular form. We can use the blue 1 to eliminate the 3 below. This gives a new matrix

$$\left[\begin{array}{ccc|c} \color{blue}{1} & 2 & 1 & 2 \\ 0 & \color{blue}{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]. \tag{2.20}$$

Thus, we have eliminated the entry in row 2 column 1. Ideally, we can use the blue 1 to eliminate also the entry in row 3 and col 1. In this particular example, this entry is already 0, so no action is required. At this point, we have 0s below the circled 1. This means that the equations corresponding to the 2nd and 3rd row do not involve x.

Now we recurse and use the blue 2 to clear out the entry in row 3 and column 2. This gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]. \tag{2.21}$$

Now, we can solve for z, then (after back substitution) for y and finally for x.

Definition 2.3.1 (Pivot elements)**:**
The **non-zero**, blue entries in eq. (2.21) are called *pivots* – the are pivotal to the execution of this procedure.

Remark:
By definition, we require that a pivot element is *non-zero*.

Consequence:
By performing the back-substitution, we find that the solution to eq. (2.21) is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}. \tag{2.22}$$

Note (Upshot)**:**
We went from $\left[A|\vec{b}\right]$ to $[U|\vec{c}]$ where U is upper triangular. Subsequently, we solved the resulting system by back substitution. Does this procedure always work? What could possibly go wrong?

Remark:
Note that the entries, which we use to clear out columns, should not vanish. Hence, how would we deal with the following augmented matrix?

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \tag{2.23}$$

While the entry at row 1 column 1 vanishes, there is a non-zero element in row 2 column 1. We can swap these two rows. Thereby, we obtain a non-zero entry at row 1 column 1. This new non-zero entry can now be used to clear out the first column, i.e. it can play the role of a pivot.

Note (Two types of failures)**:**
Let us discuss two instances, in which elimination with back-substitution fails:

1. No solution at all:

$$\left[\begin{array}{cc|c} 2 & 3 & 2 \\ 4 & 6 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3 & 2 \\ 0 & 0 & 3 \end{array}\right]. \tag{2.24}$$

   This matrix has pivot 2 and admits no solutions.

2. Infinitely many solutions:

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & 6 & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 0 & 0 \end{array}\right]. \tag{2.25}$$

   This matrix has pivot 2. Still, there are infinitely many solutions to this system of linear equations.

Consequence:
A zero in a pivot position implies either no solution or infinitely many solutions.

Note (Gaussian elimination)**:**
We proceed as follows:

1. Get a pivot in the first row and use it to clear out the column below.

2. Repeat for all other rows.

3. If you find n pivots after starting from an $n \times n$-matrix, then the system has a unique solution.

Definition 2.3.2:
A system of linear equations $A\vec{x} = \vec{b}$ is called non-singular, if it admits n pivots. Otherwise, we call it singular.

Consequence:
It follows that:

- A non-singular system $A\vec{x} = \vec{b}$ has a unique solution.

- A singular system $A\vec{x} = \vec{b}$ has either no or infinitely many solutions.

2.4 Matrix multiplication and elementary matrices

2.4.1 Elementary matrices

Observe that when we solve the system $A\vec{x} = \vec{b}$ via row operations, the vector $\vec{b}$ keeps getting transformed. We can understand this transformation by matrix multiplication. Recall that for an $n \times n$ matrix A we can write

$$A \cdot \vec{x} = \begin{bmatrix} - & \text{row 1 of A} & - \\ - & \text{row 2 of A} & - \\ & \vdots & \\ - & \text{row n of A} & - \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} (-\text{ row 1 of A }-) \cdot \vec{x} \\ (-\text{ row 2 of A }-) \cdot \vec{x} \\ \vdots \\ (-\text{ row n of A }-) \cdot \vec{x} \end{bmatrix}. \tag{2.26}$$

Example 2.4.1 (A first elementary matrix)**:**
We wonder what matrix sends $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ to $\begin{bmatrix} b_1 \\ b_2 - 3b_1 \\ b_3 \end{bmatrix}$? That is, find a 3×3 matrix E such that $E \cdot \vec{b}$ is the vector obtained by subtracting 3 times the first row from the second row. By inspection, the following matrix satisfies this property:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.27}$$

The key property of this matrix is the -3 in row 2 column 1.

Example 2.4.2 (Another elementary matrix)**:**
What matrix send $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ to $\begin{bmatrix} b_1 \\ b_2 \\ b_3 - 2b_2 \end{bmatrix}$? Convince yourself, that the following matrix achieves this:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}. \tag{2.28}$$

The key is the -2 in row 3 column 2.

Note:
This pattern generalizes by considering the $n \times n$ matrix E_{ij} which has 1's along the diagonal, a real number c in row i and column j and 0's everywhere else. Assume that $i > j$. Then it follows that

$$E_{ij} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{2.29}$$

is the vector obtained by adding c-times row j to row i. We call such matrices *elementary matrices.*

Consequence:
Let us go back to our matrix equation $A\vec{x} = \vec{b}$. Then, any elimination move changes the RHS from $\vec{b}$ to $E\vec{b}$ for some elementary matrix E. Let us 'multiply' both sides of the equation $A\vec{x} = \vec{b}$ from the left by E. Then we obtain

$$EA\vec{x} = E\vec{b}\,. \tag{2.30}$$

Thus, we would like matrix multiplication to possess the property that EA is the matrix obtained by performing the row operation corresponding to E. We will now discuss a multitude of ways to compute the product of two matrices A and B. Thereby, we will verify that this expectation is indeed satisfied.

2.4.2 Matrix multiplication

We consider an $m \times n$-matrix A and an $n \times p$ matrix B. Then the matrix product $A \cdot B$ can be defined in a multitude of ways:

- The matrix $C = AB$ has entry c_{ij} in row i and column j given by

$$c_{ij} = (\text{— row i of A —}) \cdot \begin{pmatrix} | \\ \text{column j of B} \\ | \end{pmatrix} . \tag{2.31}$$

  For instance, the entry in row 3 column 4 of c is given by

$$c_{34} = a_{31}b_{14} + a_{32}b_{24} + \cdots = \sum_{k=1}^{n} a_{3k}b_{k4} . \tag{2.32}$$

  Note that $C = AB$ is an $m \times p$ matrix.

- We restate the above as follows:

$$B = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} , \tag{2.33}$$

$$C = AB = \begin{bmatrix} A \cdot \begin{pmatrix} | \\ b_1 \\ | \end{pmatrix} & A \cdot \begin{pmatrix} | \\ b_2 \\ | \end{pmatrix} & \cdots & A \cdot \begin{pmatrix} | \\ b_p \\ | \end{pmatrix} \end{bmatrix} . \tag{2.34}$$

  Thus, the columns of AB are linear combinations of the columns of A.

- Similarly, the rows of $C = AB$ are linear combination of the rows of B.

- Note that any column of A multiplied by any row of B is an $m \times p$ matrix. For instance:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix} . \tag{2.35}$$

  Therefore, $C = AB$ is the sum of all matrices obtained by multiplying a column of A by a row of B.

Exercise:
Let us again consider the example

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix} . \tag{2.36}$$

Note that the columns are scalar multiples of each other. Likewise, the three rows are scalar multiples of each other. What does this mean geometrically?

2.4.3 Permutation matrices

Remark:
Recall that if we did not have a pivot in row 1 column 1, there was a possibility of swapping row 1 with some other row (cf. section 2.3). We will now discuss matrices which perform such swaps. For example, we could be looking for a matrix P such that

$$P \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_4 \\ b_3 \\ b_2 \end{bmatrix} . \tag{2.37}$$

This matrix P thus swaps rows 2 and 4. It is readily confirmed that the only solution to this demand is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} . \tag{2.38}$$

Thus, P is does not have 1's at the diagonal positions in rows 2 and 4. Rather, we have 1's in row 2 column 4 and row 4 column 2.

Definition 2.4.1 (Permutation matrix)**:**
Matrices of the above form are called *permutation matrices*.

Consequence:
Each row operation in Gaussian elimination can be seen either as multiplication by elementary matrices or permutation matrices. Of course, we apply a bunch of row operations successively. We can collect these operations into a single operation once we understand the composition of these operations, which – in a sense – is the most fundamental property of matrix multiplication. Let us turn to this next.

2.4.4 Properties of matrix multiplication

Note:
Recall, that we discussed various ways to multiply matrices. Each of them has their benefits in terms of the perspective they provide. One aspect, that we want to remember at all times, is that matrices act on vectors and matrices. Therefore, we can think of matrices as *functions*. Most results about matrices are obtained by interpreting matrices as functions.

Corollary 2.4.1:
Matrix multiplications is

- associative: $A(BC) = (AB)C$.

- non-commutative: $AB \neq BA$.

Remark:
The product BA might not even exist, even when AB does. Even if both products exist, they need not be equal.

Exercise:
Find matrices A, B such that:

- AB exists but BA does not exist.
- AB and BA exist but $AB \neq BA$.

Note:
Suppose that A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$. Then consider the function

$$f(\vec{x}) = A\vec{x}\,. \tag{2.39}$$

Note that $f(\vec{x})$ is an $m \times 1$ vector, i.e. an element of $\mathbb{R}^m$. Thus, A can be treated as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Any function f obtained in this way is termed a *linear transformation*. Importantly, we can compose linear transformations:

$$\mathbb{R}^p = \{\ p \times 1 \text{ vectors }\} \xrightarrow{B} \mathbb{R}^n = \{\ n \times 1 \text{ vectors }\} \xrightarrow{A} \mathbb{R}^m = \{\ m \times 1 \text{ vectors }\}\,. \tag{2.40}$$

The resulting linear transformation is given by the matrix $B \cdot A$. We will revisit this idea of matrices as functions in section 3.5.

Definition 2.4.2 (Matrix power)**:**
Consider a square matrix A and $n \in \mathbb{Z}_{>0}$. Then we define:

$$A^n := \prod_{i=1}^{p} A\,. \tag{2.41}$$

We set $A^0 := I = \text{Diag}\,(1, 1, \ldots, 1)$ the identity matrix, which has 1's along the diagonal and 0's everywhere else.

2.4.5 Matrix inverses

Note:
As mentioned before, matrices act on vectors and matrices and give rise to the notion of linear transformations. This raises the natural question if a matrix can *undo* the action of another matrix. This leads to the notion of the inverse of a matrix.

Example 2.4.3:
Let E_{32} be the 3×3 matrix that subtracts 3 times row 2 from row 3, i.e.

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}\,. \tag{2.42}$$

What is the matrix that undoes this operation? Clearly, we want to add 3 times row 2 to row 3. Consequently, the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \tag{2.43}$$

undoes what E_{32} did.

Exercise:
Convince yourself that $A \cdot E_{32} = I = E_{32} \cdot A$.

Example 2.4.4:
Let us repeat this exercise for the 4×4 permutation matrix, which swaps rows 2 and 4, i.e.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2.44}$$

In this case, you should see right away, that swapping again would get us back to where we started. Thus P undoes what P did before.

Exercise:
Convince yourself that $P^2 = I$.

Definition 2.4.3 (Inverse of matrix)**:**
A matrix A is said to be invertible if there exists a matrix B such that

$$A \cdot B = I = B \cdot A\,. \tag{2.45}$$

We then denote the matrix B as A^{-1}.

Exercise:
Convince yourself that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.46}$$

has no inverse.

Example 2.4.5:
As another example, let us consider

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}. \tag{2.47}$$

Does this matrix have an inverse? In orther words, does there exist a 2×2 matrix B such that $A \cdot B = I = B \cdot A$?

Perhaps, you already know about determinants. In this case, you can quickly tell that the answer is no. Alternatively, let us look at the column picture. Then we notice that the columns of A are scalar multiples of each other. Therefore, there is no way to obtain either $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination. Consequently, this matrix A does not have an inverse.

Claim:
If $A\vec{x} = \vec{0}$ has a solution $\vec{x} \neq \vec{0}$, then A has no inverse.

Proof
Assume that A was invertible but $A\vec{x} = \vec{0}$ had a solution $\vec{x} \neq \vec{0}$. Then $A\vec{x} = \vec{0}$ would imply $A^{-1}A\vec{x} = A^{-1}\vec{0} = \vec{0}$. Hence, since $A^{-1}A = I$, we would find $\vec{x} = 0$ which is a contradiction to our assumption. ■

Example 2.4.6:
Let us again consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}. \tag{2.48}$$

Then it holds $A \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \vec{0}$. Hence, by the above result, A is not invertible.

Corollary 2.4.2:
An $n \times n$ matrix is invertible if and only if it has n pivots.

Exercise:
Prove this corollary.

Example 2.4.7:
We now wish to compute the inverse of a 2×2 matrix. Let us consider

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}. \tag{2.49}$$

To find its inverse, we are interested in solving the following two equations:

$$A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{2.50}$$

$$A \cdot \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.51}$$

In terms of augmented matrices, we are thus looking at

$$\left[\begin{array}{cc|c} 4 & 5 & 1 \\ 3 & 4 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|c} 4 & 5 & 0 \\ 3 & 4 & 1 \end{array}\right]. \tag{2.52}$$

Both systems share the same coefficient matrix. Rather than solving them separately, we solve them together and thus consider

$$\left[\begin{array}{cc|cc} 4 & 5 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]. \tag{2.53}$$

By Gaussian elimination we find

$$\left[\begin{array}{cc|cc} 4 & 5 & 1 & 0 \\ 0 & \frac{1}{4} & -\frac{3}{4} & 1 \end{array}\right]. \tag{2.54}$$

At this point, we could use back-substitution and compute x, y, z and w. However, let us do something different instead. Namely, let us use the blue entry to clean out the column above by a row operation. This gives

$$\left[\begin{array}{cc|cc} 4 & 0 & 16 & -20 \\ 0 & \frac{1}{4} & -\frac{3}{4} & 1 \end{array}\right]. \tag{2.55}$$

Let us now rescale both pivots to 1. This gives

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 4 \end{array}\right]. \tag{2.56}$$

The columns $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 4 \end{bmatrix}$ of the right-matrix are solution to the systems eq. (2.52). Thus, the inverse of A is given by

$$A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}. \tag{2.57}$$

Definition 2.4.4:
This procedure of cleaning out columns first from left to right, top to bottom followed by right to left, bottom to top is called *Gauss-Jordan elimination.*

Consequence:
To compute A^{-1} start from the augmented matrix $[A|I]$ and use Gauss-Jordan elimination to reach $[I|B]$. Then $A^{-1} = B$.

Claim:
It holds $(AB)^{-1} = B^{-1}A^{-1}$ and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. More generally, it holds

$$\left(\prod_{i=1}^{N} A_i\right)^{-1} = \prod_{i=1}^{n} A_{n-i}^{-1}. \tag{2.58}$$

Exercise:
Prove this.

2.4.6 Transposition

Note:
Consider the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.59}$$

Its inverse is given by

$$P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{2.60}$$

The key thing is, that P^{-1} is obtained by flipping P across its diagonal. That is, we have turned the columns into rows and the rows into columns.

Definition 2.4.5 (Transposition)**:**
The transpose of an $m \times n$ matrix A is the $n \times m$ matrix obtained by changing the rows to columns and vice versa. We denote it by A^T.

Example 2.4.8:
Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}. \tag{2.61}$$

Then it holds

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}. \tag{2.62}$$

Corollary 2.4.3:
For any permutation matrix P it holds $P^{-1} = P^T$.

Exercise:

- Prove this corollary.
- Find other matrices with the property $A^{-1} = A^T$.

Note:
For any two matrix A, B (for which $A \cdot B$ exists) it holds $(AB)^T = B^T A^T$.

Definition 2.4.6:
A matrix A with $A = A^T$ is termed a *symmetric* matrix.

Exercise:
Given a matrix B, verify or falsify that BB^T is symmetric.

2.5 (P)L(D)U-Factorization

2.5.1 L(D)U-Factorization

Note:
In many applications, one needs to solve equations $A\vec{x} = \vec{b}$ where A is fixed but $\vec{b}$ could be varying. It would thus help to "remember" the elimination moves performed during Gaussian elimination, so that one does not hve to repeat this whenever $\vec{b}$ changes. This is precisely what LU-factorization accomplishes. Before we turn to the most general case, let us assume that no row exchanges are needed in the Gauss elimination.

Example 2.5.1:
Consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix} . \tag{2.63}$$

By Gauss elimination we find $E_{21}A = U$ where

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} , \tag{2.64}$$

is the elimination matrix and

$$U = \begin{bmatrix} 1 & 4 \\ 0 & -11 \end{bmatrix} , \tag{2.65}$$

Since E_{21} is invertible, we can also write

$$A = E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 0 & -11 \end{bmatrix} \equiv L \cdot U . \tag{2.66}$$

This is the lower–upper (LU) factorization of A. Namely, we have represented A as a product of a lower triangular matrix L and an upper triangular matrix U.

Note:
The analogue of this analysis of a 3×3 matrix A is the existence of elementary matrices such that

$$E_{32}E_{31}E_{21}A = U \quad \Leftrightarrow \quad A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U . \tag{2.67}$$

Remark:
In staying with a 3×3 matrix A, we can wonder if $E_{32}E_{31}E_{21}$ or $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ does a better job remembering the elimination? In order to answer this question, let us try with an example. For convenience, let us assume that the $(3,1)$-entry of A is 0 and take

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} . \tag{2.68}$$

Note that $E_{32} \cdot E_{21}$ first subtracts 2 times row 1 from row 2. Subsequently, it subtracts 4 times row 2 from row 3. The net result is therefore given by the matrix

$$E_{32} \cdot E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 8 & -4 & 1 \end{bmatrix}. \tag{2.69}$$

The inverse is given by

$$(E_{32} \cdot E_{21})^{-1} = E_{21}^{-1} \cdot E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}. \tag{2.70}$$

Note that E_{21}^{-1} does not alter row 3. In this sense, $E_{21}^{-1} \cdot E_{32}^{-1}$ does a better job at remembering the elimination process.

Claim:

- Inverses of triangular matrices are triangular.
- Products of triangular matrices are triangular.

Exercise:
Prove this statement.

Consequence:
When the elimination process does not involve row exchanges, we can write

$$A = LU\,, \tag{2.71}$$

where U is an upper triangular matrix with the pivots of A along the diagonal and L a lower triangular matrix L with 1's along the diagonal and *multipliers below the diagonal.*

Note:
In returning to our opening problem, suppose that we want to solve $A\vec{x} = \vec{b}$, where A is fixed by $\vec{b}$ varies. In this case, write $A = LU$, so that this problem is equivalent to $L\,(U\vec{x}) = \vec{b}$. Next, set $\vec{c} = U\vec{x}$. Thereby, we are left to solve two triangular systems:

$$L \cdot \vec{c} = \vec{b}\,, \qquad U \cdot \vec{x} = \vec{c}\,. \tag{2.72}$$

This is much more efficient for varying $\vec{b}$ then solving $A\vec{x} = \vec{b}$ directly.

Remark:
On a homework assignment, you will quantify the speed of Gaussian elimination. You should find that for an $n \times n$ matrix A, this process requires $\mathcal{O}(n^3)$ operations. This is why, in the computer sciences, Gauss elimination is referred to as an algorithm of $\mathcal{O}(n^3)$.

Example 2.5.2:
Let us now consider the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 6 & 7 \\ 2 & -6 & 9 \end{bmatrix} . \tag{2.73}$$

Convince yourself, that we obtain an LU decomposition of A by application of

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} , \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} , \qquad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{2.74}$$

and that this LU-decomposition is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -2 \\ 0 & -6 & 15 \\ 0 & 0 & -17 \end{bmatrix} . \tag{2.75}$$

This factorization can also be written as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -17 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} . \tag{2.76}$$

Definition 2.5.1:
Such a factorization is termed an LDU-factorization and the D refers to the diagonal middle matrix.

2.5.2 PL(D)U-Factorization

Remark:
Recall that in claiming $A = LU$ we assumed that there are no row exchanges needed in the Gauss elimination. We are now ready to generalize to arbitary matrices.

Corollary 2.5.1:
A square matrix A can be factored as

$$PA = LU \, , \tag{2.77}$$

where P is some permutation matrix (cf. section 2.4.3) and L, U are as above. This yields

$$A = P^T LU \, . \tag{2.78}$$

Recall that $P^{-1} = P^T$ is again a permutation matrix. One terms such a factorization of A a PLU-factorization.

Remark:
Recall that any matrix obtained by permuting the rows of the identity matrix is a *permutation matrix*. Consequently, permutation matrices perform permutations of the rows of a given matrix. For example, the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} . \tag{2.79}$$

performs the permutation

$$(\text{row } 1) \rightarrow (\text{row } 4) \rightarrow (\text{row } 3) \rightarrow (\text{row } 2) \rightarrow (\text{row } 1) . \tag{2.80}$$

3 Vector Spaces and Linear Subspaces

3.1 Vector Spaces

Note:
When we say *vector space*, we use the term *space* to emphasize that we are studying a collection of vectors. But not just any collection. There are constraints imposed on this collection.

Example 3.1.1:
An example of a vector space is

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, x, y \in \mathbb{R} \right\}. \tag{3.1}$$

In other words, the space $\mathbb{R}^2$ consists of all 2-dimensional vectors. Also, recall that we perform two operations with vectors:

- scaling by real numbers,
- component-wise addition.

If we scale $\vec{v} \in \mathbb{R}^2$ by a scalar $c \in \mathbb{R}$, then $c \cdot \vec{v} \in \mathbb{R}^2$. Likewise, if $\vec{v}, \vec{w} \in \mathbb{R}^2$, then $\vec{v} + \vec{w} \in \mathbb{R}^2$. More general, any linear combination of vectors in $\mathbb{R}^2$ is a vector in $\mathbb{R}^2$.

Note:
Clearly, there is nothing special about $\mathbb{R}^2$. We could have said the same thing about $\mathbb{R}^3$, i.e. the collection of vectors with 3 (real) components. This generalizes as follows.

Remark:
The following is the abstract definition of a vector space over a field F. I am presenting it here, because I believe that this level of abstraction emphasizes the important structures of a vector space in the best way. We will exemplify all of this in vector spaces over the real number $\mathbb{R}$, and much later in the course over $F = \mathbb{C}$. Therefore, in the following definition(s), you may think of F as $\mathbb{R}$, $\mathbb{C}$, or for computer implementations as $\mathbb{Q}$ (or the field extension $\mathbb{Q} + i\mathbb{Q}$).

Definition 3.1.1 (Vector space)**:**
A vector space over a field F is a triple $(V, +, \cdot)$ of a set V and operations

$$+\colon V \times V \to V\,,\ (\vec{u}, \vec{v}) \mapsto \vec{u} + \vec{v}\,, \tag{3.2}$$
$$\cdot\colon F \times V \to V\,,\ (c, v) \mapsto c \cdot \vec{v}\,. \tag{3.3}$$

which satisfy the following properties:

- Associativity of addition:
  For all $\vec{u}, \vec{v}, \vec{w} \in V$ it holds $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
- Commutativity of addition:
  For all $\vec{u}, \vec{v} \in V$ it holds $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
- Existence of neutral element of addition:
  There exists $\vec{n} \in V$ such that for all $\vec{u} \in V$ it holds $\vec{u} + \vec{n} = \vec{u}$.
- Existence of an inverse element under addition:
  For every $\vec{u} \in V$ there exists $\tilde{\vec{u}}$ such that $\vec{u} + \tilde{\vec{u}} = \vec{n}$.
- Compatibility of scalar multiplication and vector addition:
  For all $\vec{u} \in V$ and all $c_1, c_2 \in F$ it holds $c_1 \cdot (c_2 \cdot \vec{u}) = (c_1 \cdot c_2) \cdot \vec{u}$
- Neutral element of scalar multiplication:
  There exists $i \in F$ such that for all $\vec{u} \in V$ it holds $i \cdot \vec{u} = \vec{u}$.
- Distributivity laws:
  For all $c_1, c_2 \in F$ and all $\vec{u}, \vec{v} \in V$ the

$$c_1 \cdot (\vec{u} + \vec{v}) = c_1 \cdot \vec{u} + c_1 \cdot \vec{v}\,, \qquad (c_1 + c_2) \cdot \vec{u} = c_1 \cdot \vec{u} + c_2 \cdot \vec{u}\,. \tag{3.4}$$

We term $+\colon V \times V \to V$ the *vector addition* and $\cdot\colon F \times V \to V$ the *scalar multiplication*. Moreover, we term elements of V vectors and element of F scalars.

Note:
The vector space $\mathbb{R}^n$ as vector space over the field $F = \mathbb{R}$ is the triple $(\mathbb{R}^n, +, \cdot)$ with

$$+\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\,, \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) \mapsto \begin{bmatrix} u_1 +_{\mathbb{R}} v_1 \\ u_2 +_{\mathbb{R}} v_2 \\ \vdots \\ u_n +_{\mathbb{R}} v_n \end{bmatrix}, \tag{3.5}$$

$$\cdot\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\,, \left(c, \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) \mapsto \begin{bmatrix} c \cdot_{\mathbb{R}} u_1 \\ c \cdot_{\mathbb{R}} u_2 \\ \vdots \\ c \cdot_{\mathbb{R}} u_n \end{bmatrix}, \tag{3.6}$$

where $+_{\mathbb{R}}$, $\cdot_{\mathbb{R}}$ denotes addition and multiplication of real numbers, respectively.

Exercise:
Verify that $(\mathbb{R}^n, +, \cdot)$ satisfies all properties in the definition of a vector space.

Remark:
For notational simplicity, we denote $(\mathbb{R}^n, +, \cdot)$ simply as $\mathbb{R}^n$ for the rest of this course.

Example 3.1.2:
The set $(\{0\}, +, \cdot)$ is a vector space over $\mathbb{R}$ via the following operations

$$+\colon \{0\} \times \{0\} \to \{0\}\,,\ (0,0) \mapsto 0\,, \qquad \cdot\colon \mathbb{R} \times \{0\} \to \{0\}\,,\ (c,0) \mapsto 0\,. \tag{3.7}$$

We call this the trivial vector space.

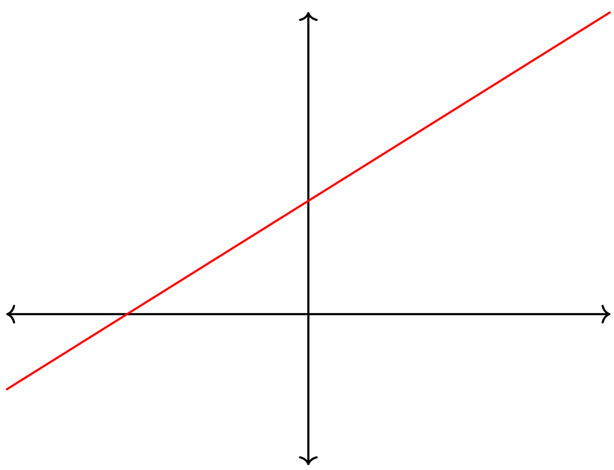
Note:
$\{0\} \subseteq \mathbb{R}^n$. This indicates, that we may want to think of the trivial vector space as a linear subspace of $\mathbb{R}^n$. More generally, we can ask if a vector space V over a field F contains linear subspaces. To this end, let us first define the notation of a linear subspace.

Definition 3.1.2 (Linear subspace)**:**
Be $(V, +, \cdot)$ a vector space over a field F. A linear subspace W of V is a subset $W \subseteq V$ such that $(W, +|_W, \cdot|_W)$ is a vector space over F.

Example 3.1.3:
For any vector space $(V, +, \cdot)$ over a field F, $\{0\}$ and V are linear subspaces.

Note:
At this point we may wonder how we can visualize a subspace W of a vector space V. To this end, we recall that for any two vectors $\vec{u}, \vec{v} \in W$ it must holds that $c_1\vec{u} + c_2\vec{v} \in W$. To fully appreciate this observation, let us exemplify its meaning by looking at examples in $V = \mathbb{R}^2$.

Example 3.1.4:
Consider the collection of points along the following red line:

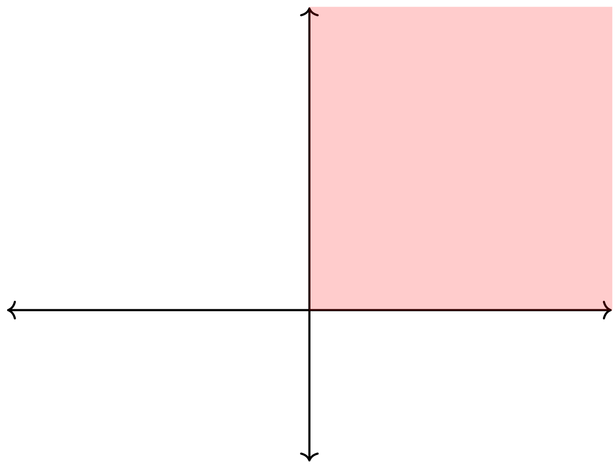


(3.8)

This clearly is a subset of $\mathbb{R}^2$. But is it a linear subspace of $\mathbb{R}^2$? The answer is *no*! Think about what happens if we scale a vector by 0. Hence, any linear subspace of $\mathbb{R}^2$ must contain $\vec{0}$.

Note:
More abstractly, $\vec{0} \in \mathbb{R}^2$ is the neutral element of the vector addition, and this remains true in any linear subspace $W \subseteq \mathbb{R}^2$.

Example 3.1.5:
As another example, let us consider the points W in the first quadrant:



(3.9)

Is W a linear subspace of $\mathbb{R}^2$? Clearly, W contains $\vec{0}$. Also, if $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$. However, we run into the following prblem. If $\vec{0} \neq \vec{v} \in W$ then $-\vec{v} \notin W$.

Example 3.1.6:
A line through the origin is a linear subspace of $\mathbb{R}^2$. Similarly, in $\mathbb{R}^n$, any line through the origin is a linear subspace. These observations lead to the following corollary.

Corollary 3.1.1:
Let $(V, +, \cdot)$ be a vector space over a field F. $W \subseteq V$ is a linear subspace of V if and only if for any two $c_1, c_2 \in F$ and any $\vec{u}, \vec{v} \in W$ it holds $c_1\vec{u} + c_2\vec{v} \in W$.

Consequence:
We can now list all linear subspaces of $\mathbb{R}^2$:

- the trivial linear subspace $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$,
- all lines through the origin,
- $\mathbb{R}^2$.

Thus, there are relatively few linear subspaces of $\mathbb{R}^2$. This shows that being a linear subspace is a *rather rigid constraint.*

Exercise:
Find all linear subspaces of $\mathbb{R}^3$. Extend this to $\mathbb{R}^n$.

Example 3.1.7:
Here is a somewhat 'exotic' example of a vector space. Let $M = \mathbb{M}(2 \times 2, \mathbb{R})$ be the set of all 2×2 matrices with real entries. Consider the following operations:

$$+\colon M \times M \to M\,,\ (A, B) \mapsto \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}, \tag{3.10}$$

$$\cdot\colon \mathbb{R} \times M \to M\,,\ (c, A) \mapsto \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} \\ c \cdot a_{21} & c \cdot a_{22} \end{bmatrix}. \tag{3.11}$$

It follows that $(M, +, \cdot)$ is a vector space over $\mathbb{R}$.

Exercise:
Let Pol_n denote the set of a polynomials with real coefficients in (the formal variable) x whose degree is *at most* n. Find operations $+_P$, $\cdot_P$ such that $(\mathrm{Pol}_n, +_P, \cdot_P)$ is a vector space over $\mathbb{R}$.

Note:
Both Pol_n and $\mathbb{R}^m$ are vector spaces over $\mathbb{R}$. In fact, there is a relation between them.

Remark:
Relations among vector spaces are encoded by so-called vector space homomorphisms, which is greek for *structure preserving maps.*

Definition 3.1.3:
Consider two vector spaces $(A, +_A, \cdot_A)$ and $(B, +_B, \cdot_B)$ over a field F. A vector space homomorphism from A to B is a map $\varphi\colon A \to B$ which satisfies for all $c \in F$ and $x, y \in A$ that

$$\varphi\left(c \cdot_A (x +_A y)\right) = c \cdot_B \varphi(x) +_B c \cdot_B \varphi(y)\,. \tag{3.12}$$

Example 3.1.8:
The canonical embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ is a vector space homomorphism. It is injective but not surjective. Likewise, the projection $\mathbb{R}^{n+1} \twoheadrightarrow \mathbb{R}^n$ is a surjective but not injective vector space homomorphism.

Definition 3.1.4:
A vector space homomorphism $\varphi\colon A \to B$ which at the same time is a bijection of the underlying sets is a vector space isomorphism. We write $A \cong B$ and then consider A and B as essentially the same vector spaces.

Example 3.1.9:
The identity $\mathbb{R}^n \xrightarrow{id} \mathbb{R}^n$ is a vector space homomorphism.

Exercise:
Show that $\mathrm{Pol}_n \cong \mathbb{R}^m$ for a suitable $m \in \mathbb{Z}_{\geq 0}$.

3.2 Column space and nullspace

3.2.1 The column space

Example 3.2.1:
Consider the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{b}\,. \tag{3.13}$$

We may ask two questions:

1. Given $\vec{b}$, does this equation have a solution?
2. What are all possible $\vec{b}$ for which the system has a solution?

We can recast these questions in the language of linear combinations:

1. Given $\vec{b}$, is there a linear combination of the columns of A that equals $\vec{b}$?
2. Can we find all the vectors that are linear combinations of the columns of A?

Therefore, we want to understand the set of linear combinations of the columns of A.

Definition 3.2.1:
For $A \in \mathbb{M}(m \times n, \mathbb{R})$, we denote the set of linear combinations of the columns of A as the *column space* $C(A)$.

Corollary 3.2.1:
$C(A)$ is a real vector space.

Exercise:
Prove this statement.

Example 3.2.2:
Let us return to the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}. \tag{3.14}$$

$C(A)$ is then a linear subspace of $\mathbb{R}^4$. $C(A)$ is more than a line, since the first two columns of A are not parallel to each other as vectors. Whether $C(A)$ is more than a plane may not be immediate at this point. We will return to this question momentarily. What should be clear at this point is that $C(A)$ is not $\mathbb{R}^4$.

Exercise:
Find $\vec{v} \in \mathbb{R}^4$ with $\vec{v} \notin C(A)$.

3.2.2 The nullspace

Note:
There is another interesting vector space attached to matrices that we have to discuss. Namely, given an $m \times n$ matrix A, we can consider the set of solution to $A\vec{x} = \vec{0}$.

Definition 3.2.2 (Nullspace)**:**
The *nullspace* of a matrix A, denoted by $N(A)$, is the set of solution to $A\vec{x} = \vec{0}$.

Corollary 3.2.2:
$N(A)$ is linear subspace of $\mathbb{R}^n$.

Proof
Any two $\vec{v}, \vec{w} \in N(A)$ satisfy $A\vec{v} = A\vec{w} = \vec{0}$. Consequently, for any $c, d \in \mathbb{R}$ we have

$$A\left(c\vec{v} + d\vec{w}\right) = Ac\vec{v} + Ad\vec{w} = cA\vec{v} + dA\vec{w} = \vec{0}\,. \tag{3.15}$$

The claim now follows from corollary 3.1.1. ■

Example 3.2.3:
Let us consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}\,. \tag{3.16}$$

Then any scalar multiple of $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is contained in $N(A)$. In particular,

$$A \cdot \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}\,, \quad \text{for } \vec{v} \in \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\,, r \in \mathbb{R} \right\}\,. \tag{3.17}$$

Consequence:
The space $N(A)$ is crucial to find *all* solutions to a linear system. Before we can discuss this important application, we first have to understand the nullspace better. In particular, we have to be able to compute it.

Example 3.2.4:
Let us consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix}\,. \tag{3.18}$$

We are interested in $N(A)$, i.e. the soutions to $A\vec{x} = \vec{0}$. We will essentially execute the same elimination procedure as before. However, since the RHS is $\vec{0}$, we will not carry it along in our computation:

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: U\,. \tag{3.19}$$

At this point, there are no more eliminations to be performed. We say that the final matrix U is in *(row) echelon form.*

Example 3.2.5:
The following matrices are *not* in (row) echelon form:

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 0 & 0 \\ 3 & 9 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.20}$$

But, the following matrices are in (row) echelon form:

$$D = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}. \tag{3.21}$$

Definition 3.2.3 (Row rank of a matrix)**:**
Any matrix A can, by use of elementary row operations, be turned into a matrix U which is in echelon form. We call the number of pivots of U the row rank $\text{rk}_R(A)$ of A.

Example 3.2.6:
Since

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: U\,. \tag{3.22}$$

and U has 2 pivots, we conclude that $\text{rk}(A) = 2$.

Note:
Again consider

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: U\,. \tag{3.23}$$

We make two observations:

- The bottom row of 0s in U tells us, that the third row of A is a linear combination of the first and second row of A.

- The columns that do not contain pivots can be expressed in terms of the columns take come before them on the left.

Definition 3.2.4 (Pivot and free columns)**:**
Let U be a matrix in echelon form. Then we distinguish two types of columns:

- Columns which have a pivot are called *pivot columns*.
- All other columns are called *free columns*.

Example 3.2.7 (Continuation)**:**
For the matrix

$$U = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.24}$$

the equations corresponding to $U\vec{x} = \vec{0}$ are given by

$$x_1 + 3x_2 + 3x_3 + 3x_4 = 0\,, \tag{3.25}$$

$$3x_3 + 6x_4 = 0\,. \tag{3.26}$$

x_2, x_4 are refered to as *free* variables.

Comment:
Let us comment on the terminology of *free* variables. Namely, if we randomly assign values to x_2 and x_4, then the values of x_1 and x_3 are uniquely determined. Therefore, we term x_1, x_3 the *pivot* variables.

Example 3.2.8 (Continuation)**:**
For example, let us assign $x_2 = 1$ and $x_4 = 0$. Then $x_3 = 0$ and $x_1 = -3$. Consequently,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)\,. \tag{3.27}$$

In fact, any scalar multiple of this vector is contained in the nullspace. Similarly, we can also try $x_2 = 0$ and $x_4 = 1$. Then, $x_3 = -2$, $x_1 = 3$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)\,. \tag{3.28}$$

These two *special solutions* enable us to describe all vectors in $N(A)$. Namely, they are all linear combinations of these special solutions! Therefore

$$N(A) = \left\{ c \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, c, d \in \mathbb{R} \right\}. \tag{3.29}$$

Note:
For an $m \times n$-matrix A with rank r, there are r pivot variables and $n - r$ free variables.

Remark:
We can reduce the echelon form further. For example for

$$U = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.30}$$

we can clean the entries above the pivots, just as we did for Gauss-Jordan elimination:

$$U = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.31}$$

This final matrix is called the *reduced (row) echelon form* (RREF) of A. All its pivots are equal to 1 and there are 0s above and below the pivots.

Exercise:
Can you see the special solutions to

$$A\vec{x} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} \cdot \vec{x} = \vec{0}, \tag{3.32}$$

in the RREF of the matrix A?

Example 3.2.9:
Let us now consider the transposed matrix A^T. For this matrix we have

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 3 & 9 & 12 \\ 3 & 12 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.33}$$

This shows $\text{rk}(A^T) = 2$ and

$$N(A) = \left\{ c \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, c \in \mathbb{R} \right\}. \tag{3.34}$$

Exercise:
Find the special solution to $A^T\vec{x} = \vec{0}$ from the RREF of A^T.

Definition 3.2.5 (Column rank of a matrix)**:**
Given a matrix A, then we can bring A by use of elementary column operations into a matrix U which is in column echelon form. We call the number of pivots of U the column rank $\text{rk}_C(A)$ of A.

Claim:
For any matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$ it holds $\mathrm{rk}_R(A) = \mathrm{rk}_C(A)$.

Proof
Neither the row nor the column rank are altered by elementary row nor column operations. By use of such elementary row and column operations, we can bring A into the form U of an identity matrix, possibly bordered by rows and columns of zero. It follows that row and column ranks coincide with the number of non-zero entries of U. ■

Remark:
The rank of a matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$ tells us how many "independent" solutions $N(A)$ contains, namely exactly $n - \mathrm{rk}(A)$. We will make precise what we mean by "independent" when we study the precise formulation of this statement, the so-called *rank-nullity theorem*.

3.2.3 All solutions to a linear system

Note:
Let us now return to the question on how to find *all* solutions to $A\vec{x} = \vec{b}$. We want an approach that allows us to tell if there are no solutions, a unique solution or infinitely many solutions. As anticipated before, we will find that the nullspace is crucial in this study.

Example 3.2.10:
We consider the linear system

$$x_1 + 3x_2 + 3x_3 + 3x_4 = b_1\,, \tag{3.35}$$
$$2x_1 + 6x_2 + 9x_3 + 12x_4 = b_2\,, \tag{3.36}$$
$$3x_1 + 9x_2 + 12x_3 + 15x_4 = b_3\,. \tag{3.37}$$

Note that row 1 + row 2 = row 3 for the left hand sides. This already tells us something about the RHS, if we are to solve this system. Namely, for instance, if $b_1 = 1$ and $b_2 = 2$, then b_3 must equal 3 if this system is to have at least one solution.

Of course, we would like elimination to discover this fact for us. To this end, we record this linear system in the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & b_1 \\ 2 & 6 & 9 & 12 & b_2 \\ 3 & 9 & 12 & 15 & b_3 \end{array}\right]. \tag{3.38}$$

Upon elimination we find

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & b_1 \\ 2 & 6 & 9 & 12 & b_2 \\ 3 & 9 & 12 & 15 & b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & b_1 \\ 0 & 0 & 3 & 6 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right]. \tag{3.39}$$

Indeed, this shows us, that this system has no solution unless $b_3 - b_1 - b_2 = 0$.

Exercise:
Find the linear subspace of $\mathbb{R}^3$ which is generated by the 4 column vectors of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} . \tag{3.40}$$

Consequence (Solvability conditions on $\vec{b}$)**:**
Consider the linear system $A\vec{x} = \vec{b}$:

- This system is solvable if and only if $\vec{b} \in C(A)$.
- If a linear combination of rows of A gives a zero row, and the same linear combination of the entries of $\vec{b}$ gives a non-zero value, then the system is not solvable.

Example 3.2.11 (Continuation)**:**
Let us continue to study the linear system eq. (3.38). However, let us proceed by using a specific vector $\vec{b}$, for which the system has a solution. We take $b_1 = 1$, $b_2 = 4$ and $b_3 = 5$. Then the system is represented by

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & 1 \\ 0 & 0 & 3 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] . \tag{3.41}$$

To find all solutions to this system, we execute three steps:

1. Set all free variables to zero and find *one* solution for the resulting system of the pivot variables. For the above system, we set $x_2 = x_4 = 0$ and find

$$x_1 + 3x_3 = 1 \,, \tag{3.42}$$
$$3x_3 = 2 \,. \tag{3.43}$$

   The unique solution to this system is given by $x_3 = \frac{2}{3}$ and $x_1 = -1$. We conclude that a particular solution to the linear system eq. (3.38) is given by

$$\vec{x}_{\text{particular}} = \begin{bmatrix} -1 \\ 0 \\ 2/3 \\ 0 \end{bmatrix} . \tag{3.44}$$

2. Find all vectors in the nullspace of A, i.e. compute $N(A)$.
3. Every solution to eq. (3.38) is then given by

$$\vec{x} = \vec{x}_{\text{particular}} + \vec{x}_{\text{null}} \,, \tag{3.45}$$

   where $\vec{x}_{\text{particular}}$ is the solution in eq. (3.44) and $\vec{x}_{\text{null}}$ is **any** vector in $N(A)$.

Exercise:
Why does that work? Hint: $A(\vec{x}_p + \vec{x}_n) = A\vec{x}_p + A\vec{x}_n$.

Example 3.2.12 (Continuation II)**:**
Every solution to eq. (3.38) is given by

$$\vec{x} \in \left\{ \begin{bmatrix} -1 \\ 0 \\ 2/3 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} . \tag{3.46}$$

Geometrically, we obtain a point and a linear subspace of $\mathbb{R}^4$ isomorphic to $\mathbb{R}^2$. This linear subspace is the nullspace $N(A)$. Thus, the complete set of solutions is an **affine** *plane in* $\mathbb{R}^4$.

Note:
The set of all solutions in eq. (3.46) is *not* a linear subspace of $\mathbb{R}^4$. It is a **translation** of the subspace $N(A) \subseteq \mathbb{R}^4$, i.e. a 'shifted' version of this linear subspace.

Corollary:
The rank r of any $A \in \mathbb{M}(m \times n, \mathbb{R})$ satisfies the inequalities

$$r \leq m\,, \qquad r \leq n\,. \tag{3.47}$$

Exercise:
Prove this statement.

Corollary (Matrices with full column rank)**:**
Consider $A \in \mathbb{M}\,(m \times n, \mathbb{R})$ with full column rank, that is $r = n$. Then the following holds true:

- There are no free variables and $N(A) = \{\vec{0}\}$.
- If there is a solution to $A\vec{x} = \vec{b}$, then this solution is unique.

$\Rightarrow$ $A\vec{x} = \vec{b}$ either has no or a unique solution.

Example 3.2.13:
Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} . \tag{3.48}$$

What is the rank of A? What is the RREF for this matrix? Convince yourself, that the RREF is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} . \tag{3.49}$$

Hence, indeed $\text{rk}(A) = 2$. It should also be obvious that this system is not always solvable. However, if it is, then this solution is unique.

Corollary (Matrices with full row rank)**:**
Consider $A \in \mathbb{M}(m \times n, \mathbb{R})$ with full row rank, that is $r = m$. Then the following holds:

- There is a pivot in every row.
- $A\vec{x} = \vec{b}$ always has (at least) one solution.

$\Rightarrow$ $A\vec{x} = \vec{b}$ either has one or infinitely many solutions.

Example 3.2.14:
Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 5 & 4 \end{bmatrix}. \tag{3.50}$$

The corresponding RREF is

$$\begin{bmatrix} 1 & 0 & -1 & -5 \\ 0 & 1 & 4 & 3 \end{bmatrix}. \tag{3.51}$$

Clearly, $A\vec{x} = \vec{b}$ is always solvable.

Corollary (Matrices with full rank)**:**
Consider $A \in \mathbb{M}(m \times n, \mathbb{R})$ with full rank, that is $r = m = n$. Then the following holds true:

- A is a square matrix,
- there is a pivot in every row and column,
- the RREF is equal to the identity matrix,

$\Rightarrow$ $A\vec{x} = \vec{b}$ always has a *unique* solution.

3.3 Linear (in)dependence, spans, basis and the dimension of vector spaces

3.3.1 Linear (in)dependence

Note:
Suppose $A \in \mathbb{M}(m \times n, \mathbb{R})$ with $m < n$. Then there are non-zero solutions to $A\vec{x} = \vec{0}$. Note that in this case we have more unknowns than equations. Thus, there will be free variables! This fact will become handy, momentarily.

Definition 3.3.1 (Linear independence)**:**
Vectors $\vec{v}_1, \ldots, \vec{v}_n$ are *linearly independent* if no linear combination gives $\vec{0}$, except the zero combination. That is, the vectors are linearly independent if

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}\,, \tag{3.52}$$

implies $c_1 = c_2 = \cdots = c_n = 0$.

Definition 3.3.2 (Linear dependence)**:**
Vectors $\vec{v}_1, \ldots, \vec{v}_n$ which are not *linearly independent* are said to be *linearly dependent.*

Example 3.3.1:
Consider the vectors $\vec{v}_1$ and $\vec{v}_2$ such that $\vec{v}_2 = 2\vec{v}_1$. Let us investigate if these vectors are linearly independent. Thus, consider $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. We note that this is solved by $c_1 = 2$ and $c_2 = -1$. Consequently, $\vec{v}_1$, $\vec{v}_2$ are *not* linearly independent.

Example 3.3.2:
How about $\vec{v}$ and $\vec{0}$, are they linearly independent? No, they are not. Namely, the equation $c_1\vec{v} + c_2\vec{0} = \vec{0}$ can be solved by $c_1 = 0$ and $c_2 = 1$.

Corollary:
Any finite set of vectors which contains $\vec{0}$ is linearly dependent.

Exercise:
Convince yourself that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.53}$$

are linearly independent. Likewise, show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.54}$$

are linearly independent.

Corollary:

Consider $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$ and the matrix $A = \begin{bmatrix} | & \cdots & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & \cdots & | \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R})$. Then:

$$\vec{v}_1, \ldots, \vec{v}_n \text{ linearly independent} \quad \Leftrightarrow \quad N(A) = \left\{\vec{0}\right\}. \tag{3.55}$$

Exercise:
Prove this corollary.

Note:
In particular, $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent iff $N(A) \neq \left\{\vec{0}\right\}$, i.e. there exists a non-zero $\vec{c}$ with $A\vec{c} = \vec{0}$.

Corollary:

Consider $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$ and the matrix $A = \begin{bmatrix} | & \cdots & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & \cdots & | \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R})$. Then the following holds true:

- $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent iff A has full column rank.
- $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent iff A does not have full column rank.

3.3.2 Spans, Basis and Dimension

Definition 3.3.3 (Span)**:**
Consider a vector space $(V, +, \cdot)$ over a field $\mathbb{F}$ and vectors $\vec{v}_1, \ldots, \vec{v}_n \in V$. Then $\text{Span}(\vec{v}_1, \ldots, \vec{v}_n)$ is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_n$:

$$\text{Span}_{\mathbb{F}}(\vec{v}_1, \ldots, \vec{v}_n) = \left\{ \sum_{i=1}^{n} c_i \cdot \vec{v}_i \, , \, c_i \in \mathbb{F} \right\} . \tag{3.56}$$

Corollary:
For the case of $\mathbb{R}^m$ we thus consider $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$. Then

$$\text{Span}_{\mathbb{R}}(\vec{v}_1, \ldots, \vec{v}_n) = \left\{ \sum_{i=1}^{n} c_i \cdot \vec{v}_i \, , \, c_i \in \mathbb{R} \right\} . \tag{3.57}$$

In particular, $\text{Span}(\vec{v}_1, \ldots, \vec{v}_n)$ is a linear subspace of $\mathbb{R}^m$.

Definition 3.3.4:
Consider a vector space $(V, +, \cdot)$ over $\mathbb{R}$. A collection of vectors $\mathcal{G} \subseteq V$ with $\text{Span}(\mathcal{G}) = V$ is termed a *generating set of* V.

Note:
Every vector space admits a generating set $\mathcal{G}$. In general, $\mathcal{G}$ need not be finite. Convince yourself that this is for example the case for $\mathbb{R}[x]$ – the polynomials in the variable x and coefficients in $\mathbb{R}$. However, the linear subspace Pol_n is 'small' and admits a finite generating set.

Example 3.3.3:
The columns of a matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$ span the column space $C(A)$. Hence, these columns form a generating set $\mathcal{G}$ for the column space $C(A)$ of A. We can wonder if there are smaller generating sets, i.e. if we could span the column space $C(A)$ with fewer columns. In general, the answer depends on the the matrix in question. However, this question leads us to the definition of a very economic generating set.

Definition 3.3.5 (Basis)**:**
Let $(V, +, \cdot)$ be a vector space. A (finite) generating set $\mathcal{G}$ which is linearly independent is termed a *basis of* V.

Example 3.3.4:
Consider the vector space $\mathbb{R}^3$. A generating set is

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} . \tag{3.58}$$

Note that $\mathcal{G}$ is not a basis since these vectors are linearly dependent. This follows for example from the fact that the following matrix does not have full column rank:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{3.59}$$

However, the following set $\mathcal{B}$ is indeed a basis of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} . \tag{3.60}$$

Note:
For the most part of this lecture, we focus on vector space which admit a finite generating set. We then have the following important statement.

Corollary:
Any collection $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is a basis of $\mathbb{R}^n$ iff $A = \begin{bmatrix} | & \cdots & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & \cdots & | \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R})$ is invertible.

Exercise:
Prove this statement.

Corollary:
Be $(V, +, \cdot)$ be a vector space over a field F which admits a finite generating set. Then every basis $\mathcal{B}$ of $(V, +, \cdot)$ is finite and all basis consist of the same number of elements.

Exercise:
Convince yourself that any two basis $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathbb{R}^m$ satisfy $|\mathcal{B}_1| = |\mathcal{B}_2|$.

Definition 3.3.6:
Be $(V, +, \cdot)$ a vector space over a field F which admits a finite generating set. We term the cardinality of a basis $\mathcal{B}$ of $(V, +, \cdot)$ the dimension $\dim_{\mathbf{F}}(V)$ of V, i.e.

$$\dim_{\mathbf{F}}(V) := |\mathcal{B}| . \tag{3.61}$$

Note:
The dimension depends on the field $\mathbb{F}$. For example, as sets we have $\mathbb{R}^2 \cong \mathbb{C}$. However, when we consider $\mathbb{R}^2$ as vector space over $\mathbb{R}$ it follows $\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$. In contrast, $\dim_{\mathbb{C}}(\mathbb{C}) = 1$.

Example 3.3.5:
Convince yourself, that

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}, \tag{3.62}$$

is not a basis of $\mathbb{R}^3$. We can however wonder if we can find a vector $\vec{v}$ such that $S' = S \cup \{\vec{v}\}$ is a basis of $\mathbb{R}^3$.

Theorem 3.3.1 (Basis Extension Theorem)**:**
Be $(V, +, \cdot)$ a finite dimensional vector space over $\mathbb{F}$. Then, any collection of $\vec{v}_1, \ldots \vec{v}_n \in V$ of linearly independent vectors can be extended to a basis $\mathcal{B}$ of V.

Example 3.3.6:
Note that $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is not a generating set of $\mathbb{R}^3$ since $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin \mathrm{Span}_{\mathbb{R}}(\mathcal{S})$.
Let us therefore consider

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}. \tag{3.63}$$

Indeed, $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

3.3.3 Computing dimension and basis of column spaces

Note:
Let us now return to talking about column spaces. Given $A \in \mathbb{M}(m \times n, \mathbb{R})$, how can we find a basis for $C(A)$ as well as its dimension? We will now try to give an answer to this question.

Example 3.3.7:
Consider

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \\ 4 & 8 & -1 & 3 \\ 4 & 8 & -1 & 3 \end{bmatrix}. \tag{3.64}$$

Let us quickly compute the row reduced form:

$$A \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -5 & -5 \end{bmatrix} \to \begin{bmatrix} \color{blue}{1} & 2 & 1 & 2 \\ 0 & 0 & \color{blue}{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.65}$$

There are two important facts to note at this stage:

- As column 2 does not possess a pivot, but column 1 does, we infer that column 2 can be written in terms of column 1.

- As column 4 does not contain a pivot, we infer that it can be written in terms of pivot columns 1 and 3.

Therefore, to generate $C(A)$, we only need to consider linear combinations of columns 1 and 3. Equivalently, we can say that columns 1 and 3 of A span $C(A)$. Even more, since they are pivot columns, they are linearly independent.

Together, these conclusions shows, that columns 1 and 3 form a basis $\mathcal{B}$ of $C(A)$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix} \right\}. \tag{3.66}$$

Note:
To get the basis elements of $C(A)$, we look at the columns of the original matrix A, not the row echelon form of A.

Exercise:
Explain why we look at the columns of the original matrix A and not its row echelon form, to get a basis of $C(A)$.

Consequence:
For any $A \in \mathbb{M}(m \times n, \mathbb{R})$, the pivot columns of A form a basis for $C(A)$. The dimension of $C(A)$ is equal to the column rank of A.

Exercise:
Pick a finite collection of vectors in $\mathbb{R}^m$ and identify a basis of the space spanned by these vectors.

3.4 Two other important vector spaces of a matrix

Definition 3.4.1 (Row space and left nullspace)**:**
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then we define:

- The row space $R(A)$ of A is the column space $C(A^T)$ of A^T.

- The left null space of A is the null space $N(A^T)$ of A^T.

Corollary:
The row space $R(A)$ and the left null space $N(A^T)$ are vector spaces:

- The row space $R(A)$ of A is a linear subspace of $\mathbb{R}^n$.

- The left null space is a linear subspace of $\mathbb{R}^m$.

Note:
Suppose $\vec{v} \in N(A^T)$. Then, by definition $A^T\vec{y} = \vec{0}$. Upon transposition, this is equivalent to $\vec{y}^T \cdot A = \vec{0}^T$. In this sense, $\vec{y}$ multiplies A from the left to give zero – hence the name *left* nullspace.

Consequence:
Strictly speaking, we may thus term the 'standard' nullspace the *right nullspace*.

Example 3.4.1:
Let us compute bases and dimensions of all these spaces in a single example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix} . \tag{3.67}$$

Let us start with the column and row space. We first recall the following general facts:

- $C(A) \subseteq \mathbb{R}^3$:
  We know that a basis is given by the pivot columns of A. The dimension of $C(A)$ matches the (column) rank r of A.

- $C(A^T) \subseteq \mathbb{R}^4$:
  This should not be too bad, since we already know how to deal with column spaces in general. However, by brute force, we are required to apply elimination to A and A^T. Luckily, it turns out that elimination of A already allows us to find a basis and the dimension of the row space of A^T.

Here is how this works. First, apply elimination to A:

$$A \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} := U . \tag{3.68}$$

The rows of U are definitely linear combinations of the rows of A, therefore

$$C(A^T) = R(A) = R(U) . \tag{3.69}$$

Furthermore, not that the non-zero rows of U are linearly independent. Therefore, since they span the row space of A, we find:

- A basis of the row space of A is given by the non-zero rows of U.

- The dimension of $R(A)$ is equal to the row rank of A.

Be mindful that row operations preserve row spaces, but not column spaces – $C(A)$ and $C(U)$ are clearly distinct. That said, let us look at the null spaces:

- $N(A) \subseteq \mathbb{R}^3$:
  This is also quite easy, given what we know. Firstly, can we span the nullspace from particular solutions? The answer is yes. Every assignment of numbers to the free variables can be interpreted as a linear combination of assignments where we set one variable to 1 and the rest to 0.

  Are these particular solutions linearly independent? Again, the answer is yes. An informal argument is to realize that the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{3.70}$$

  are linearly independent. Thus, we conclude the following:
  - A basis for $N(A)$ is given by the particular solutions.
  - The dimension of $N(A)$ is equal to the number of free variables, i.e. $n - r$ where r is the column rank of A.

- The left nullspace $N(A^T) \subseteq \mathbb{R}^4$:
  Note that this corresponds to solving

$$A^T\vec{y} = \vec{0} \quad \Leftrightarrow \quad \vec{y}^T A = \vec{0}^T \,. \tag{3.71}$$

  Since $\vec{y}^T$ is a row vector, $\vec{y}^T A$ is also a row, given by a specific linear combination of the rows of A. As an augmented matrix, we may write this thus as:

$$A = \left[\begin{array}{cccc} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]. \tag{3.72}$$

  Let us apply row eliminations. Then we find

$$A \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.73}$$

  Thus, we went from

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \to \begin{bmatrix} R_1 \\ R_1 - R_2 \\ R_3 + R_2 - 2R_1 \end{bmatrix}. \tag{3.74}$$

Which matrix does perform this transformation? Well, the following does the job:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} . \tag{3.75}$$

The third row forms an element of $N(A^T)$. Thus, for every row that becomes a zero row during elimination, we get a vector in the null space of A^T.

Remark:
In summary, to compute the four linear subspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix} , \tag{3.76}$$

we first used row eliminations:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{3.77}$$

This shows $\dim(N(A)) = 2$ and $\mathrm{rk}(A) = 2$. Explicitly, we read-off bases of $N(A)$ and $R(A)$:

$$N(A) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} , \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right) , \qquad R(A) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right) . \tag{3.78}$$

Likewise, by use of row eliminations we can transform A^T as

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{3.79}$$

From this it follows

$$N(A^T) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right) , \qquad C(A) = R(A^T) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) . \tag{3.80}$$

Note:
Be $A \in \mathbb{M}(m \times n, \mathbb{R}$. Then consider the map $\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$. Any such map can be factored by the four linear subspaces. By this we mean that there is a diagram

$$\begin{array}{ccccccc} \ker(\varphi_A) \cong N(A) & \xhookrightarrow{\varphi_K} & \mathbb{R}^n & \xrightarrow{\varphi_A} & \mathbb{R}^m & \xrightarrow{\varphi_P} & N(A^T) \cong \mathrm{coker}(\varphi_A) \\ & & \downarrow \varphi_{M_1} & & \uparrow \varphi_{M_2} & & \\ & & \mathrm{coim}(\varphi_A) \cong R(A) & \xrightarrow{\varphi_X} & C(A) \cong \mathrm{im}(\varphi_A) & & \end{array} \tag{3.81}$$

The maps φ_K and φ_{M_2} are injective. They are termed the *kernel embedding* and the *image embedding*, respectively. The maps φ_{M_1} and φ_P are surjective. They are termed the *coimage projection* and the *cokernel projection*, respectively. Crucially, the map φ_X is a vector space isomorphism, that is there exists an invertible matrix X with $A = M_2 \cdot X \cdot M_1$. This factorization is termed the *image-coimage factorization*

Remark:
This factorization exists much more generally, namely for every morphism in an Abelian category. You may encounter this if you every study category theory, which in a pedestrian fashion can be understood as a powerful tool to organize scientific programming.

Example 3.4.2:
For $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix}$ we have

$$\begin{array}{ccccccc} \mathbb{R}^2 \cong N(A) & \xhookrightarrow{\varphi_K} & \mathbb{R}^4 & \xrightarrow{\varphi_A} & \mathbb{R}^3 & \xrightarrow{\varphi_P} & N(A^T) \cong \mathbb{R}^1 \\ & & \Big\downarrow \varphi_{M_1} & & \varphi_{M_2} \Big\uparrow & & \\ & & \mathbb{R}^2 \cong R(A) & \xrightarrow{\varphi_X} & C(A) \cong \mathbb{R}^2 & & \end{array} \tag{3.82}$$

Consequence:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$ and E a matrix such that EA is an echelon form. Then the rows of E corresponding to the zero rows of A are a basis for the left null space of A. Hence

$$\dim\left(N(A^T)\right) = m - r\,, \tag{3.83}$$

where r is the (row) rank of A. Likewise,

$$\dim\left(N(A)\right) = n - r\,, \tag{3.84}$$

Thus, just the dimensions of A and its rank tell us a whole lot about the various linear subspaces associated to A. This is a special instance of the following

Theorem 3.4.1 (Rank-nullity theorem)**:**
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then it holds $\dim\left(N(A)\right) + \dim(C(A)) = n$.

Proof
$N(A)$ is a linear subspace of $\mathbb{R}^n$. Be $\mathcal{B}_{N(A)}$ a basis of $N(A)$. Then, by the basis-extension-theorem, we may extend $\mathcal{B}_{N(A)}$ to a basis of $\mathbb{R}^n$. This amounts to adding a set $\mathcal{I}$ of vectors in $\mathbb{R}^n$ to $\mathcal{B}_{N(A)}$. In particular,

$$|\mathcal{I}| = n - |\mathcal{B}_{N(A)}| = n - \dim_{\mathbb{R}}(N(A))\,. \tag{3.85}$$

It is not too hard to verify that $\mathcal{I}' = \{A\vec{v}\,,\, \vec{v} \in \mathcal{I}\}$ is a basis of $C(A)$. ■

Example 3.4.3:
For

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix}, \tag{3.86}$$

we found

$$N(A) = \mathrm{Span}_{\mathbb{R}}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right), \qquad R(A) = \mathrm{Span}_{\mathbb{R}}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right). \tag{3.87}$$

The basis of $N(A)$ is readily extended to $\mathbb{R}^4$ by adding the basis of $R(A)$. So consider

$$\mathcal{I} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}. \tag{3.88}$$

The images of these vectors under φ_A are

$$\mathcal{I}' = \left\{\begin{bmatrix} 8 \\ 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}\right\}. \tag{3.89}$$

Note that

$$\begin{bmatrix} 8 & 7 & 9 \\ 5 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}. \tag{3.90}$$

This is exactly the basis of $C(A)$ that we found above!

Corollary:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$, then it holds:

- $C(A)$ and $R(A)$ have dimension r. (This already follows from our previous finding that the row and column rank coincide.)
- $N(A)$ has dimension $n - r$.
- $N(A^T)$ has dimension $m - r$.

3.5 Linear transformations

Note:
We have already discussed the idea of a matrix as a function, namely assume that $A \in \mathbb{M}(m \times n, \mathbb{R})$ and $\vec{v} \in \mathbb{R}^n$. Then we defined a function

$$\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m \,,\; \vec{v} \mapsto A\vec{v}\,. \tag{3.91}$$

That is, $\vec{v}$ is the input and $A\vec{v}$ is the output. While one can think of $A\vec{v}$ as one vector at a time, the deeper goal is now to see what A does to the whole space!

Remark:
Matrix multiplication is linear. This is equivalent to (for all $c \in \mathbb{R}$ and $\vec{v}, \vec{w} \in \mathbb{R}^n$)

- $\varphi_A(c\vec{v}) = c \cdot A\vec{v}$
- $\varphi_A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$.

Thus, matrix multiplication fits nicely with the operations in a vector space. We now analyse functions which have the same property as matrix multiplication.

Definition 3.5.1 (Linear map)**:**
Let $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ be two vector spaces over $\mathbb{F}$. Then a function $\varphi \colon V \to W$ is called a linear map if for all $c \in \mathbb{F}$ and all $\vec{v}$, $\vec{w}$ it holds:

- $\varphi(c \cdot_V \vec{v}) = c \cdot_W \varphi(\vec{v})$,
- $\varphi(\vec{v} +_V \vec{w}) = \varphi(\vec{v}) +_W \varphi(\vec{w})$.

Claim:
Any linear map $\varphi \colon V \to W$ of vector spaces V, W over $\mathbb{F}$ satisfies $\varphi(\vec{0}) = \vec{0}$.

Proof
Since $\vec{0} = \vec{0} + \vec{0}$ it follows from linearity of φ that $\varphi(\vec{0}) = \varphi(\vec{0}) + \varphi(\vec{0}) = 2 \cdot \varphi(\vec{0})$. It follows $\vec{0} = \varphi(\vec{0})$ follows. ■

Example 3.5.1:
Is $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^3$ with

$$\varphi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_1 + x_2 \end{bmatrix} \tag{3.92}$$

a linear map? No! Because

$$2 \cdot \varphi\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \neq \varphi\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)\,. \tag{3.93}$$

Essentially, the square in the first component stops this function from being linear.

Example 3.5.2:
Consider the function $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\varphi\left(\vec{v}\right) = v_1 + 2v_2 + 3v_3\,. \tag{3.94}$$

This function is linear. In particular, we have

$$\varphi\left(\vec{v}\right) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\,. \tag{3.95}$$

Remark:
Note that perse, such a linear map is not defined by a matrix. But that does not mean we cannot find a mapping matrix. For this we focus, unless explicitly stated differently, on *finite*-dimensional vector spaces V and W.

Construction 3.5.1:
Let $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ both be finite-dimensional vector spaces over $\mathbb{F}$ and let $\varphi\colon V \to W$ be a linear map. We denote a basis of V by $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$. Therefore, every vector $\vec{x} \in V$ is expressed as a *unique* linear combination of the basis vectors in $\mathcal{B}$:

$$\vec{x} = \sum_{i=1}^{n} c_i \cdot \vec{v}_i\,, \qquad c_i \in \mathbb{F}\,. \tag{3.96}$$

Since φ is linear it follows:

$$\varphi(\vec{x}) = \varphi\left(\sum_{i=1}^{n} c_i \cdot \vec{v}_i\right) = \sum_{i=1}^{n} c_i \cdot \varphi\left(\vec{v}_i\right). \tag{3.97}$$

Hence, to compute the image of any vector $\vec{x} \in V$, it suffice to know the images of the basis vectors $\vec{v}_i$ under φ. This we can efficiently encode in the matrix

$$A_{\mathcal{B}} = \begin{bmatrix} | & & | \\ \varphi(\vec{v}_1) & \ldots & \varphi(\vec{v}_n) \\ | & & | \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R})\,, \tag{3.98}$$

where $\dim_{\mathbb{F}}(W) = m$. Namely, it then follows

$$\varphi(\vec{x}) = A_{\mathcal{B}} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\,. \tag{3.99}$$

Hence, if we agree to represent eq. (3.96) by the coefficients c_i used to express this vector (uniquely!) in the basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of V, then φ matches $\varphi_{A_{\mathcal{B}}}\colon V \to W,\ ,\ \vec{x} \mapsto A_{\mathcal{B}} \cdot \vec{x}$ with $A_{\mathcal{B}}$ in eq. (3.98).

Corollary:
Every linear map $\varphi\colon V \to W$ can be expressed as φ_A upon a choice of basis $\mathcal{B}$ of V. The mapping matrix $A = A_{\mathcal{B}}$ depends on the choice of basis!

Note:
The matrix $A_{\mathcal{B}}$ encodes properties of φ as follows:

- φ is injective iff $A_{\mathcal{B}}$ has full column rank.
- φ is surjective iff $A_{\mathcal{B}}$ has full row rank.
- φ is bijective iff $A_{\mathcal{B}}$ has full rank. In particular, $A_{\mathcal{B}}$ must be a square matrix.

Remark (Index of linear map)**:**
An important quantity of a linear map φ is its index. This is defined as

$$\operatorname{ind}(\varphi) := \dim_{\mathbb{F}}(\ker(\varphi)) - \dim_{\mathbb{F}}(\operatorname{coker}(\varphi)) = \dim_{\mathbb{F}}\left(N(A_{\mathcal{B}})\right) - \dim_{\mathbb{F}}\left(N(A_{\mathcal{B}}^T)\right) . \quad (3.100)$$

Example 3.5.3:
Consider φ a reflection in $\mathbb{R}^2$ across the line $y = x$. This a linear transformation! To find a mapping matrix, we first pick a basis of $\mathbb{R}^2$. It is particularly convenient to choose $\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. Then we have

$$\varphi(\vec{v}_1) = 1 \cdot \vec{v}_1 \,, \qquad \varphi(\vec{v}_2) = (-1) \cdot \vec{v}_2 \,. \quad (3.101)$$

Hence, in this basis, the mapping matrix is given by

$$A_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} . \quad (3.102)$$

This matrix accepts vectors (or actually their coefficients) in the basis $\mathcal{B}$ and returns the coefficient of the image vector in the standard basis $\mathcal{A} = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$. This indicates that we should be more careful. We should not only mention in what basis the input is encoded but also in what basis the output is encoded. Therefore, we write

$$A_{\mathcal{A}\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} . \quad (3.103)$$

Of course, we can also consider the matrix $A_{\mathcal{B}\mathcal{B}}$. It is not too hard to see that

$$A_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \quad (3.104)$$

The input to this matrix is the coefficients of $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. And the output is the coefficients of the image, again expressed in terms of this basis. Hence, we learn that $\varphi(1 \cdot v_1 + 0 \cdot v_2)$ is a vector, whose coefficients in the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is given as

$$A_{\mathcal{B}\mathcal{B}} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{3.105}$$

Hence, $\varphi(1 \cdot v_1 + 0 \cdot v_2) = 1 \cdot v_1 + 0 \cdot v_2$, just as expected.

We could also find the mapping matrix in the standard basis $\mathcal{A} = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. In this basis, the mapping matrix $A_{\mathcal{A}\mathcal{A}}$ is different. It is readily verified that then the mapping matrix is given by

$$A_{\mathcal{A}\mathcal{A}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{3.106}$$

We can derive this result by noting that the base change from $\mathcal{A}$ to $\mathcal{B}$ is performed by

$$T_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} , \qquad T_{\mathcal{B}\mathcal{A}}^{-1} = T_{\mathcal{A}\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{3.107}$$

Then

$$A_{\mathcal{A}\mathcal{A}} = T_{\mathcal{A}\mathcal{B}} \cdot A_{\mathcal{B}} \cdot T_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{3.108}$$

Exercise:
Consider a function $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ which rotates a vector by 45 degrees. Convince yourself that this is a linear transformation. Find the mapping matrix $A_{\mathcal{A}\mathcal{A}}$ in the standard basis $\mathcal{A}$ of $\mathbb{R}^2$. Consider a different basis $\mathcal{B}$ and find the mapping matrix $A_{\mathcal{B}\mathcal{B}}$.

4 Orthogonality

Recall the four primary subspaces attached to a matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$:

- the row space $R(A)$,
- the right null space $N(A^T)$,
- the column space $C(A)$,
- the left null space $N(A)$.

It so happens, that the the row space and the nullspace are orthogonal. Likewise the column space and the left null space are orthogonal. This is hinting at more structure underlying these four fundamental spaces. In this chapter, we wish to investigate this structure.

Convention:
Unless stated differently, from now on any vector space V is a vector space over the real numbers $\mathbb{R}$.

4.1 The notion of orthogonality

Remark:
In order to introduce a notion of orthogonality, we start in $\mathbb{R}^n$, which we may access intuitively. This leads to the notion of the "standard inner product" in $\mathbb{R}^n$, which we will generalize momentarily to define inner products also on more general vector spaces such as Pol_n and $\mathbb{M}(m \times n, \mathbb{R})$.

Example 4.1.1 ('Standard' orthogonality in $\mathbb{R}^n$)**:**
Consider the vector space $\mathbb{R}^n$ over $\mathbb{R}$. We consider the map

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\,,\ (\vec{a}, \vec{b}) \mapsto \vec{a}^T \cdot \vec{b} = \sum_{i=1}^{n} a_i \cdot b_i\,. \tag{4.1}$$

This map has the following properties:

- Linearity in the first argument, that is for all $c \in \mathbb{R}$ and $\vec{a}_1, \vec{a}_2, \vec{b} \in \mathbb{R}^n$ it holds

$$\left\langle \vec{a}_1 + c \cdot \vec{a}_2, \vec{b} \right\rangle = \left\langle \vec{a}_1, \vec{b} \right\rangle + c \cdot \left\langle \vec{a}_2, \vec{b} \right\rangle\,. \tag{4.2}$$

- Symmetry, that is for all $\vec{a}, \vec{b} \in \mathbb{R}^n$ it holds

$$\left\langle \vec{a}, \vec{b} \right\rangle = \left\langle \vec{b}, \vec{a} \right\rangle \,. \tag{4.3}$$

- $\langle \cdot, \cdot \rangle$ is positive-definite, that is for all $\vec{a} \in \mathbb{R}^n \setminus \vec{0}$ it holds

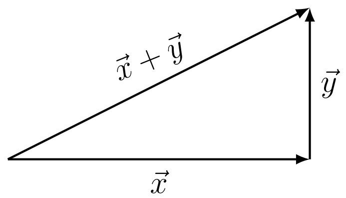
$$\langle \vec{a}, \vec{a} \rangle > 0 \,. \tag{4.4}$$

We refer to this inner product as the *standard inner product in* $\mathbb{R}^n$. This inner product gives us the following notions of length and orthogonality:

- The length of a vector $\vec{x}$ is given by $|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.
- Two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are orthogonal if and only if $\langle \vec{x}, \vec{y} \rangle = 0$.

Remark:
Why is this the "*right*" criterion? First, there is no right or wrong inner product. Any inner product gives the notion of orthogonality, and it need not be tied to our expectation on our physical surrounding.

However, if we are looking for an inner product which matches the expectation in our physical surroundings, then the above *standard inner product* does a good job. To see this, look at the right triangle:



(4.5)

Then, by Pythagoras, it holds

$$|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 \,, \tag{4.6}$$

where $|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ denotes the length of $\vec{x}$. Hence we notice that

$$|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 \qquad \Leftrightarrow \qquad \langle \vec{x}, \vec{y} \rangle = 0 \,. \tag{4.7}$$

Definition 4.1.1 (Inner product space)**:**
Be $(V, +_V, \cdot_V)$ a vector space over $\mathbb{R}$. Then a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is termed an inner product on V if and only if it satisfies the following conditions:

1. $\langle \cdot, \cdot \rangle$ is $\mathbb{R}$-linear in its first argument,
2. $\langle \cdot, \cdot \rangle$ is symmetric,
3. $\langle \cdot, \cdot \rangle$ is positive-definite.

The pair $((V, +_V, \cdot_V), \langle\cdot,\cdot\rangle)$ is termed an *inner product space* over $\mathbb{R}$. In this inner product space we define:

- The length of a vector $\vec{x}$ is given by $|\vec{x}| = \sqrt{\langle\vec{x},\vec{x}\rangle}$.
- Two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are orthogonal if and only if $\langle\vec{x},\vec{y}\rangle = 0$.

Exercise:
Verify that for any $k \in \mathbb{R}_{>0}$

$$\langle\cdot,\cdot\rangle_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\,,\ (\vec{a},\vec{b}) \mapsto k \cdot a_1 b_1 + \sum_{i=2}^{n} a_i \cdot b_i\,, \tag{4.8}$$

is an inner product. In particular, for $k \neq 1$, this inner product differs from the Standard inner product. This shows, that there are many inner products on a given vector space.

Exercise:
Define inner products on Pol_n and $\mathbb{M}(m \times n, \mathbb{R})$ and explicitly verify that these inner products satisfy the axioms of an inner product. Use these inner products to find 4 orthogonal vectors in Pol_4 and $\mathbb{M}(3 \times 3, \mathbb{R})$.

Claim:
Be $((V, +_V, \cdot_V), \langle\cdot,\cdot\rangle)$ an *inner product space* over $\mathbb{R}$. Then $\vec{0} \in V$ is orthogonal to any other vector $\vec{x} \in V$.

Proof
Since $\langle\cdot,\cdot\rangle$ is linear in the first argument, it follows from $\vec{0} + \vec{0} = \vec{0}$ that

$$\left\langle\vec{0},\vec{x}\right\rangle = \left\langle\vec{0},\vec{x}\right\rangle + \left\langle\vec{0},\vec{x}\right\rangle\,. \tag{4.9}$$

Hence $\left\langle\vec{0},\vec{x}\right\rangle = 0$ and $\vec{0}$ and $\vec{x}$ are orthogonal. ∎

Note:
We can extend the notion of orthogonality to vector spaces.

Definition 4.1.2 (Orthogonality of vector spaces)**:**
Be $S, T \subseteq V$ two linear subspaces of an inner product space $(V, \langle\cdot,\cdot\rangle)$ over $\mathbb{R}$. Then we say that S is orthogonal to T – $S \perp T$ – if every vector in S is orthogonal to every vector in T, that is

$$\left\langle\vec{s},\vec{t}\right\rangle = 0\,, \qquad \forall\vec{s} \in S \text{ and } \forall\vec{t} \in T\ . \tag{4.10}$$

Example 4.1.2:
Consider the real vector space $\mathbb{R}^2$ with the standard inner product. Moreover, let

$$S = \mathrm{Span}_{\mathbb{R}}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}\,, \qquad T = \mathrm{Span}_{\mathbb{R}}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}\,. \tag{4.11}$$

Then $S \perp T$.

Corollary:
Every subspace $S \subseteq V$ is orthogonal to the trivial subspace of V.

Claim:
Any two distinct 2-dim. linear subspaces $S, T \subseteq \mathbb{R}^3$ are *not* orthogonal.

Proof
Since S and T are distinct, they intersect for dimensional reasons in a line through the origin. Let $\vec{x}$ be a *non-zero* vector that belongs to this line of intersection. Since this vector belongs to S and T, we conclude from $\vec{x}^T\vec{x} > 0$, that S and T are not orthogonal. ■

Note:
A line and a plane in $\mathbb{R}^3$ can be orthogonal subspaces. Namely, the line through the origin normal to the plane gives such a pair of orthogonal subspaces.

Claim:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then (w.r.t. the standard inner product in $\mathbb{R}^n$) its row space $R(A)$ and (right) null space $N(A)$ are orthogonal.

Proof
Let $\vec{x} \in N(A)$. Then, by definition $A\vec{x} = \vec{0}$. More explicitly, this means

$$\begin{bmatrix} - & \text{row } 1 & - \\ - & \text{row } 2 & - \\ \vdots & & \\ - & \text{row m} & - \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.12}$$

Hence, $\text{row}_i \cdot \vec{x} = 0$ for all $1 \le i \le m$. Since this is the standard inner product in $\mathbb{R}^n$, we conclude that w.r.t. this inner product $\vec{x}$ is orthogonal to any vector $\vec{r}$ in the row space of A. ■

Consequence:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$ matrix. Then (w.r.t. the standard inner product in $\mathbb{R}^m$) its column space $C(A)$ is orthogonal to the left nullspace $N(A^T)$.

Exercise:
Proof this statement. Hint: Apply the previous result to A^T.

Remark:
Any vector space over $\mathbb{R}$ is uniquely classified by its dimension. That is, if V is a vector space over $\mathbb{R}$ of dimension $\dim_{\mathbb{R}}(V) = n$. Then $V \cong \mathbb{R}^n$. In particular, $C(A) \cong R(A)$ for any $A \in \mathbb{M}(m \times n, \mathbb{R})$.

4.2 Orthogonal complements

Note:
In fact, one can say more about the relation between the row space and the null spaces. Not only are they orthogonal to each other, they also fill out the whole space. What does this mean?

Question:
Consider a matrix $A \in \mathbb{M}(n \times 4, \mathbb{R})$ with 4 columns. Is it possible that the row space $R(A)$ is a line in $\mathbb{R}^4$ and the null space $N(A)$ is also just a line in $\mathbb{R}^4$? No! Namely, the rank-nullity theorem tells us that

$$\dim\left(R(A)\right) + \dim\left(N(A)\right) = 4\,. \tag{4.13}$$

Therefore, if the row space $R(A)$ is a line, then the nullspace $N(A)$ is forced to be 3-dimensional. Hence since $N(A) \perp R(A)$ (w.r.t. the standard inner product in $\mathbb{R}^n$), these orthogonal subspaces span $\mathbb{R}^4$. This observation leads to the notion of the orthogonal complement.

Definition 4.2.1 (Orthogonal complement)**:**
Be $(V, \langle\cdot,\cdot\rangle)$ a vector space and $S \subseteq V$ a linear subspace. The orthogonal complement of S in V is the set of all vectors $\vec{v} \in V$ which are perpendicular to S, i.e.

$$S^\perp := \{\vec{v} \in V \mid \langle \vec{s}, \vec{v}\rangle = 0 \;\; \forall \vec{s} \in S\}\ . \tag{4.14}$$

Claim:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$ matrix. Then the nullspace $N(A)$ is the orthogonal complement of the row space $R(A)$ (w.r.t. the standard inner product in $\mathbb{R}^n$).

Proof
Every vector that is orthogonal to the rows of a matrix A lies in the nullspace $N(A)$.
The converse is also true. Let us prove this by contraposition. Hence, suppose we could find a vector $\vec{v}$ that was orthogonal to the nullspace $N(A)$ but is not included in the row space of A. Then we could use this vector to construct a new matrix A' which is the same as A but contains this vector $\vec{v}$ as an additional row. The rank of this matrix A' would then be one more than that of A, but $N(A) = N(A')$. This is impossible by the rank-nullity theorem! ■

Exercise:
Prove that $N(A^T)$ is the orthogonal complement of $C(A)$.

Remark:
W.r.t. the standard inner product in $\mathbb{R}^n$, the have:

- $N(A)$ is the orthogonal complement of $R(A)$.
- $N(A^T)$ is the orthogonal complement of $C(A)$.

Claim:
Let $S, T \subseteq V$ two linear subspaces of an inner product space $(V, \langle\cdot,\cdot\rangle)$. Suppose that T is the orthogonal complement of S in $(V, \langle\cdot,\cdot\rangle)$. Then every vector $\vec{v} \in V$ can be written *uniquely* in the form

$$\vec{v} = \vec{v}_S + \vec{v}_T \,, \tag{4.15}$$

where $\vec{v}_S \in S$ and $\vec{v}_T \in T$.

Proof
It should be clear that for any vector $\vec{v} \in V$ such a decomposition exists. We therefore suffice it to show that this decomposition is unique. We assume the contrary, i.e. let us assume that

$$\vec{v} = \vec{v}_S + \vec{v}_T = \vec{w}_S + \vec{w}_T \,, \tag{4.16}$$

for distinct $\vec{v}_S, \vec{w}_S \in S$ and distinct $\vec{v}_T, \vec{w}_T \in S$. But this is equivalent to

$$S \ni \vec{v}_S - \vec{w}_S = \vec{v}_T - \vec{w}_T \in T \,, \tag{4.17}$$

Since $S \cap T = \left\{\vec{0}\right\}$ we conclude that $\vec{v}_S = \vec{w}_S$ and $\vec{v}_T = \vec{w}_T$ contrary to our assumption. Hence, the decomposition is unique, as claimed. ■

Consequence:
Let $S \subseteq V$ a linear subspace of an inner product space $(V, \langle\cdot,\cdot\rangle)$. Suppose that $T = S^\perp$ is the orthogonal complement of S in $(V, \langle\cdot,\cdot\rangle)$. Then, since every vector $\vec{v} \in V$ has a unique decomposition $\vec{v} = \vec{v}_S + \vec{v}_T$ with $\vec{v}_S \in S$ and $\vec{v}_T \in T$, we have an isomorphism

$$S \oplus S^\perp := \left\{(s,t) \,|\, s \in S \,,\, t \in S^\perp\right\} \to V \,,\, (s,t) \mapsto s + t \,. \tag{4.18}$$

We term $S \oplus S^\perp$ the direct sum of S and $S^\perp$. Thus, we have found $S \oplus S^\perp \cong V$.

Example 4.2.1:
Consider the line $S = \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^2$ with standard inner product. Then we have $S^\perp \cong \mathbb{R}$ and $\mathbb{R}^2 \cong S \oplus S^\perp \cong \mathbb{R} \oplus \mathbb{R}$. Explicitly

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \end{bmatrix} . \tag{4.19}$$

Example 4.2.2:
Consider the vector space or square $n \times n$-matrices. Then the subspaces of symmetric and of anti-symmetric $n \times n$-matrices are orthogonal complements. This follows from

$$A = \left(\frac{A + A^T}{2}\right) + \left(\frac{A - A^T}{2}\right) . \tag{4.20}$$

4.3 Orthogonal projections

4.3.1 Orthogonal projection onto a line

Note:
We now wish to discuss how we actually write a vector as a sum of vectors coming from orthogonal complements. Recall

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \end{bmatrix}, \tag{4.21}$$

where the first vector is contained in the row space of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and the second in its nullspace. Observe, that we project $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ on the row space and nullspace to obtain the summands. This motivates the discussion of projections.

Remark:
In the following discussion, we focus on the Standard inner product in $\mathbb{R}^n$. The results easily generalize to arbitrary inner products.

Construction 4.3.1:
We begin by considering the simplest case, a projection onto a line. Then our challenge is the following: Given a line L through the origin in the direction $\vec{a}$, find the point $\vec{p}$ on the line, which is closest to $\vec{b}$.

Standard Euclidean geometry tells us, that that point $\vec{p}$ is obtained by dropping a line from $\vec{b}$ in the direction perpendicular to the line L. But how do we actually find $\vec{p}$?

Clearly, it must hold $\vec{p} = \hat{x} \cdot \vec{a}$ for some $\hat{x} \in \mathbb{R}$. Furthermore, $\vec{b} - \vec{p} = \vec{b} - \hat{x} \cdot \vec{a}$ is orthogonal to $\vec{a}$. Therefore,

$$0 = \vec{a}^T \cdot \left(\vec{b} - \hat{x} \cdot \vec{a}\right) \quad \Leftrightarrow \quad \hat{x} = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}}. \tag{4.22}$$

Consequently, it holds

$$\vec{p} = \left(\frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}}\right) \cdot \vec{a}. \tag{4.23}$$

Remark:
The vector $\vec{b} - \vec{p}$ is ofentimes referred to as *error vector* and is therefore denoted as $\vec{e}$.

Claim:
This projection is a linear transformation.

Proof

We construct a matrix P which sends $\vec{b}$ to $\vec{p}$. From

$$\vec{p} = \vec{a} \cdot \left(\frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}}\right), \tag{4.24}$$

we can see that the following matrix

$$P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}, \tag{4.25}$$

indeed satisfies $P\vec{b} = \vec{p}$. ■

Exercise:
Compute the matrix P for $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then repeat this exercise for $\vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note:
The rank of the matrix P is 1. Its column space is the line we are projection upon.

Example 4.3.1:
Let us compute the projection matrix for projection onto the line determined by $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Hence, we plug in the above formula and find

$$P = \frac{\vec{a}\vec{a}^T}{1^2 + 2^2 + 3^2} = \frac{1}{14} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}. \tag{4.26}$$

Next, let us compute the projection of $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Hence, we simply compute

$$\vec{p} = P\vec{b} = \frac{1}{14} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}. \tag{4.27}$$

Note:
Scaling $\vec{a}$ does not change the projection matrix P.

Claim:
Projecting twice is the same as projecting once: $P^2 = P$.

Exercise:
Prove this statement.

Question:
If P is the projection matrix on the line L through the origin in the direction $\vec{a}$, then what is $I - P$? For any vector $\vec{b}$, it holds

$$(I - P) \cdot \vec{b} = \vec{b} - P\vec{b} = \vec{b} - \vec{p} = \vec{e}. \tag{4.28}$$

This is the error vector perpendicular to the line L. But note also that

$$(I - P)^2 = I^2 - 2P + P = I - P. \tag{4.29}$$

Consequently, $I - P$ is a projection. It is the projection onto the subspace orthogonal to the line through $\vec{a}$.

Exercise:
Interpret the equation $I = P + (I - P)$. Hint: Orthogonal complements.

4.3.2 Orthogonal projection onto subspaces

Lemma 4.3.1:
Be $A \in \mathbb{M}(m \times n, \mathbb{R}$. Then the following two statements are equivalent:

- A has linearly independent columns.
- $A^T A$ is invertible.

Proof
We prove that A and $A^T A$ have the same nullspaces:

- Be $\vec{x} \in N(A)$. Then $A\vec{x} = \vec{0}$. But then $A^T A\vec{x} = \vec{0}$, i.e. $\vec{x} \in N(A^T A)$.
- Conversely, let $\vec{x} \in N(A^T A)$. Then $A^T A\vec{x} = \vec{0}$. We multiply by $\vec{x}^T$ on the LHS:

$$0 = \vec{x}^T A^T A\vec{x} \quad \Leftrightarrow \quad (A\vec{x})^T (A\vec{x}) = \vec{0}\,. \tag{4.30}$$

  This means that $\langle A\vec{x}, A\vec{x}\rangle = 0$. But, by the properties of the inner product, this is only possible if $A\vec{x} = \vec{0}$, i.e. $\vec{x} \in N(A)$.

Now we can show the stated equivalence:

- If A has linearly independent columns, then $N(A) = \{\vec{0}\}$. Then, by our previous observation, $N(A^T A) = \{\vec{0}\}$. Consequently, the square matrix $A^T A$ is invertible.
- Conversely, if $A^T A$ is invertible, then $N(A^T A) = \{\vec{0}\}$. But then, also $N(A) = \{\vec{0}\}$ and A has linearly independent columns. ■

Question (Projections onto general subspaces)**:**
Given linearly independent vectors $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^m$, how do we find the linear combination

$$\vec{p} = \widehat{x}_1\vec{a}_1 + \cdots + \widehat{x}_n\vec{a}_n \tag{4.31}$$

which is closest to a given vector $\vec{b}$?

Note:
The case $n = 1$ is projection onto a line. The case $n = 2$ is projection onto a plane.

Construction 4.3.2:
Let consider the matrix

$$A = \left[\begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{array}\right] \in \mathbb{M}(m \times n, \mathbb{R})\,. \tag{4.32}$$

By assumption, A has linearly independent columns. We are now looking for a vector $\vec{p} \in C(A)$ which is closest to $\vec{b}$. Let us set

$$\widehat{\vec{x}} = (\widehat{x}_1, \dots, \widehat{x}_n)\ . \tag{4.33}$$

The 'right' vector $\widehat{\vec{x}}$ is defined by the property that $\vec{b} - A\widehat{\vec{x}}$ is orthogonal to $C(A)$. This is equivalent to saying that

$$A^T \cdot \left(\vec{b} - A\widehat{\vec{x}}\right) = \vec{0}\,. \tag{4.34}$$

We can rewrite this as

$$A^T A\widehat{\vec{x}} = A^T\vec{b}\,. \tag{4.35}$$

The matrix $S = A^T A \in \mathbb{M}(n \times n, \mathbb{R})$ is, by our previous lemma, invertible since A has linearly independent columns. Therefore, we can write

$$\widehat{\vec{x}} = \left(A^T A\right)^{-1} \cdot A^T\vec{b}\,. \tag{4.36}$$

Consequently, the projection $\vec{p}$ and projection matrix P are given by

$$\vec{p} = P\vec{b}\,, \qquad P = A \cdot \left(A^T A\right)^{-1} \cdot A^T\,. \tag{4.37}$$

Note that A by itself is (in general) not invertible! Hence, you cannot write $(A^T A)^{-1}$ as $A^{-1} \cdot (A^T)^{-1}$.

Example 4.3.2:
Let us find the projection matrix for projection onto the plane in $\mathbb{R}^3$, which is given by $x - 2y + z = 0$. First, we need to find the matrix A. To this end, we pick two linearly independent vectors in $x - 2y + z = 0$, say

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}. \tag{4.38}$$

Then we find

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}. \tag{4.39}$$

Note that A is *not* invertible! From this, the projection matrix follows as

$$P = \frac{1}{6} \cdot \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}. \tag{4.40}$$

Note:
Properties of projection matrices include the following:

1. P is symmetric:

$$P^T = \left(A(A^TA)^{-1}A^T\right)^T = A\left((A^TA)^{-1}\right)^{-1}A = P\,. \tag{4.41}$$

2. $P^2 = P$:

$$P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A = A(A^TA)^{-1}A = P\,. \tag{4.42}$$

3. P does not depend on the choice of vectors that make up A:
   This should be surprising given the expression for P. Yet, it is obvious that projection onto a subspace does not depend on the basis of the subspace.

4. $I - P$ is the matrix for projection onto the orthogonal complement of $C(A)$:

$$\vec{b} = P\vec{b} + (I - P)\cdot\vec{b}\,. \tag{4.43}$$

5. $P\vec{b} = \vec{b}$ if $\vec{b} \in C(A)$:
   This is clear geometrically. Algebraically, it follows by using $\vec{b} = A\vec{c}$. Then we see:

$$A(A^TA)^{-1}A^T\vec{b} = A(A^TA)^{-1}A^TA\vec{c} = A\vec{c} = \vec{b}\,. \tag{4.44}$$

6. $P\vec{b} = \vec{0}$ if $\vec{b} \in N(A^T)$:
   Again, this is clear geoemtrically. Algebraically, we note that $b \in N(A^T)$ means $A^T\vec{b} = \vec{0}$. Hence

$$A(A^TA)^{-1}A^T\vec{b} = \vec{0}\,. \tag{4.45}$$

7. The rank of the projection matrix P matches $\dim(C(A))$, which is the rank of A.

4.4 Application: Least square approximation

Remark:
Motivated by orthogonal complements, we discussed projections in the previous section. Given a point $\vec{b} \in \mathbb{R}^n$ and a linear subspace $S = \mathrm{Span}_{\mathbb{R}}(\vec{a}_1, \ldots, \vec{a}_k) \subseteq \mathbb{R}^n$, we have projected $\vec{b}$ onto the point $\vec{p} \in S$ which is closest to $\vec{b}$.

In this section, we use these insights for a real world problem. Namely, suppose we are given points $\vec{b}_1, \ldots, \vec{b}_k \in \mathbb{R}^2$ and we want to fit a line to these points, which best describes/approximates these points. We will see that our insights from the previous section allow us to achieve this very goal.

Example 4.4.1:
Consider $\vec{b}_1 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \in \mathbb{R}^2$. Can we find $C, D \in \mathbb{R}$ such that the line $L(C,D) = \left\{ \begin{bmatrix} t \\ C + D \cdot t \end{bmatrix} \mid t \in \mathbb{R} \right\}$ is closest to these three points? Let us therefore see if we can find a line which contains all these points:

- $\vec{b}_1 \in L(C,D)$ iff $C + D \cdot 0 = 6$,
- $\vec{b}_2 \in L(C,D)$ iff $C + D \cdot 1 = 0$,
- $\vec{b}_3 \in L(C,D)$ iff $C + D \cdot 2 = 0$.

We are thus trying to find $\vec{x} = \begin{bmatrix} C \\ D \end{bmatrix}$ which solves $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}. \tag{4.46}$$

But this equation has no solution!

Remark:
Let us reinterpret the task. We consider $C(A)$ and $\vec{b}$. Of course, we can project $\vec{b}$ onto $C(A)$. As in the previous section, we denote this projection by $A\widehat{\vec{x}}$. This is the best approximation of $\vec{b}$ by $C(A)$, in that it minimizes the error vector $\vec{e} = \vec{b} - A\widehat{\vec{x}}$. Therefore, the vector $\widehat{\vec{x}}$ informs us on the straight line which is closest to the points $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$.

To measure the "distance" of a line parametrized by $\vec{x}$ to the given points, we consider

$$l_e(\vec{x}) = \left\langle \vec{b} - A\vec{x}, \vec{b} - A\vec{x} \right\rangle. \tag{4.47}$$

Note that $l_e(\vec{x})$ is minimal for $\vec{x} = \widehat{\vec{x}}$. This allows us to systematically find the best line by the *method of the least square* or the *least square approximation.*

Example 4.4.2:
Let us continue with the previous example. There we have

$$l_e(\vec{x}) = (C-6)^2 + (C+D)^2 + (C+2D)^2. \tag{4.48}$$

Our task is thus to minimize the function

$$l_e \colon \mathbb{R}^2 \to \mathbb{R}, \ (C,D) \mapsto 3C^2 - 12C + 6CD + 5D^2 + 36. \tag{4.49}$$

We thus consider the Jacobian matrix

$$J \colon \mathbb{R}^2 \to \mathbb{R}^2, \ (C,D) \mapsto \begin{bmatrix} \left(\frac{\partial l_e}{\partial C}\right)(C,D) \\ \left(\frac{\partial l_e}{\partial D}\right)(C,D) \end{bmatrix} = \begin{bmatrix} 6C - 12 + 6D \\ 6C + 10D \end{bmatrix}. \tag{4.50}$$

The Jacobian matrix vanishes at all extrema of the function l_e, i.e. at its (local) minima, saddle points and maxima. We notice that the vanishing of J is equivalent to

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} . \tag{4.51}$$

Remark:
The analytics does *by no means guarantee* that the zeros of the Jacobian matrix are (local) minima. The type of local extrema is found by analysing the Hessian matrix. To argue that a local extremum is a global extremum, a further argument is required, which analyses the behaviour of the function away from the local extremum.

Note:
Recall eq. (4.35). It says that the best $\widehat{\vec{x}}$ satisfies the equation $A^T A\widehat{\vec{x}} = A^T\vec{b}$. In the above example we have

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} , \qquad \vec{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} . \tag{4.52}$$

It follows

$$A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} , \qquad A^T\vec{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} . \tag{4.53}$$

This reproduces exactly the analytic criterion eq. (4.51).

Corollary:
The partial derivatives of $l_e(\vec{x})$ vanish iff $A^T A\vec{x} = A^T\vec{b}$.

Example 4.4.3:
The unique solution to *eq.* (4.51) is $C = 5$ and $D = -3$, i.e. $\widehat{\vec{x}} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$. We then have $l_e(\widehat{\vec{x}}) = 6$. To see that this is minimum, we consider the Hessian

$$\begin{aligned} &H \colon \mathbb{R}^2 \to \mathbb{M}(2 \times 2, \mathbb{R}) , \\ &(C, D) \mapsto \begin{bmatrix} \left(\frac{\partial^2 l_e}{\partial C^2}\right)(C, D) & \left(\frac{\partial^2 l_e}{\partial C \partial D}\right)(C, D) \\ \left(\frac{\partial^2 l_e}{\partial D \partial C}\right)(C, D) & \left(\frac{\partial^2 l_e}{\partial D^2}\right)(C, D) \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 10 \end{bmatrix} . \end{aligned} \tag{4.54}$$

As we will learn later in the course, the matrix $\begin{bmatrix} 6 & 6 \\ 6 & 10 \end{bmatrix}$ is positive definite. This proves that $\widehat{\vec{x}}$ is a *local* minimum. Since we found that $l_e(\vec{x})$ in eq. (4.48) has a unique local extremum, there are no other local minima or maxima. To argue that $\widehat{\vec{x}}$ is the *global* minimum, it remains to compare $l_e(\widehat{\vec{x}}) = 6$ to

$$\lim_{C,D\to\infty} l_e(\vec{x}) = \infty . \tag{4.55}$$

It follows that indeed, we have found the global minimum of eq. (4.48). We have thus found that the line $L(5,-3) = \left\{ \begin{bmatrix} t \\ 5 - 3 \cdot t \end{bmatrix} \mid t \in \mathbb{R} \right\}$ is closest to the points

$$\vec{b}_1 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{4.56}$$

Exercise:
Draw an image of the situation. Compute the error vector for the orthogonal projection of $\vec{b}$ onto $C(A)$.

Note:
This situation generalizes. In experiments, we often measure a quantity over and over again. For example, we could have 100 points in $\mathbb{R}^2$, which we intend to explain by one shifted (straight) line. In general, those measured points are not located on a perfectly straight line. It is then our task to find the straight line closest to all these points.

Consequence:
To fit m points $\{p_i = (t_i, b_i)\}_{1 \le i \le m} \in \mathbb{R}^2$ to a straight line $L(C,D) = \left\{ \begin{bmatrix} t \\ C + D \cdot t \end{bmatrix} \mid t \in \mathbb{R} \right\}$, we are looking at the equations

$$C + Dt_1 = b_1\,, \tag{4.57}$$
$$C + Dt_2 = b_2\,, \tag{4.58}$$
$$\vdots \tag{4.59}$$
$$C + Dt_m = b_m\,. \tag{4.60}$$

Equivalently, we are looking at

$$A\vec{x} = \vec{b}, \qquad \vec{x} = \begin{bmatrix} C \\ D \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}. \tag{4.61}$$

The closest line minimizes the size of the error vector, i.e. $l_e(\vec{x}) = \left\langle \vec{b} - A\vec{x}, \vec{b} - A\vec{x} \right\rangle$, i.e. solves $A^T A\widehat{\vec{x}} = A^T\vec{b}$. This we can work out explicitly. Namely

$$A^T A = \begin{bmatrix} m & \sum_{i=1}^m t_i \\ \sum_{i=1}^m t_i & \sum_{i=1}^m t_i^2 \end{bmatrix}, \qquad A^T\vec{b} = \begin{bmatrix} \sum_{i=1}^m b_i \\ \sum_{i=1}^m t_i b_i \end{bmatrix}. \tag{4.62}$$

In a specific problem, i.e. fitting given points to a line, the numbers t_i and b_i are given. We can then work out these matrices and find the best solution as

$$\widehat{\vec{x}} = (A^T A)^{-1} \cdot A^T\vec{b}\,. \tag{4.63}$$

The error vector $\left\langle A\widehat{\vec{x}} - \vec{b}, A\widehat{\vec{x}} - \vec{b} \right\rangle$ gives a measure for how good this fit is. The smaller, the better the line describes the given data. In particular, if all points are on a line, then $\left\langle A\widehat{\vec{x}} - \vec{b}, A\widehat{\vec{x}} - \vec{b} \right\rangle$ vanishes.

Remark:
Note that A is not invertible, so $(A^T A)^{-1}$ cannot be written as $A^{-1} \cdot (A^T)^{-1}$ because in general neither A nor A^T are invertible.

Note:
This strategy is not limited to fitting straight lines. For example, suppose that we are given $m > 3$ points $\vec{b}_i = (t_i, b_i)$. Then we can fit a parabola

$$P(C, D, E) = \left\{ \begin{bmatrix} t \\ C + Dt + Et^2 \end{bmatrix} \mid t \in \mathbb{R} \right\}, \tag{4.64}$$

to these points. We are then looking at

$$C + Dt_1 + Et_1^2 = b_1 , \tag{4.65}$$
$$C + Dt_2 + Et_2^2 = b_2 , \tag{4.66}$$
$$\vdots \tag{4.67}$$
$$C + Dt_m + Et_m^2 = b_m . \tag{4.68}$$

Equivalently, we are looking at

$$A\vec{x} = \vec{b}, \qquad \vec{x} = \begin{bmatrix} C \\ D \\ E \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \\ 1 & t_m & t_m^2 \end{bmatrix}. \tag{4.69}$$

4.5 Orthonormal bases and Gram-Schmidt

Note:
So far, when we discussed projections P onto a linear subspace $S \subseteq \mathbb{R}^n$ we have considered a basis of S, i.e. $S = \mathrm{Span}_{\mathbb{R}}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k)$ and $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$ is a linearly independent family of vectors. However, the computation of the projections and the projection matrices becomes easier in some basis. This is what we are going to discuss now.

Remark:
Recall that the projection matrix for projection onto S is given by

$$\vec{p} = P\vec{b}, \qquad P = A \cdot \left(A^T A\right)^{-1} \cdot A^T . \tag{4.70}$$

The expression for P simplifies provided that $A^T A = I$. We may thus wonder for which basis of the subspace S this condition is satisfied. Recall that

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_k \\ | & | & & | \end{bmatrix}. \tag{4.71}$$

Hence, the entries of A^TA are the inner products of the basis vectors $\vec{a}_1$ (w.r.t. to $\langle\cdot,\cdot\rangle_{\text{Std}}$). We conclude that $A^TA = I$ if and only if for all $1 \le i, j \le k$ with $i \neq j$ it holds

$$\langle\vec{a}_i, \vec{a}_i\rangle_{\text{std}} = \vec{a}_i^T\vec{a}_i = 1\,, \qquad \langle\vec{a}_i, \vec{a}_j\rangle_{\text{std}} = \vec{a}_i^T\vec{a}_j = 0\,. \tag{4.72}$$

This gives rise to the following notion.

Definition 4.5.1 (Orthonormal basis)**:**
Be V an inner product space and $S \subseteq V$ a linear subspace with basis $\mathcal{B}_S = \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_k\}$. The basis $\mathcal{B}_S$ is said to be orthonormal if for all $1 \le i, j \le k$ with $i \neq j$ it holds

$$\langle\vec{a}_i, \vec{a}_i\rangle = 1\,, \qquad \langle\vec{a}_i, \vec{a}_j\rangle = 0\,. \tag{4.73}$$

Corollary:
Be $Q \in \mathbb{M}(m \times n, \mathbb{R})$. Then the following holds true:

- If the column of Q are orthonormal, then $Q^TQ = I$.
- If Q is square, i.e. $m = n$, then $Q^T = Q^{-1}$.

Definition 4.5.2:
If $Q \in \mathbb{M}(n \times n, \mathbb{R})$ with $Q^TQ = I$. Then we term Q are orthogonal matrix.

Corollary:
The columns of an orthogonal matrix are orthonormal. Likewise, the (transpose of the) rows of an orthogonal matrix are orthonormal.

Example 4.5.1:
All rotation and permutation matrices are orthogonal. For example:

- Rotation matrix: $R = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$.
- Permutation matrix: $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Exercise:
Verify that these matrices are orthogonal, i.e. verify that the transposed matrix is the inverse.

Example 4.5.2:
Another important class of matrices which are orthogonal are reflections. Consider a **unit** vector $\vec{u}$. The reflection matrix about $\vec{u}$ is given by

$$Q = I - 2\vec{u}\vec{u}^T\,. \tag{4.74}$$

It follows that Q is symmetric, i.e.

$$Q^T = \left(I - 2\vec{u}\vec{u}^T\right)^T = I - 2\vec{u}\vec{u}^T = Q\,. \tag{4.75}$$

Furthermore, we have

$$Q^T Q = \left(I - 2\vec{u}\vec{u}^T\right) \cdot \left(I - 2\vec{u}\vec{u}^T\right) = I - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T = I = QQ^T \,. \tag{4.76}$$

So Q is indeed an orthogonal matrix. As an example, consider $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{4.77}$$

This is indeed the expected matrix for reflection at the x-axis.

Note:
A base change by an orthogonal matrix has a very important property – it preserves the (standard) inner product, and thereby lengths and angles! Here is the proof.

Claim:
Consider the inner product space $(\mathbb{R}^n, \langle , \rangle_{\text{Std}})$ and an orthogonal matrix $Q \in \mathbb{M}(n \times n, \mathbb{R})$. Then for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ it holds

$$\langle Q\vec{x}, Q\vec{y}\rangle_{\text{Std}} = \langle \vec{x}, \vec{y}\rangle_{\text{Std}} \,. \tag{4.78}$$

Proof
By definition of the standard inner product in $\mathbb{R}^n$ it holds

$$\langle Q\vec{x}, Q\vec{y}\rangle_{\text{Std}} = (Q\vec{x})^T \cdot (Q\vec{y}) = \vec{x}^T (Q^T Q)\vec{y} = \vec{x}^T \vec{y} = \langle \vec{x}, \vec{y}\rangle_{\text{Std}} \,. \tag{4.79}$$

In the second equality we have used the defining property of the orthogonal matrix Q, namely $Q^T Q = I$. ■

Note:
We will have more to say about base changes with orthogonal matrices later in the course. For now, let us return to our application. Namely, to describe the projection onto a linear subspace $S \subseteq \mathbb{R}^n$. Instead of considering an arbitrary basis of S, say $S = \text{Span}_{\mathbb{R}}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k)$ with linearly independent $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$, we wish to consider an orthonormal basis. This is always possible by the Gram-Schmidt procedure, as we will discuss momentarily. For the time being, suffice it to assume the existence of an orthonormal basis of S, i.e.

$$S = \text{Span}_{\mathbb{R}}(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k) \,, \qquad \{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k\} \text{ orthonormal basis of } S \,. \tag{4.80}$$

This replaces the matrix A by

$$Q = \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \ldots & \vec{q}_k \\ | & | & & | \end{bmatrix} . \tag{4.81}$$

In particular, $A^T A$ becomes $Q^T Q = I$ and the projection formuli simplify:

$$\hat{\vec{x}} = Q^T \vec{b} \,, \qquad \vec{p} = Q\hat{\vec{x}} \,, \qquad P = QQ^T \,. \tag{4.82}$$

There are no matrices to invert any more! This is the key simplification that we achieve with orthonormal basis.

Example 4.5.3:
Let us exemplify this with the projection matrix onto the plane in $\mathbb{R}^3$ given by $x-2y+z=0$. We already discussed this in example 4.3.2. This time, we will pick an orthonormal basis of this plane. Namely

$$\vec{q}_1 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{4.83}$$

Then we find

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.84}$$

As before, the projection matrix follows as

$$P = \frac{1}{6} \cdot \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}. \tag{4.85}$$

Remark:
Given the significance of orthonormal basis, we can ask two important questions:

- Does every linear subspace S of an inner product space have an orthonormal basis?
- If yes, how do we find such a basis?

The answer to the first question is yes. Namely, any such subspace admits a basis. And the Gram-Schmidt procedure, by answering the second question, provides an algorithmic procedure to find an orthonormal basis.

Construction 4.5.1:
Let us exemplify the Gram-Schmidt procedure by looking at a 3-dimensional linear subspace S, i.e. $S = \mathrm{Span}_{\mathbb{R}}\left(\vec{a}, \vec{b}, \vec{c}\right)$. We first intend to construct three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ which are an orthogonal basis of S. At the end we will normalize them to form an orthonormal basis of S. We perform the following steps:

- Take $\vec{A} = \vec{a}$.
- Next consider $\vec{b}$ and subtract its projection along $\vec{A}$. This gives

$$\vec{B} = \vec{b} - \frac{\left\langle \vec{A}, \vec{b} \right\rangle}{\left\langle \vec{A}, \vec{A} \right\rangle} \cdot \vec{A}. \tag{4.86}$$

  In particular $\left\langle \vec{A}, \vec{B} \right\rangle = 0$.

- Now consider $\vec{c}$. To construct $\vec{C}$ which is orthogonal to both $\vec{A}$ and $\vec{B}$ we subtract the projections of $\vec{c}$ along $\vec{A}$ and along $\vec{B}$:

$$\vec{C} = \vec{c} - \frac{\left\langle \vec{A}, \vec{c} \right\rangle}{\left\langle \vec{A}, \vec{A} \right\rangle} \cdot \vec{A} - \frac{\left\langle \vec{B}, \vec{c} \right\rangle}{\left\langle \vec{B}, \vec{B} \right\rangle} \cdot \vec{B}\,. \tag{4.87}$$

  Indeed, $\left\langle \vec{A}, \vec{C} \right\rangle = \left\langle \vec{B}, \vec{C} \right\rangle = 0$.

- Finally, normalize these vectors, i.e. devide $\vec{A}$, $\vec{B}$ and $\vec{C}$ by their lengths.

Note:
This generalizes to any finite family of vectors. For example, if there was also a vector $\vec{d}$ above, then we would form $\vec{D}$ by subtracting from $\vec{d}$ the projections along $\vec{A}$, $\vec{B}$ and $\vec{C}$.

Example 4.5.4:
In going back to example 4.3.2 once more, i.e. the projection matrix onto the plane in $\mathbb{R}^3$ given by $x - 2y + z = 0$. We have taken a basis

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \tag{4.88}$$

previously. We construct an orthonormal basis by first computing:

$$\vec{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{B} = \vec{a}_2 - \frac{\left\langle \vec{A}, \vec{a}_2 \right\rangle}{\left\langle \vec{A}, \vec{A} \right\rangle} \cdot \vec{A} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{4.89}$$

Now normalize these vectors, then we find

$$\vec{q}_1 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \tag{4.90}$$

This is exactly the basis used in example 4.5.3.

Example 4.5.5:
Here is another example. Consider the following basis of $\mathbb{R}^3$:

$$\vec{a} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \qquad \vec{c} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}. \tag{4.91}$$

By applying the Gram-Schmidt procedure, we find

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \vec{q}_2 = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \qquad \vec{q}_3 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{4.92}$$

Recall that the original basis A and the new basis Q are related by a base change. Explicitly, it holds here

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{18} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix} = QR\,. \tag{4.93}$$

This pattern holds more generally.

Corollary:
Consider the inner product space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. Then, starting with linearly independent vectors $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^n$, the Gram-Schmidt procedure constructs orthonormal vectors $\vec{q}_1, \dots, \vec{q}_n \in \mathbb{R}^n$. Let A be the matrix with columns $\vec{a}_i$ and Q the matrix with columns $\vec{q}_i$. Then $A = QR$ with $R = Q^T A$ an upper triangular matrix.

Corollary:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. Consider $\mathbb{R}^n$ with the standard inner product. Then it follows that every such matrix A can be written as $A = QR$ where $Q \in \mathbb{M}(n \times n, \mathbb{R})$ is orthogonal w.r.t. the standard inner product in $\mathbb{R}^n$ and R is an upper triangular matrix. This is termed a QR decomposition/factorization of A.

Note:
Similarly to the LU-factorization, the QR-factorization is key to efficiently perform computation in linear algebra. Let us illustrate the use of the QR-factorization for the least square approximation. Recall that this amounts to solving

$$A^T A \hat{\vec{x}} = A^T \vec{b}\,. \tag{4.94}$$

Now use $A = QR$. Then we find

$$R^T R \hat{\vec{x}} = R^T Q^T Q R \hat{\vec{x}} = (QR)^T QR \hat{\vec{x}} = (QR)^T \vec{b} = R^T Q^T \vec{b}\,. \tag{4.95}$$

Since R^T is invertible (it is a base change matrix), we conclude that this is equivalent to

$$R \hat{\vec{x}} = Q^T \vec{b}\,. \tag{4.96}$$

Since R is upper triangular, we can use back-substitution to solve this equation efficiently and fast. The real cost are the operations in the Gram-Schmidt procedure.

4.6 Application: Fourier series

Note:
We will now, for the only time in this course, leave the terrain of finite-dimensional vector spaces in this course. Namely, we will use our knowledge about orthogonality to perform linear algebra in two infinite dimensional vector spaces. Generally speaking, we have to be careful which results generalize from finite-dimensional linear algebra to infinite-dimensional linear algebra. For example, we might not be able to write matrices and vectors, but the ideas about orthogonality still do apply.

Definition 4.6.1:
We consider the vector space

$$V = \{f\colon [0,2\pi] \to \mathbb{R} \,|\; f \text{ measurable and square integrable}\}\ , \tag{4.97}$$

that is $f\colon [0,2\pi] \to \mathbb{R}$ belongs to V if and only if the following exists and is finite

$$(f,f) = \int_0^{2\pi} |f(x)|^2 \, dx\,. \tag{4.98}$$

Note:
The $f\colon [0,2\pi] \to \mathbb{R}\,,\, x \mapsto f(x) = \sin(x)$ satisfies $(f,f) = \pi$, i.e. $f \in V$. This follows as follows. We first use $\sin(x)^2 + \cos(x) = 1$ to see

$$(f,f) = \int_0^{2\pi} \sin(x)^2 dx = \int_0^{2\pi} \left(1 - \cos(x)^2\right) dx = 2\pi - \int_0^{2\pi} \cos(x)^2 dx\,. \tag{4.99}$$

Hence

$$\int_0^{2\pi} \sin(x)^2 dx + \int_0^{2\pi} \cos(x)^2 dx = 2\pi\,. \tag{4.100}$$

By periodicity of $\sin(x)$ and $\cos(x)$, we also have

$$\int_0^{2\pi} \sin(x)^2 dx = \int_0^{2\pi} \cos(x)^2 dx\,. \tag{4.101}$$

Hence

$$\int_0^{2\pi} \sin(x)^2 dx + \int_0^{2\pi} \sin(x)^2 dx = 2\pi\,. \tag{4.102}$$

This implies $(f,f) = \pi$.

Definition 4.6.2:
We define an inner product for $f, g \in V$ by

$$(f,g) = \int_0^{2\pi} f(x)g(x)dx\,. \tag{4.103}$$

Consequence:
Any two $f, g \in V$ have finite length. Therefore, it follows form the *Schwarz inequality*

$$|\,(f,g)\,|^2 \leq (f,f) \cdot (g,g) \;, \tag{4.104}$$

that also $f+g$ belongs to V. Namely,

$$\begin{aligned}(f+g, f+g) &= (f,f) + 2\,(f,g) + (g,g) \\ &\leq (f,f) + 2\sqrt{(f,f)} \cdot \sqrt{(g,g)} + (g,g) < \infty \,.\end{aligned} \tag{4.105}$$

Note:
In honour of the German mathematician David Hilbert, inner product spaces are also termed a *Prähilbertraum* – pre-Hilbert space. Hence, $(V, (\cdot,\cdot))$ is a pre-Hilbert space. It actually has more structure. Namely, the inner product induces a length for all vectors – also called a *norm*. A pre-Hilbert space in which every Cauchy sequence converges w.r.t. to this norm is called a Hibert space. The above space $(V, (\cdot,\cdot))$ satisfies this "completeness relation" and is therefore a Hilbert space.

Hilbert spaces are of fundamental interest in quantum mechanics and quantum field theory. For example, in quantum mechanics, we have the following dictionary between mathematics and physics:

Physics	Mathematics
State of a quantum system	(Equivalence class of) vector in Hilbert space $\mathcal{H}$
Measurement on quantum system	(Special) operator ($\sim$ linear map) $\mathcal{H} \to \mathcal{H}$
Possible measurement values	Eigenvalues of these (special) operators

Remark:
There is nothing special about our integrals ranging from 0 to 2π. We can study square integrable spaces such as $[0,1]$ or $(-\infty, \infty)$ just as well.

Lemma 4.6.1:
For any two $x, y \in \mathbb{R}$ it holds $\sin(x) \cdot \sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$.

Proof

We use a central property of the exponential function, namely

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \,, \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \,. \tag{4.106}$$

Then it follows

$$\begin{aligned}\sin(x) \cdot \sin(y) &= \frac{1}{4i^2} \cdot \left[e^{i(x+y)} + e^{-i(x+y)} - e^{i(x-y)} - e^{-i(x-y)}\right] \\ &= \frac{1}{2} \cdot \left(\cos(x-y) - \cos(x+y)\right) \,.\end{aligned} \tag{4.107}$$

■

Consequence:
The function $\sin(m \cdot x)$ and $\sin(n \cdot x)$ are orthogonal in V iff $m \neq n$.

Proof
We consider

$$(\sin(m \cdot x), \sin(n \cdot x)) = \frac{1}{2} \cdot \int_0^{2\pi} \cos((m-n)x) - \cos((m+n)x)\, dx\,. \tag{4.108}$$

This vanishes iff $m \neq n$. ■

Note:
Similarly, one can show that for any two $x, y \in \mathbb{R}$ it holds $\cos(x) \cdot \cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$. As a consequence, it follows that $\cos(m \cdot x)$ and $\cos(n \cdot x)$ are orthogonal in V iff $m \neq n$.

Consequence:
Consider the set

$$\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots\}\,. \tag{4.109}$$

Any two distinct functions in this list are orthogonal. It would be nice to get a basis of V out of these. This is essentially the idea behind the Fourier series.

Definition 4.6.3:
The *Fourier series* of a function $f(x)$ is its expansion

$$f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \dots\,. \tag{4.110}$$

Note:
As $\sin(x)$ and $\cos(x)$ are periodic, our function f must be periodic as well.

Remark:
Let us turn the tables around. Given a choice of coefficients a_i, we may wonder if the resulting series is the Fourier series of a function $f \in V$. As preparation for this, let us introduce the vector space of these coefficients.

Definition 4.6.4:
We consider the vector space

$$W = \left\{ \vec{v} = (v_1, v_2, v_3, \dots) \mid v_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} v_i^2 < \infty \right\}. \tag{4.111}$$

We define an inner product for $\vec{v}, \vec{w} \in W$ by $\vec{v} \cdot \vec{w} = \sum_i v_i w_i$.

Example 4.6.1:
The vector $\vec{v} = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \dots\right)$ has the feature that $\vec{v} \cdot \vec{v} = 2$. Thus $\vec{v} \in W$.

Note:
W is another example of a *Hilbert spaces.* If $\vec{v}, \vec{w} \in W$, then it again follows from the *Schwarz inequality* that also $v + w \in W$.

Remark:
We now have to answer the question, which Fourier series are actually honest functions. To this end, let us compute the length of a function $f \in V$ from its expansion. For this, let us use the orthonormal basis

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \dots \right\} . \tag{4.112}$$

Then we have

$$\begin{aligned}(f,f) &= \int_0^{2\pi} \left(\frac{a_0}{\sqrt{2\pi}} + \frac{a_1}{\sqrt{\pi}} \cos(x) + \frac{b_1}{\sqrt{\pi}} \sin(x) + \dots \right)^2 dx \\ &= \int_0^{2\pi} \left(\frac{a_0^2}{2\pi} + \frac{a_1^2}{\pi} \cos^2(x) + \frac{b_1^2}{\pi} \sin^2(x) + \frac{a_2^2}{\pi} \cos^2(2x) + \dots \right)^2 dx \\ &= a_0^2 + a_1^2 + b_1^2 + a_2^2 + b_2^2 + \dots .\end{aligned} \tag{4.113}$$

This implies $f \in V$ if and only if its vector of coefficients belongs to W.

Note:
Given $f \in V$, we term the vector

$$\vec{v}(f) = (a_0, a_1, b_1, a_2, b_2, \dots) \in W , \tag{4.114}$$

formed from the coefficients of the Fourier series of f, the *Fourier transform* of f. In this sense, we have just found a 1-to-1 correspondance between function $f \in V$ and their Fourier transforms. Put differently, for all $f \in V$ it is possible to recover the function f uniquely from its Fourier transform. This is the so-called *Fourier inversion theorem.* Let us mention again, that there is nothing special about our integrals ranging over $[0, 2\pi]$. It is for example possible to generalize to $(-\infty, \infty)$, which is common to formulate the Fourier inversion theorem.

Example 4.6.2:
Let us consider the function

$$f\colon [0, 2\pi] \to \mathbb{R} , \, x \mapsto f(x) = \begin{cases} 1 & 0 \le x < \pi \\ -1 & \pi \le x \le 2\pi \end{cases} . \tag{4.115}$$

and compute its Fourier series:

$$f(x) = a_0 + \sum_{k \ge 1} a_k \cos(kx) + \sum_{k \ge 1} b_k \sin(kx) . \tag{4.116}$$

We can identity the coefficients by apply our knowledge of inner products. Namely, in order to find a_k, we simply compute the inner product with $\cos(kx)$:

$$\int_0^{2\pi} f(x)\cos(kx)dx = \int_0^{2\pi}\left(a_0 + \sum_{k\geq 1} a_k \cos(kx) + \sum_{k\geq 1} b_k \sin(kx)\right) dx\,. \tag{4.117}$$

By orthogonality, we then find

$$\begin{aligned}\int_0^{2\pi} f(x)\cos(kx)dx &= a_k \int_0^{2\pi} \cos^2(kx)dx = \frac{1}{2}\cdot a_k \cdot 2\pi\,,\\ \Leftrightarrow \qquad a_k &= \frac{1}{\pi}\cdot\int_0^{2\pi} f(x)\cos(kx)dx\,.\end{aligned} \tag{4.118}$$

Similarly, we can find

$$b_k = \frac{1}{\pi}\cdot\int_0^{2\pi} f(x)\sin(kx)dx\,, \qquad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx\,. \tag{4.119}$$

So in particular, a_0 is the average value of f on $[0, 2\pi]$. Let us apply this to the function in eq. (4.115). Since this function is odd, the cosine terms in the Fourier series vanish. Moreover, we readily confirm $b_k = \frac{4}{\pi k}$. Thus

$$a_0 + \sum_{k\geq 1} a_k \cos(kx) + \sum_{k\geq 1} b_k \sin(kx) = \frac{4}{\pi}\cdot\left[\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \ldots\right]. \tag{4.120}$$

Note that this series vanishes at $x = 0$, which is different from $f(0) = 1$! This is because $f(x)^2$ is not continous at $x = 0$!

Note:
In general, you want to be careful to compare a function to a series expansion. Another example of this sort is the Taylor expansion. For "well-behaved" functions, these series expansions coincide with the original functions. But this is not true in general.

Remark:
The above Fourier series for the function f in eq. (4.115) is reliable at $x = \frac{\pi}{2}$. Therefore, we find

$$f\left(\frac{\pi}{2}\right) = 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right). \tag{4.121}$$

This is equivalent to the famous *Leibniz formula for* π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\,. \tag{4.122}$$

Remark:
We have come a long way in terms of computing π using series. Here is an example:

$$\frac{4}{\pi} = \frac{1}{882} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(4^n n!)^4} \cdot \frac{(4n)!}{882^{2n}} \cdot (1123 + 21460 \cdot n)\,. \tag{4.123}$$

Note:
Let us conclude this discussion, by analysing the Fourier coefficient computation with our knowledge about inner products. Given an orthonormal basis $\vec{v}_1, \dots, \vec{v}_n$ of an inner product space $(V, \langle \cdot, \cdot \rangle)$, we can express any $\vec{v} \in V$ as

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n\,. \tag{4.124}$$

The coefficients c_i are simply given by

$$\langle \vec{v}_i, \vec{v} \rangle = \sum_{j=1}^{n} c_j \, \langle \vec{v}_i, \vec{v}_j \rangle = c_i\,. \tag{4.125}$$

In our computation of Fourier series, we did exactly the same thing. Namely, for

$$f(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{k \geq 1} \frac{a_k \cos(kx)}{\sqrt{\pi}} + \sum_{k \geq 1} \frac{b_k \sin(kx)}{\sqrt{\pi}}\,, \tag{4.126}$$

we computed for $k \geq 1$ the inner products

$$a_k = (f(x), \cos(kx))\,, \qquad b_k = (f(x), \sin(kx))\,. \tag{4.127}$$

5 Determinants

In this section we go back to the question of inverses of matrices. We already found that we can find the inverse of $A \in \mathbb{M}(n \times n, \mathbb{R})$ by Gauss-Jordan elimination. Also, this process fails exactly when A has no inverse.

We will now extend this analysis by studying determinants. For $A \in \mathbb{M}(n \times n, \mathbb{R})$, the determinant is a real number i.e. $\det(A) \in \mathbb{R}$. This number tells us immediately if a matrix is invertible or not. Namely, we will find that A is invertible iff $\det(A) \neq 0$. In extending, we can even find formuli for A^{-1}.

On a more theoretical level, it must be noted that the determinant is by itself a remarkable function with interesting properties. In fact, these properties uniquely fix the determinant. This is a feature addressed formally as *universal properties* in category theory. While we will not touch upon this category theory point, we will elaborate why these properties uniquely fix the determinant. As a consequence, knowing these properties is as good as knowing an explicit mapping rule for the determinant. This in turn is beneficial in practical computations, for which abstract arguments can replace or at least simplify brute force computations.

5.1 The Properties of Determinants

Note:
In this section we consider fixed but arbitary $n \in \mathbb{Z}_{>0}$. Then the determinant is a map $\det\colon \mathbb{M}(n \times n, \mathbb{R}) \to \mathbb{R}$. We define it by its properties.

Definition 5.1.1 (Determinant)**:**
The determinant $\det\colon \mathbb{M}(n \times n, \mathbb{R}) \to \mathbb{R}$ is a map with the following properties:

1. The determinant is linear in all rows of A (one says it is multi-linear):

$$\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_{k-1}^T \\ \vec{a}_k^T + \vec{b}_k^T \\ \vec{a}_{k+1}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_{k-1}^T \\ \vec{a}_k^T \\ \vec{a}_{k+1}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) + \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_{k-1}^T \\ \vec{b}_k^T \\ \vec{a}_{k+1}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right). \tag{5.1}$$

2. The determinant is alternating in the rows of A:

$$\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_i^T \\ \vdots \\ \vec{a}_j^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_j^T \\ \vdots \\ \vec{a}_i^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right). \tag{5.2}$$

3. The determinant of the identity matrix is 1, i.e. $\det(I) = 1$.

Note:
It is not automatic, that a map with these properties does even exist. Neither does it follow immediately that this map is unique. To see that it exists and is unique, we will use the above properties to derive rules for how to compute the determinant of a given matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$. This will estabish both the existence and the uniqueness of this map.

Corollary 5.1.1:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ with two identical rows. Then $\det(A) = 0$.

Proof
We denote the two identical rows of A as $\vec{a}^T$. When we exchange those rows, the matrix A remains unchanged. However, since the determinant is alternating in the rows of A, the sign of the determinant changes:

$$\det(A) = \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}^T \\ \vdots \\ \vec{a}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}^T \\ \vdots \\ \vec{a}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = -\det(A)\,. \tag{5.3}$$

The only real number with this property is 0. Hence $\det(A) = 0$. ∎

Corollary 5.1.2:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. Subtracting a multiple of one row of A from another row of A leaves $\det(A)$ unchanged.

Proof
W.l.o.g. let us assume that the relevant rows are the first two of A. Since the

determinant is linear in the rows of A, it follows

$$\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T - \lambda \cdot \vec{a}_1^T \\ \vec{a}_3^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vec{a}_3^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) - \lambda \cdot \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_1^T \\ \vec{a}_3^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) \tag{5.4}$$

The last matrix has two identical rows. Hence, by corollary 5.1.1, its determinant vanishes which proves our claim. ■

Note:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ with (at least) one zero row. Then $\det(A) = 0$. Namely, we add any other row of A to this zero row. By corollary 5.1.2, this leaves the determinant unchanged. The resulting matrix has now two identical rows and by corollary 5.1.1, the determinant vanishes.

Corollary 5.1.3:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. We rescale its rows by $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \setminus 0$ to obtain A'. Then

$$\det(A') = \prod_{i=1}^{n} \lambda_i \cdot \det(A) \,. \tag{5.5}$$

Proof
This follows from the multi-linearity of the determinant in the rows of A. ■

Corollary 5.1.4:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ be an upper (or lower) triangular matrix. Then

$$\det(A) = \prod_{i=1}^{n} a_{ii} \,. \tag{5.6}$$

Proof
We distinguish two cases:

- All diagonal entries of A are non-zero:
  By elementary row operations, we can bring A into diagonal form. By corollary 5.1.2, the determinant remains unchanged. By corollary 5.1.3, it follows then

$$\det(A) = \prod_{i=1}^{n} a_{ii} \cdot \det(I) \,. \tag{5.7}$$

  The normalization of the determinant states $\det(I) = 1$. Hence, the claim follows.

- At least one diagonal entry of A vanishes:
  Then A is singular. We can again perform elementary row operations, and the determinant remains unchanged by corollary 5.1.2. However, since A is singular, this process leads to a *zero row*. The determinant vanishes then as consequence of corollary 5.1.1. This matches the product of the diagonal entries, since at least one diagonal entry was assumed to vanish.

This completes the proof. ■

Convention:
By using corollary 5.1.3, corollary 5.1.4 we can compute the determinant for any matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ as follows:

- Employ elementary row transformations to bring A into upper trianguar form U. This process may involve row exchanges. Let their number by N. Then, by corollary 5.1.3 and the alternatingness of the determinant, if follows

$$\det(A) = (-1)^N \cdot \det(U)\,. \tag{5.8}$$

- Use corollary 5.1.4 to infer $\det(U)$ and thereby $\det(A)$.

Note:
This proves existence and uniqueness of the determinant. Even more, it shows that if we consider a function $f\colon S \to \mathbb{R}$ which is multi-linear and alternating in the rows of A, but not normalized as the determinant, then $f(A) = c \cdot \det(A)$ for a suitable constant $c \in \mathbb{R}$, which is given by $c = f(I)$. This observation allows us to prove the following.

Corollary 5.1.5:
Let $A, B \in \mathbb{M}(n \times n, \mathbb{R})$. Then $\det(AB) = \det(A) \cdot \det(B)$.

Proof
We distinguish two cases:

- B is singular:
  Then also AB is singular and $\det(AB) = 0 = \det(B)$.

- B non-singular:
  Then $\det(B) \neq 0$. We can therefore consider the map

$$f\colon \mathbb{M}(n \times n, \mathbb{R}) \to \mathbb{R}\,,\; A \mapsto \frac{\det(AB)}{\det(B)}\,. \tag{5.9}$$

  This map f is multilinear and alternating in the rows of A. Furthermore, for $A = I$ we have $f(I) = 1$. Hence, this map has the defining properties of the determinant of A and it follows $f(A) = \det(A)$.

This completes the proof. ■

Note:
This generalizes to $\det\left(\prod_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \det(A_i)$.

Corollary 5.1.6:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then the following holds true:

- If A is singular, then $\det(A) = 0$.
- If A is invertible, then $\det(A) \neq 0$.

Proof

- If A is singular, then at least one of its pivots vanishes. Consequently, we can bring A into an upper triangular form, but at least one diagonal entry vanishes. It follows from corollary 5.1.4 that $\det(A) = 0$.
- If A is invertible, then none of its pivots vanish. Hence, we bring A into an upper tringular form U for which all diagonal entries are non-zero. It follows from corollary 5.1.4 that $\det(A) \neq 0$. ■

Corollary 5.1.7:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then $\det(A) = \det(A^T)$.

Proof
We first note the following:

- Consider a permutation matrix $P \in \mathbb{M}(n \times n, \mathbb{R})$. All pivots of P are 1. Hence $\det(P) = \pm 1$. Furthermore, $P \cdot P^T = I$ implies $\det(P) \cdot \det(P^T) = 1$. It follows $\det(P) = \det(P^T)$.
- Consider an upper triangular matrix $U \in \mathbb{M}(n \times n, \mathbb{R})$. Then, since U and U^T have the same pivots, it follows $\det(U) = \det(U^T)$. Similarly, for a lower triangular matrix $L \in \mathbb{M}(n \times n, \mathbb{R})$ it holds $\det(L) = \det(L^T)$.

With this we can now prove the general statement. To this end, consider $A \in \mathbb{M}(n \times n, \mathbb{R})$. If A is singular, then so is A^T and it follows $\det(A) = \det(A^T)$. However, if A is non-singular we can express it by a PLU-decomposition as $A = PLU$. Hence

$$\det(A) = \det(P) \cdot \det(L) \cdot \det(U)\,. \tag{5.10}$$

Upon transposition, we then find $A^T = U^T L^T P^T$. Hence

$$\det(A^T) = \det(U^T) \cdot \det(L^T) \cdot \det(P^T) = \det(P) \cdot \det(L) \cdot \det(U) = \det(A)\,. \tag{5.11}$$

This completes the proof. ■

Note:
As a consequence of this result, we notice that the determinant is multi-linear and alternating in the columns of A.

5.2 Applications of determinants

5.2.1 Symbolic computation of matrix inverses

In this section we wish to solve $A\vec{x} = \vec{b}$ algebraically, and not by elimination. This will involve quotients of certain determinants.

Note:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ and $\vec{b} \in \mathbb{R}^n$. We wish that $\vec{x} \in \mathbb{R}^n$ with

$$A\vec{x} = \vec{b}, \tag{5.12}$$

exists iff $\det(A) \neq 0$. We thus proceed under the assumption that $\det(A) \neq 0$. We write

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix}. \tag{5.13}$$

Then we notice that

$$A \cdot \left[\begin{array}{c|cccc} | & 0 & 0 & \dots & 0 \\ | & 1 & 0 & \dots & 0 \\ \vec{x} & 0 & 1 & \ddots & \vdots \\ | & \vdots & \ddots & \ddots & 0 \\ | & 0 & \dots & 0 & 1 \end{array}\right] = \begin{bmatrix} | & | & & | \\ \vec{b} & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix}. \tag{5.14}$$

Consequently,

$$\det(A) \cdot \underbrace{\det\left(\left[\begin{array}{c|cccc} | & 0 & 0 & \dots & 0 \\ | & 1 & 0 & \dots & 0 \\ \vec{x} & 0 & 1 & \ddots & \vdots \\ | & \vdots & \ddots & \ddots & 0 \\ | & 0 & \dots & 0 & 1 \end{array}\right]\right)}_{=x_1} = \underbrace{\det\left(\begin{bmatrix} | & | & & | \\ \vec{b} & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix}\right)}_{:=\det(B_1)}. \tag{5.15}$$

Hence, since we assumed $\det(A) \neq 0$, we can write

$$x_1 = \frac{\det(B_1)}{\det(A)}, \tag{5.16}$$

where B_1 is the matrix obtained by replacing the first columns of A by $\vec{b}$.

Consequence:
Cramer's rule Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ with $\det(A) \neq 0$ and $\vec{b} \in \mathbb{R}^n$. The unique vector $\vec{x} \in \mathbb{R}^n$ with

$$A\vec{x} = \vec{b}, \tag{5.17}$$

satisfies

$$x_i = \frac{\det(B_i)}{\det(A)}\,, \tag{5.18}$$

where $B_i \in \mathbb{M}(n \times n, \mathbb{R})$ is obtained by replacing the i-th column of A by $\vec{b}$.

Note:
For matrices with numbers as entries, Cramer's rule is inefficient. But for symbolic operations, Cramer's rule can be useful.

Example 5.2.1:
Consider the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,, \tag{5.19}$$

where $a, b, c, d \in \mathbb{R}$ are arbitrary but fixed real numbers. We assume that A is invertible.

We wish to find the inverse of this matrix from Cramer's rule. To this end, we definitely need to compute a number of determinants, in particular $\det(A)$. Without loss of generality, we may assume that a is a pivot of A, and thus $a \neq 0$. Then, by subtracting $\left(\frac{c}{a}\right)$-times the first row from the second, we find

$$A \to \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix}. \tag{5.20}$$

Since this is an upper triangular matrix, it follows that $\det(A) = a \cdot \left(d - \frac{bc}{a}\right) = ad - bc$.

Next we compute the entries of $A^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. Hence, $A \cdot \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and by Cramer's rule, we have

- $\alpha = \frac{\det\left(\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}\right)}{\det(A)} = \frac{d}{\det(A)}$,
- $\gamma = \frac{\det\left(\begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix}\right)}{\det(A)} = -\frac{\det\left(\begin{bmatrix} c & 0 \\ a & 1 \end{bmatrix}\right)}{\det(A)} = -\frac{c}{\det(A)}$.

Similarly, for β and δ we have $A \cdot \begin{bmatrix} \beta \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Cramer's rule now gives

- $\beta = \frac{\det\left(\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}\right)}{\det(A)} = -\frac{\det\left(\begin{bmatrix} 1 & d \\ 0 & b \end{bmatrix}\right)}{\det(A)} = -\frac{b}{\det(A)}$,
- $\delta = \frac{\det\left(\begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix}\right)}{\det(A)} = \frac{a}{\det(A)}$.

So overall,

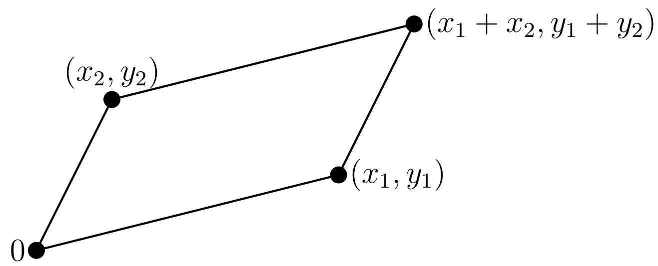
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{5.21}$$

Note:
While inverses of matrices may or may not exist, we can in general say the following:

- In general, an analytic expression for the inverse of $A \in \mathbb{M}(n \times n, \mathbb{R})$ is hard to remember. The above (2×2)-case may the be solve exception to that rule.
- A closed analytic expression for A^{-1} does exist in terms of the cofactors. We will discuss cofactors momentarily.

5.2.2 Areas and volumes

Note:
Let us consider a parallelogram with corners $(0,0)$, (x_1, y_1), (x_2, y_2) and (x_1+x_2, y_1+y_2):



(5.22)

I claim that its area is given by

$$A = \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}. \tag{5.23}$$

We establish this fact by considering the area A as a function

$$A \colon \mathbb{M}(2 \times 2, \mathbb{R}) \to \mathbb{R}, \tag{5.24}$$

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \mapsto \text{area of parallelogram eq. (5.22)}. \tag{5.25}$$

To verify that A is the determinant, it suffices to verify that A satisfies the three defining properties of determinants:

- Property 3:
  If $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = I$, then the parallelogram eq. (5.22) is a square with area 1.
- Property 2:
  If we exchange the rows, then the parallelogram remains the same as collection of points. On the other hand, the determinant changes sign, indicating whether the edges form a right-handed tuple $(\det(A) > 0)$ or a left-handed tuple $(\det(A) < 0)$. This sign information, we embrace in the area of the parallelogram.

- Property 1:
  The area of the parallelogram associated to

$$\begin{bmatrix} x_1 & y_1 \\ \lambda x_2 + x_2' & \lambda y_2 + y_2' \end{bmatrix}, \tag{5.26}$$

  is the sum of the areas of the parallelograms $\begin{bmatrix} x_1 & y_1 \\ \lambda x_2 & \lambda y_2 \end{bmatrix}$ and $\begin{bmatrix} x_1 & y_1 \\ x_2' & y_2' \end{bmatrix}$.

Remark:
Whilst this proof may seem exotic – we could have simply done with basic geometry – it will allow us to extend this result to arbitrary dimension. Before we get to this, let us point out the following result.

Claim:
The area of the triangle T in $\mathbb{R}^3$ with corners (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by

$$A(T) = \frac{1}{2} \cdot \det\left(\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}\right). \tag{5.27}$$

Proof
Let us consider

$$\Delta_1 = (x_1 - x_3, y_1 - y_3), \qquad \Delta_2 = (x_2 - x_3, y_2 - y_3). \tag{5.28}$$

These are two of the three sides of the triangle in a shift coordinate system, in which (x_3, y_3) is considered the origin. Hence, by applying the above results, we conclude that

$$A(T) = \frac{1}{2} \cdot \det\left(\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}\right). \tag{5.29}$$

This completes the proof. ■

Exercise:
By an explicit computation one can show that

$$\det\left(\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}\right) = \det\left(\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right). \tag{5.30}$$

Note:
To see how these results generalize to $\mathbb{R}^n$, we first note that a triangle with corners $(0, 0)$, (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ can be described as the convex hull

$$\begin{aligned} T &= \mathrm{Conv}\left\{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right\} \\ &= \left\{a \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, a, b \in \mathbb{R}_{\geq 0} \text{ and } a + b \leq 1\right\}. \end{aligned} \tag{5.31}$$

Similarly, the parallelogram in eq. (5.22) can be described as

$$\begin{aligned} P &= \operatorname{Conv}\left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \right\} \\ &= \left\{ a \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + c \cdot \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, a, b, c \in \mathbb{R}_{\geq 0} \text{ and } a + b + c \leq 1 \right\} . \end{aligned} \tag{5.32}$$

Corollary:
For any two $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^2$, the area of the convex hull $T = \operatorname{Conv}\{\vec{a}_1, \vec{a}_2\}$ is given by

$$A(T) = \frac{1}{2} \cdot \det\left(\begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \right) . \tag{5.33}$$

and for $P = \operatorname{Conv}\{\vec{a}_1, \vec{a}_2, \vec{a}_1 + \vec{a}_2\}$ it holds

$$A(P) = \det\left(\begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \right) . \tag{5.34}$$

Note:
In two dimensions, we talk about the area of a triangle, a parallelogram etc. However, the established wording for the equivalent quantity in 3 dimension is volume. For example, we talk about the volume of a bottle, whereas the area of a bottle is not clearly defined. It requires to make reference to the surface, bottom, ... of the bottle. With this terminology in mind, let me generalization our results to $\mathbb{R}^n$ by replacing *area* by *volume*.

Consequence:
Let $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$. Then the *volume* of the hyper-parallelogram

$$P = \operatorname{Conv}\left\{ \vec{a}_1, \ldots, \vec{a}_n, \sum_{i=1}^{n} \vec{a}_i \right\} \tag{5.35}$$

is given by

$$V(P) = \det\left(\begin{bmatrix} | & \cdots & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & \cdots & | \end{bmatrix} \right) . \tag{5.36}$$

Cross and triple product

These notions are special to three dimensions. I will not disuss them here in detail, but additional information is for example available in Strang's book on pages 279ff.

5.3 Three ways to compute determinants

In the previous section we have convinced ourselves that the determinant exists and is unique. Moreover, we already derived quite a few useful properties for computing determinants. We will now extend this study in order to find rules by which we can easily compute the determinant.

5.3.1 The Pivot formula

Remark:
This pivot formula follows directly from section 5.1. Namely, given $A \in \mathbb{M}(n \times n, \mathbb{R})$, we can write

$$A = P \cdot L \cdot U\,. \tag{5.37}$$

L is lower triangular and has 1s along the diagonal, so its determinant is 1. $\det(P) = (-1)^N$ with N the number of row exchanges required to bring A into the form LU. Finally, on the diagonal of U we list the pivots of A. Hence

$$\det(A) = \det(P) \cdot \det(U) = (-1)^N \cdot \prod_{i=1}^{n} U_{ii}\,. \tag{5.38}$$

Example 5.3.1:
Let us compute the determinant of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}. \tag{5.39}$$

We readily find a PLU decomposition of A:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ \frac{1}{3} & \frac{-2}{3} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = LU\,. \tag{5.40}$$

In this case, we have $P = I$ and hence $\det(P) = 1$, which corresponds to $N = 0$ above. Furthermore, $\det(L) = 1$. This indeed only leaves to find the determiant of U. Hence

$$\det(A) = \det(U) = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} = 4\,. \tag{5.41}$$

5.3.2 The Big Formula for Determinants

Example 5.3.2:
Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{5.42}$$

Then one can compute the determinant as follows:

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \,. \tag{5.43}$$

This is an instance of the big formula for the determinant. It generalizes as follows

Claim:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. We consider the symmetric group S_n. Then

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \,. \tag{5.44}$$

Proof
We have to show that this quantity has the three defining properties of the determinant. Linearity in the rows is clear. It is alternating in the rows as a consequence of $\operatorname{sgn}(\sigma)$. Finally, for the identity matrix we have

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} = \det(I) \cdot 1 \cdot \dots \cdot 1 = 1. \tag{5.45}$$

This completes the proof. ■

Note:
For $A \in \mathbb{M}(n \times n, \mathbb{R})$, this big formula consists of $n!$ terms, of which half have positive and the remaining ones negative sign. The total number of terms increases sharply with n.

Example 5.3.3:
Let us again compute the determinant of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} . \tag{5.46}$$

With the big formula we then find

$$\det(A) = 8 + 0 + 0 - 0 - 2 - 2 = 4 \,. \tag{5.47}$$

5.3.3 Determinant by cofactors

Note:
Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} . \tag{5.48}$$

Then one can compute the determinant as follows:

$$\begin{aligned}\det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \\ &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) .\end{aligned} \tag{5.49}$$

The three quantities in parentheses are called *cofactors*. We can understand them clearer by writing this finding as

$$\begin{aligned}\det(A) = \det\left(\begin{bmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33}\end{bmatrix}\right) - \det\left(\begin{bmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33}\end{bmatrix}\right) \\ + \det\left(\begin{bmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{bmatrix}\right) .\end{aligned} \tag{5.50}$$

We thus understand the cofactors as determinants of 2×2 ”submatrices“ of A. These submatrices M_{1j} are obtained by crossing out row 1 and column j from A. This leads to the following observation.

Corollary 5.3.1 (Cofactor expansion)**:**
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then

$$\det(A) = \sum_{j=1}^{n} a_{1j} \cdot C_{1j} \qquad C_{1j} = (-1)^{1+j} \cdot \det(M_{1j}) . \tag{5.51}$$

Note:
In fact, we can expand the determinant about any row and any column of A in this spirit. For example, expanding about the i-th row is given by

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot C_{ij} \qquad C_{ij} = (-1)^{i+j} \cdot \det(M_{ij}) . \tag{5.52}$$

M_{ij} is obtained by crossing out the i-th row and the j-th column.

6 Eigenvalues and Eigenvectors

The primary topic of this chapter are eigenvalues and eigenvectors. The former are certain special numbers, wheres the latter are certain special vectors.

6.1 Basic properties of eigenvalues and eigenvectors

Remark:
We work with square matrices A. Recall that any $n \times n$ matrix A may be thought of as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. In other words, A is a function that takes $\vec{x} \in \mathbb{R}^n$ as input and outputs $A\vec{x} \in \mathbb{R}^n$. We typically expect $A\vec{x}$ to be quite "different" from $\vec{x}$. That said, there are certain non-zero vectors with a special property.

Definition 6.1.1 (Eigenvector and eigenvalue)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. A *non-zero* vector $\vec{x} \in \mathbb{R}^n$ with the property $A\vec{x} = \lambda\vec{x}$ – $\lambda \in \mathbb{R}$ – is called an *eigenvector* of A. λ is then termed the *eigenvalue* of $\vec{x}$.

Note:
One immediate goal is to compute eigenvalues and eigenvectors in an efficient manner. But before that, let us discuss a few examples.

Example 6.1.1 (Eigenvalues of reflections)**:**
Consider the reflection about a 2-dimensional linear subspace $S \subset \mathbb{R}^3$ with mapping matrix A. What are the eigenvalues and eigenvectors of A?

Observe that if $\vec{x}$ belongs to S, then $A\vec{x} = \vec{x}$. Thus, every non-zero vector in S is an eigenvector of A with eigenvalue 1. Furthermore, observe that any non-zero vector $\vec{x}$ orthogonal to S satisfies $A\vec{x} = -1\vec{x}$, i.e. those are eigenvectors of S with eigenvalue -1. Since $S \oplus S^\perp \cong \mathbb{R}^3$, these are all the eigenvectors of A. Consequently, the eigenvalues of A are ± 1.

Example 6.1.2:
The eigenvalues of real matrices do not have to exist. To this end consider the following matrix, which corresponds to a rotation by 90 degrees in the plane $\mathbb{R}^2$:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \tag{6.1}$$

Then it holds

$$A \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = (-i) \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} , \qquad A \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} . \tag{6.2}$$

Hence, this matrix has complex eigenvalues $\pm i$ and complex values eigenvectors.

We will uncover the reasons behind this momentarily. In anticipation of such complex eigenvalues and eigenvectors, we extend the previous definition of eigenvectors and eigenvalues as follows.

Definition 6.1.2 (Complex eigenvectors and eigenvalues)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R}) \subset \mathbb{M}(n \times n, \mathbb{C})$. A *non-zero* vector $\vec{x} \in \mathbb{C}^n$ with the property $A\vec{x} = \lambda\vec{x}$ – $\lambda \in \mathbb{C}$ – is called an *eigenvector* of A. λ is termed the *eigenvalue* of $\vec{x}$.

Note:
Since any $A \in \mathbb{M}(n \times n, \mathbb{R})$ can naturally be understood as element of $\mathbb{M}(n \times n, \mathbb{C})$, it makes sense to think of the eigenvalues and eigenvectors as complex valued. However, there are interesting cases, in which all eigenvalues of a matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ are real. Most importantly, we will eventually show, that this is true for symmetric matrices, i.e. matrices with $A = A^T$. In such cases, it makes sense to think of the eigenvalues and eigenvectors are real valued. With this application in mind, let us formulate two notations of the so-called eigenspace.

Definition 6.1.3 (Eigenspace)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$ and $\lambda \in \mathbb{C}$ an eigenvalue of A. Then we term

$$\mathrm{Eig}\left(A, \lambda\right) \equiv \mathrm{Eig}_{\mathbb{C}}\left(A, \lambda\right) := \left\{\vec{x} \in \mathbb{C}^n \,|\, A\vec{x} = \lambda\vec{x}\right\} \,, \tag{6.3}$$

the *complex eigenspace* of A to the eigenvalue λ (or for short the λ-eigenspace of A). In case $\lambda \in \mathbb{R}$, we define

$$\mathrm{Eig}_{\mathbb{R}}\left(A, \lambda\right) := \left\{\vec{x} \in \mathbb{R}^n \,|\, A\vec{x} = \lambda\vec{x}\right\} \,, \tag{6.4}$$

and term it the *real eigenspace* of A to the eigenvalue λ.

Example 6.1.3:
For a reflection $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ about a plane $S \subseteq \mathbb{R}^3$ it holds:

- $\mathrm{Eig}_{\mathbb{R}}\left(A, 1\right) = \left\{\vec{x} \in \mathbb{R}^3 \,|\, A\vec{x} = \vec{x}\right\} = S$,
- $\mathrm{Eig}_{\mathbb{R}}\left(A, -1\right) = \left\{\vec{x} \in \mathbb{R}^3 \,|\, A\vec{x} = -\vec{x}\right\} = S^{\perp}$.

Consequently, for reflections, we can identify the eigenvectors, eigenvalues and eigenspaces from simple geometric intuition. Is this true more generally?

Note:
In the previous example, the 1-eigenspace $\mathrm{Eig}_{\mathbb{R}}\left(A, 1\right)$ has dimension 2 while the 0-eigenspace $\mathrm{Eig}_{\mathbb{R}}\left(A, 0\right)$ has dimension 1. The dimension of $\mathbb{R}^3$ is 3. We will return to this aspect very soon.

Example 6.1.4:
Consider the permutation matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) \,. \tag{6.5}$$

Thus, A takes a vector in $\mathbb{R}^2$ as input and spits out the vectors obtained by swapping the components of the input. Geometrically, A performs a reflection at the line $y = x$. Again, we can use this to find the eigenvalues and eigenvectors. Namely, all vector $\vec{x}$ on the line $x = y$ are of the form

$$\vec{x} = \begin{bmatrix} a \\ a \end{bmatrix} \in \mathbb{R}^2 , \tag{6.6}$$

and satisfy $A\vec{x} = \vec{x}$. Consequently:

$$\mathrm{Eig}_{\mathbb{R}}(A, 1) = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} \in \mathbb{R}^2 \,|\, a \in \mathbb{R} \right\} . \tag{6.7}$$

Are there any other eigenvalues? Indeed, namely observe the line normal to the line of reflection. It forms the eigenspace of A to the eigenvalue -1:

$$\mathrm{Eig}_{\mathbb{R}}(A, -1) = \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} \in \mathbb{R}^2 \,|\, a \in \mathbb{R} \right\} . \tag{6.8}$$

Note that both eigenspaces are of dimension 1 and that $1 + 1 = 2 = \dim(\mathbb{R}^2)$.

Question:
How do we compute eigenvalues and eigenvectors in general? We are trying to solve $A\vec{x} = \lambda\vec{x}$, except that we know neither λ nor $\vec{x}$.

Suppose that $\vec{x}$ is an eigenvector. Then $A\vec{x} = \lambda\vec{x}$. Equivalently, $A - \lambda I$ has non-zero nullspace, i.e. $A - \lambda I$ is singular. It then follows that $\det(A - \lambda I) = 0$.

Definition 6.1.4 (Characteristic polynomial)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then

$$\mathrm{ch}_A(\lambda) := \det(A - \lambda I) \in \mathbb{R}[\lambda] \subset \mathbb{C}[\lambda] , \tag{6.9}$$

is called the characteristic polynomial of A.

Note:
If $A \in \mathbb{M}(n \times n, \mathbb{R})$, then $\mathrm{ch}_A(\lambda)$ is a polynomial of degree n. At this point, you want to recall the fundamental theorem of algebra.

Theorem 6.1.1 (Fundamental theorem of algebra)**:**
Let $p \in \mathbb{C}[\lambda]$ be a polynomial of degree n. Then p has, counted with multiplicity, exactly n zeros. That is, there are $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $a \in \mathbb{C}$ such that

$$p = a \cdot \prod_{i=1}^{n} (\lambda - \lambda_i) , \tag{6.10}$$

but the λ_i need not be pairwise distinct.

Exercise:
What does this theorem tell you about the eigenvalues of a matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$?

Consequence:
Once we know all zeros of $\mathrm{ch}_A(\lambda)$, i.e. all eigenvalues of A, then we find the corresponding eigenspace by $\mathrm{Eig}(A, \lambda) = N(A - \lambda I)$.

Example 6.1.5:
Consider the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) . \tag{6.11}$$

Then it is readily verified that $\mathrm{ch}_A(\lambda) = \lambda^2 - 8\lambda + 15$. Note that

- 8 is the sum of the entries along the diagonal of A – the so-called *trace* tr(A),
- 15 is the determinant of A.

We note that

$$\mathrm{ch}_A(\lambda) = \lambda^2 - \mathrm{tr(A)} \cdot \lambda + 15 = (\lambda - 5) \cdot (\lambda - 3) \ . \tag{6.12}$$

Hence, the eigenvalues of A are 3 and 5. It is then readily verified that

$$\begin{aligned} \mathrm{Eig}_{\mathbb{R}}(A, 3) &= \left\{ c \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} , \, c \in \mathbb{R} \right\} , \\ \mathrm{Eig}_{\mathbb{R}}(A, 5) &= \left\{ c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \, c \in \mathbb{R} \right\} , \end{aligned} \tag{6.13}$$

Exercise:
Convince yourself that the matrices $A, B \in \mathbb{M}(2 \times 2, \mathbb{R})$ with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} , \tag{6.14}$$

have the same eigenspaces. Their eigenvalues are $-1, 1$ and $3, 5$, respectively. Moreover $B = A + 4 \cdot I$.

Show that if $B = A + c \cdot I$, then the eigenvalues of B are obtained by adding c to the eigenvalues of A. The eigenspaces remain unchanged.

Example 6.1.6:
Let us again come back again to note that real matrices need not have real eigenvalues nor eigenvectors. We already discussed the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) . \tag{6.15}$$

Convince yourself that $\mathrm{ch}_A(\lambda) = \lambda^2 + 1$. This polynomial has no real zeros, but complex zeros ± 1.Geometrically, we could have forseen this as rotations do not scale any non-zero vector in $\mathbb{R}^2$. By computing $N(A \pm i \cdot I)$, it is readily verified that $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} i \\ 1 \end{bmatrix}$ are eigenvectors of A.

Example 6.1.7:
Let us compute the eigenvalues and eigenspaces of

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) \,. \tag{6.16}$$

Observe that A is triangular and its eigenvalues are simply the entries on the diagonal. Thus, in this case, 3 is the only eigenvalue. We readily verify that

$$\mathrm{Eig}_{\mathbb{R}}(A, 3) = \left\{ c \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \,, c \in \mathbb{R} \right\} \,. \tag{6.17}$$

Thus, the 3-eigenspace of this matrix A is 1-dimensional.

Definition 6.1.5 (Algebraic and geometric multiplicity of an eigenvalue)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$, $\mathrm{ch}_A(\lambda)$ its characterstic polynomial and λ and eigenvalue. Then we define:

- The algebraic multiplicity $\mu_{\mathrm{alg}}(A, \lambda)$ of λ is the order of vanishing of ch_A at λ.
- The geometric multiplicity $\mu_{\mathrm{geo}}(A, \lambda)$ of λ is $\dim_{\mathbb{R}} (\mathrm{Eig}_{\mathbb{R}}(A, \lambda))$.

Example 6.1.8:
Let us exemplify these notions:

- In the previous example, we thus have $\mu_{\mathrm{alg}}(A, 3) = 2$ and $\mu_{\mathrm{geo}}(A, 3) = 1$.
- As another example consider the $n \times n$ identity matrix I. Then $\mu_{\mathrm{alg}}(I, 1) = n$ and $\mu_{\mathrm{geo}}(I, 1) = n$.

We will come back to these observations when we discuss diagonalizations.

Definition 6.1.6:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. The sum of the entries along the diagonal of A is termed the trace $\mathrm{tr}(A)$ of A.

Claim:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then the following hold true:

- The sum of the eigenvalues of A equals $\mathrm{tr}(A)$.
- The product of all eigenvalues of A equals $\det(A)$.
- Be $k \geq 0$. Then, the eigenvalues of A^k are obtained by raising the eigenvalues of A to the k-th power.
- If A is invertible, then λ^{-1} is an eigenvalue of A^{-1} iff λ is an eigenvalue of A.

Exercise:
Prove these statements.

Claim:
Eigenvectors of distinct eigenvalues are linearly independent.

Proof
We suffice it to give the proof for two vectors. Let us thus consider vectors $\vec{v}_1$ and $\vec{v}_2$ which are eigenvectors to A with distinct eigenvalues λ_1 and λ_2. Then we consider

$$0 = c_1\vec{v}_1 + c_2\vec{v}_2\,. \tag{6.18}$$

Let us now solve for c_1 and c_2. To this end, we perform two distinct steps:

- Multiply *eq.* (6.18) from the left with A. This gives

$$0 = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2\,. \tag{6.19}$$

- Multiply *eq.* (6.18) with λ_1. This gives

$$0 = c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2\,. \tag{6.20}$$

Now consider the difference of eq. (6.19) and eq. (6.20):

$$0 = c_2(\lambda_1 - \lambda_2)\vec{v}_2\,. \tag{6.21}$$

Since λ_1 and λ_2 are distinct and $\vec{v}_2$ non-zero (defining property of eigenvectors!), we conclude $c_2 = 0$. Consequently, eq. (6.18) implies $c_1 = 0$ and it follows that $\vec{v}_1$, $\vec{v}_2$ are linearly independent. ■

6.2 Diagonalizing matrices

6.2.1 The notation of diagonalizability

Note (Motivation)**:**
Why is it nice to have a basis of eigenvectors of $A \in \mathbb{M}(n \times n, \mathbb{R})$? Here is one reason. Say we want to compute $A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$. If we have a basis of eigenvectors $\vec{v_1}, \ldots, \vec{v_n}$ we may express $\vec{x}$ as

$$\vec{x} = c_1\vec{v}_1 + \ldots c_n\vec{v}_n\,. \tag{6.22}$$

Then we find

$$A\vec{x} = c_1A\vec{v}_1 + \ldots c_nA\vec{v}_n = c_1\left(\lambda_1\vec{v}_1\right) + \cdots + c_n\left(\lambda_n\vec{v}_n\right)\,. \tag{6.23}$$

The last expression involves ordinary multiplication and not matrix multiplication! So it is less cumbersome than computing $A\vec{x}$.

Definition 6.2.1:
If a matrix possesses a basis of eigenvectors, it is said to be *diagonalizable*.

Remark:
Diagonalizability is a very important property of matrices.

Note:
Suppose $A \in \mathbb{M}(n \times n, \mathbb{R})$ has a basis of eigenvectors $\vec{x}_1, \ldots \vec{x}_n$. Put these vectors into the columns of the eigenvector matrix X. Let us then compute AX:

$$\begin{aligned} A \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n \end{array} \right] &= \left[\begin{array}{cccc} A\vec{x}_1 & A\vec{x}_2 & \ldots & A\vec{x}_n \end{array} \right] \\ &= \left[\begin{array}{cccc} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \ldots & \lambda_n\vec{x}_n \end{array} \right] \\ &= \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n \end{array} \right] \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} . \end{aligned} \tag{6.24}$$

By assumption, the columns of X are a basis of $\mathbb{R}^n$. Thus X is invertible. So in particular, we find

$$X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} . \tag{6.25}$$

Definition 6.2.2:
The matrix Λ is called the *eigenvalue matrix*.

Remark:
The equality $X^{-1}AX$ has a geometric meaning. We will come back to this very soon.

Example 6.2.1:
Consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) . \tag{6.26}$$

Let us find X and Λ such that $X^{-1}AX = \Lambda$. To this end, we first find the characteristic polynomial of A:

$$\mathrm{ch}_A(\lambda) = (1-\lambda)(3-\lambda) - 8 = (\lambda+1)(\lambda-5) . \tag{6.27}$$

Thus, the eigenvalues are -1 and 5. It is readily verified that the eigenspaces are

$$\begin{aligned} \mathrm{Eig}_{\mathbb{R}}(A, -1) &= \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} , \\ \mathrm{Eig}_{\mathbb{R}}(A, 5) &= \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} . \end{aligned} \tag{6.28}$$

Consequently, we conclude

$$X = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} , \qquad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} . \tag{6.29}$$

Corollary 6.2.1:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$ such that its characteristic polynomial $\mathrm{ch}_A(\lambda)$ has n distinct real zeros, then A is diagonalizable.

Proof
Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be the distinct eigenvalues and $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ corresponding eigenvectors. Then $\{\vec{v}_1, \dots, \vec{v}_n\}$ are linearly independent. Hence, these eigenvectors form a basis of $\mathbb{R}^n$. ■

Consequence:
Under these assumptions, all (real) eigenspaces of A are 1-dimensional linear subspaces of $\mathbb{R}^n$.

Note (Exponentiation of diagonalizable matrices)**:**
Consider a diagonalizable matrix A, i.e. $A = X\Lambda X^{-1}$ for a diagonal matrix Λ. Then

$$A^k = \left(X\Lambda X^{-1}\right) \cdot \left(X\Lambda X^{-1}\right) \cdot \dots \cdot \left(X\Lambda X^{-1}\right) . \tag{6.30}$$

Since matrix multiplication is associative, we ignore those brackets and then, there is a massive cancellation leading to

$$A^k = X\Lambda^k X^{-1} . \tag{6.31}$$

Even more, since Λ is a diagonal matrix, computing Λ^k is very easy!

Example 6.2.2:
Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} . \tag{6.32}$$

We are interested in computing A^k. To this end, let us diagonalize A. We first find

$$\mathrm{ch}_A(\lambda) = \lambda^2 - \lambda - 1 . \tag{6.33}$$

There are thus two distinct real eigenvalues

$$\lambda_1 = \frac{1+\sqrt{5}}{2} , \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} . \tag{6.34}$$

Exercise:
Use $\lambda_1 + \lambda_2 = 1$ and $\lambda_1\lambda_2 = -1$ to conclude that

$$\begin{aligned} \mathrm{Eig}\,(A, \lambda_1) &= \mathrm{Span}\left\{\begin{bmatrix} -\lambda_2 \\ 1 \end{bmatrix}\right\} , \\ \mathrm{Eig}\,(A, \lambda_2) &= \mathrm{Span}\left\{\begin{bmatrix} -\lambda_1 \\ 1 \end{bmatrix}\right\} . \end{aligned} \tag{6.35}$$

Example 6.2.3 (Continuation)**:**
We thus have

$$X = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \Lambda^{-1} = \frac{1}{\lambda_2 - \lambda_1} \cdot \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}. \tag{6.36}$$

Thus we have

$$\begin{aligned} A^k &= \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \cdot \frac{1}{\lambda_2 - \lambda_1} \cdot \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}, \\ &= \frac{1}{\lambda_2 - \lambda_1} \cdot \begin{bmatrix} \lambda_1^{k-1} - \lambda_2^{k-1} & \lambda_2^k - \lambda_1^k \\ \lambda_1^k - \lambda_2^k & \lambda_2^{k+1} - \lambda_1^{k+1} \end{bmatrix}. \end{aligned} \tag{6.37}$$

We have used that $\lambda_1 \cdot \lambda_2 = \det(A) = -1$, to simplify the expression.

Note:
The above matrix A allows us to compute the Fibonacci numbers:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 5 \\ 8 \end{bmatrix} \xrightarrow{A} \dots . \tag{6.38}$$

Hence, we identify the k-th Fibonacci number F_k as $F_k = \frac{\lambda_2^k - \lambda_1^k}{\lambda_2 - \lambda_1}$ from

$$A^k \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_2^k - \lambda_1^k}{\lambda_2 - \lambda_1} \\ \frac{\lambda_2^{k+1} - \lambda_1^{k+1}}{\lambda_2 - \lambda_1} \end{bmatrix}. \tag{6.39}$$

Definition 6.2.3:
Two matrices $A, B \in \mathbb{M}(n \times n, \mathbb{R})$ are said to be *similar* if there exists an invertible $P \in \mathbb{M}(n \times n, \mathbb{R})$ satisfying $A = PBP^{-1}$.

Claim:
If A and B are similar, then they have the same eigenvalues.

Proof
We compute the characteristic polynomial:

$$\begin{aligned} \mathrm{ch}_A(\lambda) &= \det(A - \lambda I) = \det\left(PBP^{-1} - \lambda I\right) \\ &= \det\left(P(B - \lambda I)P^{-1}\right) \\ &= \det(B - \lambda I). \end{aligned} \tag{6.40}$$

This completes the proof. ■

Consequence:
If A and B are similar and A is diagonalizable, then also B is diagonalizable.

Exercise:
Prove this statement.

6.2.2 Failure of matrices to be diagonalizable

Remark:
Given $A \in \mathbb{M}(n \times n, \mathbb{R})$, we associated to an eigenvalue $\lambda \in \mathbb{R}$ two integers:

- The algebraic multiplicity $\mu_{\text{alg}}(A, \lambda)$ is the order of vanishing of $\text{ch}_A(\lambda)$ at λ.
- The geometric multiplicity $\mu_{\text{geo}}(A, \lambda)$ is the dimension $\text{Eig}_{\mathbb{R}}(A, \lambda)$.

Note:
It always holds $\mu_{\text{geo}}(A, \lambda) \leq \mu_{\text{alg}}(A, \lambda)$. Equality however is special.

Theorem 6.2.1:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$ with eigenvalues $\lambda_i \in \mathbb{R}$. A is diagonalizable if and only if

$$\mu_{\text{geo}}(A, \lambda_i) = \mu_{\text{alg}}(A, \lambda_i)\,, \tag{6.41}$$

holds true for all eigenvalues. Thus, if and only if this condition is satisfied, there exists an invertible $S \in \mathbb{M}(n \times n, \mathbb{R})$ and a diagonal matrix $D \in \mathbb{M}(n \times n, \mathbb{R})$ with $A = SDS^{-1}$.

Remark:
Similarly, a matrix $A \in \mathbb{M}(n \times n, \mathbb{C})$ is diagonalizable if and only if

$$\mu_{\text{geo}}(A, \lambda_i) = \mu_{\text{alg}}(A, \lambda_i)\,, \tag{6.42}$$

for all its eigenvalues. In constrast to the above theorem, the matrices S and D are then in general complex valued.

Example 6.2.4:
Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.43}$$

It has an eigenvalue 3 with algebraic multiplicity 2 and geometric multiplicity 1. Hence, A is not diagonalizable.

6.3 Diagonalizability and linear transformations

Question:
In the previous section, we discussed factorizations $A = X\Lambda X^{-1}$ where Λ is a diagonal matrix. We now want to understand what this factorization means by connecting it to linear transformations. A natural question to ask is as follows: "How would one have forseen the fact that $A = X\Lambda X^{-1}$ whenever A has a basis of eigenvectors?" The natural perspective is that of linear transformations.

Note:
Say we want to compute $A\vec{v}$ for some vector $\vec{v} \in \mathbb{R}^n$. Let X be a matrix formed by our basis of eigenvectors. One may think of the eigenvectors as giving a new coordinate system. What do the components of $X^{-1}\vec{v}$ represent?

They give the precise linear combination of the eigenvectors that equals $\vec{v}$. One refers to these components as the *coordinates in the basis of eigenvectors*.

Now we apply our linear transform corresponding to A. On the basis of eigenvectors, this linear transform simply scales the axes. Hence, in this new coordinate frame, the linear transform is given by a diagonal matrix of eigenvalues, which we called Λ. Thus, $\Lambda X^{-1}\vec{v}$ gives the coordinates of the linear transform applied to $\vec{v}$.

When we want to return to our original coordinate system, we multiply by X on the left to undo what X^{-1} did. Thus, the linear transform we are studying sends $\vec{v}$ to $X\Lambda X^{-1}\vec{v}$. Thus, the corresponding matrix of transformation must be $X\Lambda X^{-1}$!

Consequence:
"Every" linear transform has its own preferred choice of basis it wants to be understood in. This distinguished basis is given by the basis of eigenvectors. In this basis, the matrix of linear transformation is diagonal.

Example 6.3.1:
Consider reflection across a line. Note that I did not specify the coordinate frame and hence did not give this line an equation. So, if we are to write the matrix of transformation, we have numerous choices. But there are some choices that are easier than others.

Pick the line of the reflection to be the x-axis and the line perpendicular to it to be the y-axis. In this coordinate frame, the linear transform is given by

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.44}$$

Example 6.3.2:
Similarly, if we consider projection onto a plane, we may take a coordinate system where the x and y-axes are in the plane and the z-axis is orthogonal to the plane. In this coordinate frame, the matrix of transformation is given by

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{6.45}$$

Exercise:
Write down the matrix of projection onto the plane $x+y+z=0$ by computing $X\Lambda X^{-1}$ for the appropriate X and Λ.

6.4 Applications

6.4.1 Markov matrices and processes

Note:
We now turn our attention to Markov matrices. Our goal is to model a random process in which a system transitions from one state to another in discrete time steps.

Assume that at each time step, there are n-states a system could be in. At time k, we model the system as a vector $\vec{x}_k \in \mathbb{R}^n$, whose components represent the probability of being in each of the n states. We denote the initial state by $\vec{x}_0$.

Definition 6.4.1:
We term a vector $\vec{x}_i$ whose components are non-negative and sum up to 1 a *probability vector.*

Example 6.4.1:
Let us model the evolution of population in a city and its suburbs, where migration to and from the city occurs. We assume that

$$\vec{x}_0 = \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}, \tag{6.46}$$

i.e. 60% live in the city and 40% in the suburbs. Say each year 5% of the city dwellers move to the suburbs and 3% of the suburbanites move to the city. The rest stay. We represent the population after k years/k steps as

$$\vec{x}_k = \begin{bmatrix} c_k \\ s_k \end{bmatrix}. \tag{6.47}$$

The migration information translate to the following matrix equation:

$$\begin{bmatrix} c_{k+1} \\ s_{k+1} \end{bmatrix} = \vec{x}_{k+1} = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \cdot \begin{bmatrix} c_k \\ s_k \end{bmatrix} \equiv M \cdot \vec{x}_k\,. \tag{6.48}$$

Then we see

$$\vec{x}_0 = \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix} \to \vec{x}_1 = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix} \to \vec{x}_2 = \begin{bmatrix} 0.56 \\ 0.44 \end{bmatrix} \to \vec{x}_3 = \begin{bmatrix} 0.47 \\ 0.53 \end{bmatrix} \to \dots \tag{6.49}$$

In particular, $\vec{x}_k = M^k \cdot \vec{x}_0$. It turns out that

$$\lim_{k\to\infty} \vec{x}_k = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}. \tag{6.50}$$

Thus, in the long run, 37.5% of the population will be living in the city, whereas 62.5% will be in the suburbs.

Definition 6.4.2:
A *Markov matrix* is a square matrix M whose columns are probability vectors. A *Markov chain* is a sequence of probability vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$ such that

$$\vec{x}_{k+1} = M\vec{x}_k \,, \tag{6.51}$$

for a Markov matrix M. We refer to the limit $\lim_{k\to\infty} \vec{x}_k$ – if it exists – as the *steady state vector.*

Note:
A steady state vector *necessarily* has the property $M\vec{x} = \vec{x}$, i.e. satisfies $(M-I)\vec{x} = \vec{0}$. Thus any steady state vector is an eigenvector to the eigenvalue 1.

Example 6.4.2:
In the previous example, we compute

$$(M-I)\vec{x} = \begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \,. \tag{6.52}$$

This shows $0.03x_2 = 0.05x_1$. We also know $x_1 + x_2 = 1$. Thus $x_1 = 0.735$ and $x_2 = 0.625$. This is exactly what was stated above.

Example 6.4.3:
Suppose we are interested in changes in voter preferences during each election cycle – say, among demoncrats, republicants and liberals (DRL). We list the shifts in thie order from left to right and top to bottom:

$$M = \begin{bmatrix} 0.70 & 0.10 & 0.30 \\ 0.20 & 0.80 & 0.30 \\ 0.10 & 0.10 & 0.40 \end{bmatrix} \,. \tag{6.53}$$

Hence, the 0.20 says that 20% of the supporters of deomracts transition to the republicans. Yet again, we may ask for a steady state vector.

Theorem 6.4.1:
If M is a Markov matrix, then there exists a vector $\vec{x} \neq \vec{0}$ such that $M\vec{x} = \vec{x}$.

Proof
We need to show that 1 is always an eigenvalue of M. In other words, we need to show that $M - I$ is singular. But note that

$$\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix} \cdot (M-I) = \begin{bmatrix} 0 & \ldots & 0 \end{bmatrix} \,. \tag{6.54}$$

The left nullspace of $M-I$ is thus non-trivial. Hence, the cokernel of φ_{M-I} is non-trivial. Therefore $M - I$ is not bijective and $M - I$ must be singular. Hence, as claimed, 1 is an eigenvalue of M. ■

Consequence:
We are thus always guaranteed that a candidate for a steady state vector does exist, i.e. a vector $\vec{x} \neq 0$ with $M\vec{x} = \vec{x}$. Is this sufficient to conclude the existence of a steady state vector, i.e. to conclude that the limit $\lim_{k\to\infty} \vec{x}_k$ does exist?

Question:
We can thus ask the following questions:

- Does the steady state vector always have non-negative entries?
- Under what conditions is the steady state vector unique?
- Does the Markov chain attached to M always settle to a steady state vector?

Note:
The existence of $\vec{x} \neq 0$ with $M\vec{x} = \vec{x}$ does not imply that the Markov process eventually settles into a steady state. Consider the matrix

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{6.55}$$

This Markov matrix goes from state 1 to state 2 and vice versa with probability 1. Thus, there cannot be a steady state under these circumstances.

The real issue here is that $\lambda = -1$ is an eigenvalue and it is equal in modulus to 1. We will now try to establish, that for a *unique* steady state vector to exist, every eigenvalue other than 1 must be strictly less than 1 in modulus.

Lemma 6.4.1:
If M is a Markov matrix, then M^k is a Markov matrix.

Exercise:
Prove this statement.

Claim:
Be $M \in \mathbb{M}(n \times n, \mathbb{R})$ a Markov matrix. Then M cannot have an eigenvalue λ with $|\lambda| > 1$.

Proof

Let us assume the contrary, i.e. suppose M is a Markov matrix which has an eigenvalue λ with $|\lambda| > 1$. This means that there exists a vector $\vec{v}$ with

$$M\vec{v} = \lambda\vec{v} . \tag{6.56}$$

Hence, $M^n\vec{v} = \lambda^n\vec{v}$, which implies

$$|M^n\vec{v}| = |\lambda|^n \cdot |\vec{v}| . \tag{6.57}$$

Since $|\lambda| > 1$, the length of the vector on the right grows to ∞ for $n \to \infty$. Thus, to mirror this on the LHS, also the entries of M^n have grow very large for $n \to \infty$. This contradicts with M being a Markov matrix. Namely, the entries of every column of M are non-negative and add to 1. ■

Theorem 6.4.2 (Perron-Frobenius)**:**
If $M \in \mathbb{M}(n \times n, \mathbb{R})$ is a *positive* (i.e. all entries strictly positive) Markov matrix, then $\lambda = 1$ is the unique largest eigenvalue and the corresponding eigenvector is the unique steady state eigenvector.

Remark:
You may think that being a positiv Markov matrix is a rather rigid requirement. In fact, one can establish the following stronger result.

Claim:
If $M \in \mathbb{M}(n \times n, \mathbb{R})$ is a Markov matrix such that some power M^k is positive, then the Perron-Frobenius theorem applies to M.

Note:
Consider a positive Markov matrix $M \in \mathbb{M}(n \times n, \mathbb{R})$. What can we say about the columns of M^k in the long run? We claim that they look more and more like the steady state vector. This is because of the following: For any matrix, we can extract the i-th column by multiplication with the appropriate standard basis vector. So for instance, the first column of M^k is given by

$$M^k \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{6.58}$$

Let us interpret $\vec{x}_0 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$ as the initial state vector. Moreover, since M is positive, we know that there is a unique steady-state vector and, by definition of the steady state vector, it is given as $\lim\limits_{k\to\infty} \left(M^k \vec{x}_0\right)$. Hence, as k increases, the vector $M^k \vec{x}_0$ converges to the steady state vector.

Consequence:
One can obtain a very good approximation of the steady state vector of positive Markov matrices from computing matrix powers.

6.4.2 Page rank algorithm

Note:
Our next topic concerns searching on the web. Namely, given a search string, how should the search engine determine the order in which to rank the output? A naive approache is as follows:

1. Keep an index of all web pages.
2. Respond to a query by browsing through the index and list the webpages according to the number of times the search query appears on that webpage.

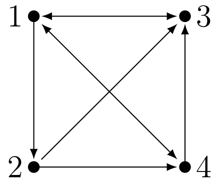
We can agree that this approach is not very smart. It is fairly easy to abuse the system to have a completely unimportant webpage appear as the first search result. Naive as it may sound, this was exactly the approach used by search engines in the 90s such as Altavista & Lycos. In a very simplistic viewpoint, Sergey Brin and Larry Page realized that the world wide web was a democracy, where someone linking to your webpage was a vote for your webpage. Thus, their idea was to rank the webpages according to the number of votes:

- If I create a webpage A and link to webpage B, that means I consider B relevant.
- Also, if B is considered important and it links to C, then it asserts, that C is important as well. Thus, B transfers its authority to C.

In the following, we want to quantify importance.

Question:
Given n interlinked webpages, rank them in order of importance. To this end, assign the pages importance scores $x_1, x_2, \ldots, x_n \geq 0$. Our insight is to use the existing link structure of the web to determine these scores.

Example 6.4.4:
Let us consider a simplified version of the world-wide-web:



(6.59)

Thus, there are 4 webpages. Each is represented by a node and a directed edge from node i to j represents a hyperlink on webpage i to j.

According to our model, each page transfers its importance *evenly* to the pages it links to. For instance, node 1 passes $\frac{1}{3}$ of its importance score to each of the three nodes it links to. Let us encode this information as a system of equations:

$$\begin{aligned} x_1 &= 1 \cdot x_3 + \frac{1}{2} x_4 \,, \\ x_2 &= \frac{1}{3} x_1 \,, \\ x_3 &= \frac{1}{3} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_4 \,, \\ x_4 &= \frac{1}{3} x_1 + \frac{1}{2} x_2 \,. \end{aligned} \tag{6.60}$$

As a matrix equation, this says

$$\vec{x} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \cdot \vec{x} \,. \tag{6.61}$$

Thus, the importance vector is an eigenvector of a certain Markov matrix! In this case, it is the unique steady state vector

$$\vec{x} = \frac{1}{31} \cdot \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}. \tag{6.62}$$

It might appear a tiny bit magical, that such a vector indeed exists, especially as our rules for deriving importance appear contrived and self-referrential.

Note:
Here are two alternative approaches:

- Instead of finding the 1-eigenvector, one could start with a random assignment of importance scores and then update them according to our rules. Hence, one could start for example with

$$\vec{x}_0 = \frac{1}{4} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{6.63}$$

  Then, we multiply with A repeatedly. Then, we already find

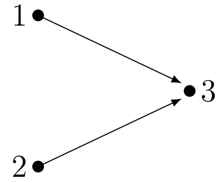
$$A^8 \vec{x}_0 \sim \cdot \begin{bmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{bmatrix}. \tag{6.64}$$

- Brin and Page considered the "random surfer model". This involves a guy starting on a webpage and clicking one of the hyperlinks on the webpage *uniformly at random*. This creates a Markov chain whose Markov matrix is the same as the one given earlier. The components of the steady state vector of this matrix can now be interpreted as the amount of time one spends on a certain webpage. Or you could think of its components as giving you the probability of ending on a certain webpage in the long run.

Irrespective of the perspective we pick, we definitely get a sense of importance.

Remark:
Here are two potential issues:

1. Webpages that do not have any hyperlink:
   Consider the following network:



(6.65)

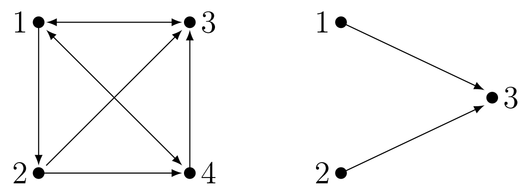
This leads to the Markov matrix

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}. \tag{6.66}$$

In this case $M^2 = 0$ and the importance vector is identically zero, which does not really reflect the above network appropriately.

An easy fix to this problem is to turn the third column to $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}^T$. This means, that anybody who is on webpage 3 ”restarts“ their browsing experience by picking a website uniformly at random.

2. Disconnected components:
   Let us consider the following network with two disconnectec components:



(6.67)

While our approach can compare the importance of webpage in a component, it seems to not be helpful in comparing webpages belonging to different components.

We would like our the 1-eigenspace of the associated Markov matrix to be 1-dimensional. But convince yourself, that if the web has r components, then the 1-eigenspace must be at least r-dimensional. This does not play well with the fact that we would really like a unique steady state vector.

Here is the simple and ingenious solution by Brin and Page: We replace the Markov matrix M by the new matrix

$$G = (1-p) \cdot M + p \cdot \frac{1}{n} \cdot \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}. \tag{6.68}$$

The matrix on the right is called the *teleportation matrix*. n is the total number of webpages and $0 \leq p \leq 1$ a probability.

The probabilistic interpretation is as follows: With probability $1 - p$, we follow the random surfer model from before and with probability p, we open a random webpage amongst all possible webpages. Google originally chose $p = 0.15$.

The new matrix G above is referred to as the *Google matrix*. It is still Markov. Even more important, it is *positive*. Thus, it is guaranteed to have a unique steady state vector, the so-called *Page-Rank vector*.

Example 6.4.5:
For $p = 0.15$, we find for eq. (6.67) that

$$G = \begin{bmatrix} 0.0214286 & 0.0214286 & 0.871429 & 0.446429 & 0.0214286 & 0.0214286 & 0.0214286 \\ 0.304762 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 \\ 0.304762 & 0.446429 & 0.0214286 & 0.446429 & 0.0214286 & 0.0214286 & 0.0214286 \\ 0.304762 & 0.446429 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 \\ 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.304762 \\ 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.304762 \\ 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.871429 & 0.871429 & 0.304762 \end{bmatrix} . \tag{6.69}$$

By computing the eigenvector to the eigenvalue 1, which is normalized such that its entries add up to 1, we find the unique steady-state vector/the importance vector:

$$\begin{bmatrix} 0.20979 \\ 0.0810366 \\ 0.164541 \\ 0.11559 \\ 0.0913204 \\ 0.0913204 \\ 0.246401 \end{bmatrix} . \tag{6.70}$$

This is the *Page-Rank vector*.

6.4.3 Systems of ordinary differential equations

Example 6.4.6:
Let us consider two functions

$$u_1 \colon \mathbb{R} \to \mathbb{C}\,,\, t \mapsto u_1(t)\,, \qquad u_2 \colon \mathbb{R} \to \mathbb{C}\,,\, t \mapsto u_1(t)\,. \tag{6.71}$$

We encode their behaviour with time by the following equations:

$$\begin{aligned} u_1'(t) &= -1 \cdot u_1(t) + 2 \cdot u_2(t), , \\ u_2'(t) &= 1 \cdot u_1(t) - 2 \cdot u_2(t), . \end{aligned} \tag{6.72}$$

You can consider this to be a continuous analog of Markov chains discussed in the previous section. In particular, we need to provide the initial condition, say $\vec{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

To understand $u(t)$ as a function of time, we will need to understand the eigenvalues and eigenvectors of the matrix formed by the coefficients. In the case at hand, this matrix is

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.73}$$

Thus, we are interested in solving

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} . \tag{6.74}$$

We know how to proceed if A is a diagonal matrix by simply utilizing the fact that the solution to the differential equation

$$f'(t) = \lambda \cdot f(t) , \tag{6.75}$$

is given by $f(t) = C \cdot e^{\lambda t} + D$. Thus, if we want to solve the system for a non-diagonal matrix A, then we should perhaps try and alter the system so that it becomes diagonal. Hence, let us compute the eigenvalues and eigenvectors of A. Since A is singular, we know that $\lambda_1 = 0$ is an eigenvalue. Since the trace is -3, we know that $\lambda_2 = -3$ is the other eigenvalue. For the eigenspaces, we find

$$\text{Eig}\,(A, 0) = \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} , \qquad \text{Eig}\,(A, -3) = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} . \tag{6.76}$$

Let us set

$$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} , \qquad \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} . \tag{6.77}$$

Then, a quick check shows that $\vec{u} = c_1 \cdot e^{\lambda_1 t} \cdot \vec{x}_1$ satisfies $\vec{u}'(t) = A\vec{u}(t)$. Similarly, $\vec{u} = c_2 \cdot e^{\lambda_2 t} \cdot \vec{x}_2$ satisfies the same differential equation. It follows, that any linear combination of these two special solutions does satisfy this differential equation. By plugging in the values $\lambda_1 = 0$ and $\lambda_2 = -3$, it follows

$$\vec{u} = c_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} . \tag{6.78}$$

Finally, we use the initial condition to compute c_1 and c_2. This gives $c_1 = c_2 = \frac{1}{3}$.

Remark:
As time goes to infinity, the term involving e^{-3t} shrinks to 0. Thus, $\vec{u}(\infty) = \frac{1}{3}\vec{x}_1$. One says the system approaches a steady state.

Note:
It is instructive to compare the solution of the continuous version $\vec{u}'(t) = A \cdot \vec{u}(t)$ with its discrete analogue $\vec{u}_{k+1} = A \cdot \vec{u}_k$. In the former, a general solution is given by

$$\vec{u}(t) = c_1 e^{\lambda_1 t} \vec{x}_1 + c_2 e^{\lambda_2 t} \vec{x}_2 , \tag{6.79}$$

and in the latter, a general solution is given by

$$\vec{u}_k = c_1 \lambda_1^k \vec{x}_1 + c_2 \lambda_2^k \vec{x}_2 . \tag{6.80}$$

Remark:
Let us discuss various aspects of the solutions to $\vec{u}'(t) = A\vec{u}(t)$:

- Stability:
  A is stable if $\vec{u}(t)$ approaches 0 as $t \to \infty$. Under what condition on the eigenvalues of A are we guaranteed stability? Clearly, we want the exponentials to decay. That happens precisely when *the real part of all eigenvalues is negative.* It is useful to recall that $\left|e^{a+ib}\right| = |e^a|$.

- Steady state:
  Under what condition on the eigenvalues does $\vec{u}(t)$ approach a fixed vector as $t \to \infty$? We want $\lambda_1 = 0$ and all other eigenvalues to have negative real part.

- Decoupling:
  Assume that A is diagonalizable, i.e. $A = X\Lambda X^{-1}$ with Λ the (diagonal) eigenvalue matrix. Thus, we have
  $$\vec{u}'(t) = A\vec{u}(t) \qquad \Leftrightarrow \qquad \vec{u}'(t) = X\Lambda X^{-1}\vec{u}(t)\,. \tag{6.81}$$
  We set $\vec{v}(t) = X^{-1}\vec{u}(t)$. Then, we see
  $$\vec{u}'(t) = A\vec{u}(t) \qquad \Leftrightarrow \qquad X\vec{v}'(t) = X\Lambda\vec{v}(t) \qquad \Leftrightarrow \qquad \vec{v}'(t) = \Lambda\vec{v}(t)\,. \tag{6.82}$$
  Since Λ is diagonal, the equation $\vec{v}'(t) = \Lambda\vec{v}(t)$ describes a system of n independent ordinary differential equation. One says, the coupled system $\vec{u}'(t) = A\vec{u}(t)$ becomes uncoupled or decouples.

  The advantage of uncoupling the system is, that each of its equations is of the form $v_i'(t) = \lambda_i v_i(t)$ which can be solved readily. This is exactly how one obtains the solution in general form.

Note:
If the matrix A is not diagonalizable, then one has to work harder to solve the system. This is what we will turn to now. Our intention is to write the solution of $\vec{u}'(t) = A\vec{u}(t)$ as $\vec{u}(t) = e^{A\cdot t}\cdot\vec{u}(0)$. This is simply to mimic what happens in the one variable case. For this, we first have to make sense of the expression $e^{A\cdot t}$.

Definition 6.4.3 (Matrix exponential)**:**
We define
$$e^{A\cdot t} = \sum_{k=0}^{\infty}\frac{(At)^k}{k!} = I + A\cdot t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\,. \tag{6.83}$$

Remark (Differentiation)**:**
We may wonder what happens when we differentiate e^{At} with respect to t:
$$\begin{aligned}\left(\frac{d}{dt}\left(e^{At}\right)\right)(t) &= A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \dots\\ &= A\cdot\left(I + A\cdot t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right)\\ &= A\cdot e^{At}\,.\end{aligned} \tag{6.84}$$

Note: We interchange differentiation and the infinite sum. This is not always allowed. You want to revise your calculus class and absolute convergence.

Note:
The eigenvalues of e^{At} are closely related to those of A. Namely, suppose that $\vec{x}$ is an eigenvector of A with eigenvalue λ. Then it holds

$$\begin{aligned} e^{At}\vec{x} &= \left(I + A \cdot t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right) \cdot \vec{x} \\ &= \left(1 + \lambda \cdot t + \frac{\lambda^2t^2}{2!} + \frac{\lambda^3t^3}{3!} + \dots\right) \cdot \vec{x} \\ &= e^{\lambda t} \cdot \vec{x}\,. \end{aligned} \tag{6.85}$$

Hence, the eigenvalues of e^{At} are given by $e^{\lambda t}$ as λ ranges over all eigenvalues of A.

Example 6.4.7:
Let us compute e^{At} when A is diagonalizable, i.e. $A = X\Lambda X^{-1}$ with diagonal eigenvalue matrix Λ. Then:

$$\begin{aligned} e^{At}\vec{x} &= I + A \cdot t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots \\ &= I + X\Lambda X^{-1} \cdot t + \frac{\left(X\Lambda X^{-1}\right)^2 t^2}{2!} + \frac{\left(X\Lambda X^{-1}\right)^3 t^3}{3!} + \dots \\ &= X \cdot \left(I + \Lambda \cdot t + \frac{\Lambda^2t^2}{2!} + \frac{\Lambda^3t^3}{3!} + \dots\right) X^{-1} \\ &= X \cdot e^{\Lambda t} X^{-1}\,. \end{aligned} \tag{6.86}$$

Example 6.4.8:
Let us return to our opening example and see if $e^{At}\vec{u}(0)$ indeed does give the same solution as obtained earlier. Recall that we considered

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}, \qquad X = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}. \tag{6.87}$$

Consequently, it holds

$$X^{-1} = \frac{-1}{3} \cdot \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}. \tag{6.88}$$

Now an easy computation shows

$$e^{At}\vec{u}(0) = X \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix} \cdot X^{-1} \cdot \vec{u}(0) = \frac{1}{3} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}e^{-3t} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{6.89}$$

This is indeed the same answer as before.

Example 6.4.9:
Let us discuss another example. Let us try and solve the equation $y''(t)-2y'(t)+y(t)=0$ for given initial values $y(0)$ and $y'(0)$. We realize that, even though we have been given one equation, there are secretly two equations. To see this, let us set $\vec{u}(t)=\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Then, we have

$$\begin{aligned} \left(\frac{d}{dt}y\right)(t) &= y'(t)\,, \\ \left(\frac{d}{dt}y'\right)(t) &= y''(t) = 2y'(t)-y(t)\,. \end{aligned} \tag{6.90}$$

This we can encode in the equation

$$\left(\frac{d}{dt}\vec{u}\right)(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \cdot \vec{u}(t) \equiv A \cdot \vec{u}(t)\,. \tag{6.91}$$

Note that the matrix A is *not* diagonalizable. Namely, the characteristic polynomial has repeated root 1 and the 1-eigenspace is spanned only by [11]. That is, the algebraic and geometric multiplicity do not coincide, showing that A is not diagonalizable.

Hence, we compute e^{At} from its definition as infinite series. The key fact which makes this computation easy is $(A-I)^2=0$. We use this by writing

$$e^{At} = e^{It}\cdot e^{(A-I)t} = e^{It}\cdot(I+(A-I)\cdot t) = e^t \cdot \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}\,. \tag{6.92}$$

Therefore, the general solution is given by

$$\vec{u}(t) = e^t \cdot \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \cdot \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}\,. \tag{6.93}$$

And thus $y(t)$ is given by the first component of $\vec{u}(t)$.

Remark:
In general, for two matrices A and B it holds

$$e^A \cdot e^B \neq e^B \cdot e^A \neq e^{A+B}\,! \tag{6.94}$$

Example 6.4.10:
Let us compute e^{At} for $A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. To this end, we note that

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\,, \qquad A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\,, \qquad A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\,. \tag{6.95}$$

From this it follows $e^{At}=\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$. This is a rotation matrix, i.e. an element of the group $SU(2)$. We observe the following:

- Our A is skey-symmetric and e^{At} is orthogonal. This holds in general.
- The eigenvalues of A are i and $-i$. The eigenvalues of e^{At} are e^{it} and e^{-it}.

Remark:
If you study Lie groups, you will find that the matrix A is an element of the Lie algebra $\mathfrak{su}(2)$ and that there is a map, called the *exponential map*, $\mathfrak{su}(2) \to SU(2)$. It is given by exponentiation and allows to describe elements of $SU(2)$ by elements of the Lie algebra $\mathfrak{su}(2)$. This insight from group theory is particular important when it comes to representations of groups, which are, for example, employed in qantum mechanics and quantum field theory frequently.

Note:
Just like e^x is never zero, we have the analogous fact for matrices: e^{At} always has the inverse e^{-At}.

6.5 Eigenvalues and eigenvectors of real, symmetric matrices

6.5.1 The spectral theorem

Note:
We now focus on eigenvalues and eigenvectors of real, symmetric matrices. We already discussed special instances of such matrices, namely reflections and projections. In both cases, we had a basis of eigenvectors. Even more – we had a basis of *orthogonal* eigenvectors! We can thus wonder if this is true in general and what can be said about the eigenvalues. It turns out, a whole lot!

Remark:
Let us see what diagonalizability implies in the context of symmetric matrices, i.e $S \in \mathbb{M}(n \times n, \mathbb{R}$ with $S = S^T$. Say $S = X\Lambda X^{-1}$. Then $S^T = \left(X^{-1}\right)^T \Lambda^T X^T$. Since $S = S^T$, we may hope that $X^{-1} = X^T$, or equivalently $X^T X = I$. This in turn means that X better be orthogonal! Indeed, diagonalization acquires a really nice form in the setting of symmetric matrices.

Theorem 6.5.1 (Spectral theorem)**:**
Every symmetric matrix $S \in \mathbb{M}(n \times n, \mathbb{R})$ has real eigenvalues. It admits a factorization

$$S = Q\Lambda Q^T \,, \tag{6.96}$$

with the eigenvalues $\lambda_1, \ldots, \lambda_n$ of S along the diagonal of Λ (all other entries of Λ are zero). Furthermore, there is a basis of $\mathbb{R}^n$ formed from orthonormal eigenvectors of S. Such eigenvectors, with eigenvalues $\lambda_1, \ldots, \lambda_n$, form the columns of Q. In particular, Q is orthogonal, i.e. $Q^T Q = I = QQ^T$.

Remark:
Let us emphasize again, that for a real, symmetric matrix $S \in \mathbb{M}(n \times n, \mathbb{R})$, it holds:

- All eigenvalues of S are real.
- There exists a basis of $\mathbb{R}^n$ formed from *orthonormal* eigenvectors of S.

Example 6.5.1:
Let us study an example. We consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}). \tag{6.97}$$

Then we find $\mathrm{ch}_A(\lambda) = -(\lambda - 5) \cdot (\lambda + 1)^2$. Hence, the eigenvalues of A are 5, -1. By use of the Gram-Schmidt procedure, we find

$$\begin{aligned} \mathrm{Eig}\,(A, -1) &= \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\} \\ \mathrm{Eig}\,(A, 5) &= \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \end{aligned} \tag{6.98}$$

This leads to the following orthogonal basis of $\mathbb{R}^3$ from eigenvectors of A:

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{6.99}$$

We can normalize this basis, to find

$$\mathcal{B}_0 = \left\{ \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{6.100}$$

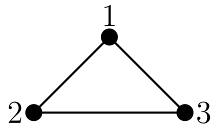
This is an orthonormal basis of $\mathbb{R}^3$ from eigenvectors of A.

Exercise:
Let Q be the matrix whose columns are the above 3 eigenvectors of A. Check that

$$Q^T A Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}. \tag{6.101}$$

Note:
No inverses are needed for the diagonalization of symmetric matrices, but we make use of the Gram-Schmidt procedure.

Note:
Symmetric matrices appear in many important applications. One is the Hessian matrix that allows to investigate the type of local extrema of a function $f\colon \mathbb{R}^n \to \mathbb{R}$. Another instance are *adjacency matrices* in graph theory. If node i connects to node j, then we record a 1 in column i row j. Otherwise, we record a 0. For example, the graph



(6.102)

gives the *adjacency matrix*

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} . \tag{6.103}$$

Consequently, adjancency matrices are always symmetric. Thus, the spectral theorem applies and we conclude that all eigenvalues are real. This bit is very important when studying connectivity properties of *sparse graphs.* The keyword you want to look up is the *spectral gap.* These questions are to be studied when designing robust networks, which should be so-called *expander graphs.*

Claim:
Be $S \in \mathbb{M}(n \times n, \mathbb{R})$ a symmetric matrix. Then its eigenvalues are real.

Proof
Suppose S is symmetric and $S\vec{x} = \lambda\vec{x}$. A priori, $\vec{x}$ and λ might be complex valued. Complex conjugation yields $S\overline{\vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}$ where we used $\overline{S} = S$ since $S \in \mathbb{M}(n \times n, \mathbb{R})$. Now, transposition gives $\overline{\vec{x}}^T \cdot S = \overline{\vec{x}}^T \cdot \overline{\lambda}$ where we used that S is symmetric, i.e. $S^T = S$. Right multipication with $\vec{x}$ gives

$$\overline{\vec{x}}^T \cdot S \cdot \vec{x} = \overline{\vec{x}}^T \cdot \overline{\lambda} \cdot \vec{x} . \tag{6.104}$$

But note, that left multiplication of $S\vec{x} = \lambda\vec{x}$ with $\overline{\vec{x}}$ gives

$$\overline{\vec{x}}^T \cdot S \cdot \vec{x} = \overline{\vec{x}}^T \cdot \lambda \cdot \vec{x} . \tag{6.105}$$

Hence, by comparing eq. (6.104) and eq. (6.105), we find

$$\overline{\lambda} \cdot \vec{x}^T \vec{x} = \lambda \cdot \vec{x}^T \vec{x} . \tag{6.106}$$

Recall that we assumed that $\vec{x}$ is an eigenvector. Hence, $\vec{x} \neq 0$ and thus $\overline{\vec{x}}^T \cdot \vec{x} \in \mathbb{R}_{>0}$. Therefore $\lambda = \overline{\lambda}$, which completes this proof. ■

Claim:
Be $S \in \mathbb{M}(n \times n, \mathbb{R})$ a symmetric matrix, $\vec{x}$, $\vec{y}$ two eigenvectors of S with different eigenvalues. Then $\vec{x}^T\vec{y} = 0$, i.e. $\vec{x} \perp \vec{y}$.

Proof

Suppose that $S\vec{x} = \lambda_1\vec{x}$ and $S\vec{y} = \lambda_2\vec{y}$ with $\lambda_1 \neq \lambda_2$. Then

$$\lambda_1 \cdot \vec{x}^T \cdot \vec{y} = (\lambda_1\vec{x})^T \cdot \vec{y} = (S\vec{x})^T \cdot \vec{y} = \vec{x}^T S^T \vec{y} = \vec{x}^T S \vec{y} = \lambda_2 \cdot \vec{x}^T \cdot \vec{y}\,. \tag{6.107}$$

Thus, since $\lambda_1 \neq \lambda_2$, we must have $\vec{x}^T\vec{y} = 0$ and $\vec{x} \perp \vec{y}$. ■

Note:
We will omit the argument which implies that any symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$ does admit enough eigenvectors to form a basis of $\mathbb{R}^n$. Rather, we take this as faith. Then, in summary, our observations imply that we can write

$$S = Q \cdot \Lambda Q^T\,, \tag{6.108}$$

with $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_i \in \mathbb{R}$ are the eigenvalues of S and the columns of Q are eigenvectors of S which furnish an orthonormal basis of $\mathbb{R}^n$.

Consequence (Interpretation of the spectral theorem)**:**
Let us write $S = Q \cdot \Lambda Q^T$ explicitly:

$$S = \begin{bmatrix} \vec{q}_1 & \ldots & \vec{q}_n \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \cdot \begin{bmatrix} \vec{q}_1^T \\ \vdots \\ \vec{q}_n^T \end{bmatrix} = \sum_{i=1}^{n} \lambda_i \cdot \left(\vec{q}_i \vec{q}_i^T\right). \tag{6.109}$$

Note that, since $\vec{q}_i$ is a vector of length 1, the matrices $\vec{q}_i \cdot \vec{q}_i^T$ are projection matrices. Hence, the spectral theorem says, that any symmetric, real matrix is a linear combination of projection matrices.

6.5.2 Definitness of matrices

Note:
We encountered the definiteness of matrices when studying the type of local extrema of functions $f\colon \mathbb{R}^n \to \mathbb{R}$. Namely, we found that a local extremum is a local minimum/maximum if the Hessian matrix of f is positive/negative definite. Other applications include the study of quadrics, e.g. ellipses/parabola. We develop the notion of definiteness of matrices before we exemplify its applications to the topics of quadrics.

Definition 6.5.1:
A symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$ with eigenvalues $\lambda_i \in \mathbb{R}$ is termed

- positive semi-definite iff all $\lambda_i \geq 0$,
- positive definite iff all $\lambda_i > 0$,
- negative semi-definite iff all $\lambda_i \leq 0$,
- negative definite iff all $\lambda_i < 0$,

- indefinite, if there is (at least) one positive and one negative eigenvalue.

Comment:
We will focus on positive definite matrices. But many of the following statements extend negative (semi-)definite matrices.

Note:
Our first task is to be able to tell when a matrix is positive-definite. One potential approach would be to compute the roots of the characteristic polynomial and then check the signs. However, this approach is not very smart. Namely, computing the roots of polynomials is not an easy task, expecially as numerical approaches are prone to error. So we would ideally like to avoid computing the roots. After all, we are solely interested in their signs. Luckily for us, there are various ways.

Example 6.5.2:
Let us consider $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})$. The eigenvalues are $\lambda_1 = \det(S)$ and $\lambda_2 = \mathrm{tr}(S)$. Hence, this matrix is positive definite iff

$$a + c > 0 \qquad \text{and} \qquad ac - b^2 > 0\,. \tag{6.110}$$

Equivalently, we can write ($ac > b^2$ requires that a, c have the same sign and it is positive since $a + c > 0$)

$$a > 0 \qquad \text{and} \qquad ac - b^2 > 0\,. \tag{6.111}$$

Note that A is the determinant of the top-left submatrix $\tilde{S} = [a]$ of S and that $ac - b^2$ is the determinant of S. Alternatively, we can take the viewpoint of pivots:

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \to \begin{bmatrix} a & b \\ 0 & \frac{ac-b^2}{a} \end{bmatrix}\,. \tag{6.112}$$

We thus see that positivity of the pivots a and $\frac{ac-b^2}{a}$ guarantees that all eigenvalues are positive as well. Both of these criteria do apply to real, symmetric matrices.

Corollary 6.5.1:
For a symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$, the following are equivalent:

- S is positive definite,
- all upper left determinants of S are positive,
- all pivots of S are positive.

Example 6.5.3:
Consider the matrix

$$S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{6.113}$$

The upper left determinants of S are 2, 3 and 4. Thus, we see that this matrix is positive definite. Alternatively, we find the row echelon form of S:

$$S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} \end{bmatrix} . \tag{6.114}$$

We thus see that all pivots are positive. Consequently, S is positive definite.

Note:
Let us look at the eigenvector equation $S\vec{x} = \lambda\vec{x}$. Then

$$\vec{x}^T S\vec{x} = \lambda \cdot \vec{x}^T \vec{x} . \tag{6.115}$$

So, for a positive definite matrix S, the RHS is positive. In fact, this is true for all non-zero vectors $\vec{x}$ if S is positive definite.

Corollary 6.5.2:
For a symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$, the following are equivalent:

- S is positive definite,
- $\vec{x}^T S\vec{x} > 0$ for all $\vec{x} \in \mathbb{R}^n \setminus \vec{0}$.

Example 6.5.4:
Let us consider $\vec{x}^T = \begin{bmatrix} x & y \end{bmatrix}$ and $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})$. Then we conclude

$$\vec{x}^T S\vec{x} = ax^2 + 2bxy + cy^2 . \tag{6.116}$$

Let us discuss what the positivity of this expression means for all $(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$. W.l.o.g., let us assume $y \neq 0$. Then we have

$$a\left(\frac{x}{y}\right)^2 + 2b \cdot \left(\frac{x}{y}\right) + c > 0 . \tag{6.117}$$

let us set $z = \frac{x}{y}$. Then, what do we know about a, b, c if the parabola $az^2 + bz + c$ is positive for all $z \in \mathbb{R}$? Since this parabola must be concave up, we get that $a > 0$. Since we do not want any roots to $az^2 + bz + c = 0$, we must have $4\left(b^2 - ac\right) < 0$. Note that these are precisely the conditions that guarantee positive definiteness! Of course, this analysis becomes a little more challenging for S of larger dimension.

Remark:
Observe the surprising fact, that if S and T are positive definite, then so is $S+T$. Realize, that proving this fact by way of positivity of pivots or upper left determinants is nearly impossible. But with our third criterion, it is completely straightforward. Indeed, for any non-zero $\vec{x}$ we find

$$\vec{x}^T\left(S + T\right)\vec{x} = \vec{x}^T S\vec{x} + \vec{x}^T T\vec{x} > 0 . \tag{6.118}$$

Note:
There is yet another criterion for definiteness. Let $A \in \mathbb{M}(m \times n, \mathbb{R})$ a rectangular, real matrix. Then $S = A^T A \in \mathbb{M}(n \times n, \mathbb{R})$ is symmetric. It turns out that S is positive definite, provided A has linearly independent columns. Namely, under this assumption we know $A\vec{x} \neq 0$ for any $\vec{x} \neq 0$. Consequently,

$$\vec{x}^T S \vec{x} = \vec{x}^T A^T A \vec{x} = (A\vec{x})^T (A\vec{x}) = |A\vec{x}|^2 > 0\,. \tag{6.119}$$

Consequence:
We have established that for symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$, the following are equivalent:

- S is positive definite,
- all upper left determinants of S are positive,
- all pivots of S are positive,
- $\vec{x}^T S \vec{x} > 0$ for all $\vec{x} \neq 0$,
- $S = A^T A$ for some real matrix A with linearly independent columns.

Note:
We may wonder how we can express a symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$ as $S = A^T A$. To this end we note that $S = LDL^T$ is the symmetric version of the LU-factorization (cf. section 2.5). Furthermore, since S is positive definite, we know that D has positive entries λ_i along the diagonal. Therefore, we can consider $\sqrt{D} := \operatorname{diag}\left(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}\right)$. Now, for $A^T = L\sqrt{D}$ we find

$$A^T A = L \cdot \sqrt{D} \cdot \sqrt{D} \cdot L^T = LDL^T = S\,. \tag{6.120}$$

Definition 6.5.2:
For a symmetric, positive definite $S \in \mathbb{M}(n \times n, \mathbb{R})$, we term $A = \left(L\sqrt{D}\right)^T$ the *Cholesky factor* and $S = A^T A$ the *Cholesky decomposition* of S.

Remark:
The *Cholesky factor* is triangular but involves square roots. The latter can at times be undesirable.

Note:
We can also use the eigenvalue matrix of S instead of D. Say $S = Q\Lambda Q^T$, where Q is orthogonal. Since S is positive definite, Λ has positive entries and hence we can take square roots. Consequently

$$S = Q\Lambda Q^T = \left(Q\sqrt{\Lambda}Q^T\right)^T \cdot \left(Q\sqrt{\Lambda}Q^T\right)\,. \tag{6.121}$$

Thus, with $A = Q\sqrt{\Lambda}Q^T$, we again find $S = A^T A$.

Example 6.5.5:
Let us consider

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{6.122}$$

Convince yourself, with any of the above tests, that S is indeed positive definite. In particular, we find

$$\vec{x}^T S \vec{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2\,. \tag{6.123}$$

Since S is positive definite, this quantity is positive whenever $(x_1, x_2, x_3) \neq (0,0,0)$. To see this, we want to write this quantity as a sum of squares. If we achieve $S = A^T A$, then indeed we have $\vec{x}^T S \vec{x} = |A\vec{x}|^2$. Explicitly:

- If you use $S = LDL^T$, then you find

$$\vec{x}^T S \vec{x} = 2\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}x_3^2\,. \tag{6.124}$$

- If you use $S = Q\Lambda Q^T$, then you find

$$\vec{x}^T S \vec{x} = \lambda_1 \cdot \left(\vec{q}_1^T \vec{x}\right)^2 + \lambda_2 \cdot \left(\vec{q}_2^T \vec{x}\right)^2 + \lambda_3 \cdot \left(\vec{q}_3^T \vec{x}\right)^2\,. \tag{6.125}$$

Remark:
As already mentioned, closely related to positive definite matrices are *positive semi-definite matrices.*, for which the eigenvalues are constrained to be non-negative. By replacing all ">0" above by "≥ 0", you can find criteria for semi-definiteness. In particular, positive semi-definite matrices possess a factorization $S = A^T A$ where A is allowed to have linearly dependent columns.

6.5.3 Application: Quadratic forms

Note:
The decomposition $S = Q\Lambda Q^T$ has a meaning for quadratic forms. This is what we turn to next.

Example 6.5.6:
Let us consider the ellipse

$$E = \left\{(x,y) \in \mathbb{R}^2\,,\, ax^2 + 2bxy + cy^2 = 1\right\}\,. \tag{6.126}$$

Note that we can write

$$ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \equiv \vec{x}^T S \vec{x}\,. \tag{6.127}$$

When S is positive definite, then E is indeed an ellipse, otherwise not. It thus follows, that there is a relation between

- 2×2 positive definite matrices,
- ellipses in $\mathbb{R}^2$.

Exercise:
Verify that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives a circle.

Example 6.5.7:
Let us now find the axes of the tilted ellipse given by

$$E = \left\{(x,y) \in \mathbb{R}^2 \,,\, 5x^2 + 8xy + 5y^2 = 1\right\} . \tag{6.128}$$

To this end we first notice that

$$5x^2 + 8xy + 5y^2 = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \equiv \vec{x}^T S \vec{x} . \tag{6.129}$$

If is readily verified that the eigenvalues are $1, 9$. Moreover, we have

$$\begin{aligned} \mathrm{Eig}\,(S,1) &= \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} , \\ \mathrm{Eig}\,(S,9) &= \mathrm{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} . \end{aligned} \tag{6.130}$$

We know that these two eigenvectors are orthogonal as S is symmetric. Let us make them orthonormal. Then we have

$$S = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \equiv Q \Lambda Q^T . \tag{6.131}$$

Thereby, we find

$$5x^2 + 8xy + 5y^2 = 1 \cdot \left(\frac{-x+y}{\sqrt{2}}\right)^2 + 9 \cdot \left(\frac{x+y}{\sqrt{2}}\right)^2 . \tag{6.132}$$

If we introduce new coordinates by

$$X := \frac{x+y}{\sqrt{2}} , \qquad Y := \frac{-x+y}{\sqrt{2}} , \tag{6.133}$$

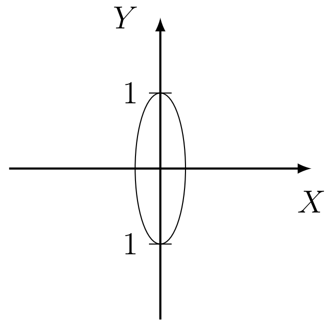
then we can view the ellipse as

$$\tilde{E} = \left\{(X,Y) \in \mathbb{R}^2 \,,\, 9X^2 + Y^2 = 1\right\} . \tag{6.134}$$

Hence, the original ellipse E is really $\tilde{E}$ after a certain coordinate transformation, namely

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{\sqrt{2}} \\ \frac{-x+y}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \equiv R \cdot \begin{bmatrix} x \\ y \end{bmatrix} . \tag{6.135}$$

Note that R is a rotation of the x-y plane by $\frac{\pi}{4}$ clockwise. Hence, E becomes $\tilde{E}$ upon rotation by 45 degrees clockwise:



(6.136)

Consequence:
The eigenvectors give us the directions of the major and minor axis of the ellipise. For this very reason, the factorization for $S = Q\Lambda Q^T$ is sometimes referred to as the *principal axis theorem*.

Remark:
In general, $\vec{x}^T S\vec{x} = 1$ describes an ellipsoid in $\mathbb{R}^n$, provided that S is positive definite.

7 Further topics

7.1 Singular value decomposition (SVD)

Note:
Now we move on to discuss a factorization for matrices that is inspired by diagonalization. It will be referred to as the *singular value decomposition* of a matrix. Unlike diagonalization, SVD works for all matrices and not just square matrices. Furthermore, the so-called singular values actually possess a meaning from the statistical view point ($\rightarrow$ principal component analysis).

Definition 7.1.1 (Singular value decomposition)**:**
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then the singular value decomposition of A is given by

$$A = U \cdot \Sigma \cdot V^T \,, \tag{7.1}$$

where

- U contains an *orthonormal basis* for $\mathbb{R}^m$ such that $\vec{u}_1, \ldots, \vec{u}_r$ is a basis for the column space $C(A)$ and $\vec{u}_{r+1}, \ldots, \vec{u}_m$ is a basis for the left nullspace $N(A^T)$ of A,
- V contains *orthonormal vectors* $\vec{v}_1, \ldots, \vec{v}_n$ such that $\vec{v}_1, \ldots, \vec{v}_r$ is a basis for the row space $R(A)$ while $\vec{v}_{r+1}, \ldots, \vec{v}_n$ is a basis of $N(A)$,
- $A\vec{v}_i = \sigma_i \cdot \vec{u}_i$ for $1 \leq i \leq r$. The σ_i are called the *singular values* of A. In particular σ_i is the length of the vector $A\vec{v}_i$.
- $\Sigma \in \mathbb{M}(m \times n, \mathbb{R})$ has non-zero entries only along the main diagonal, namely

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \sigma_r & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \,, \qquad \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_r > 0 \,. \tag{7.2}$$

Note:
You may want to recall the image-coimage factorization discussed earlier.

Remark:
The singular value decomposition of $A \in \mathbb{M}(m \times n, \mathbb{R})$ can be written as

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T \,. \tag{7.3}$$

Hence, the singular value decomposition expresses A as a sum of r matrices of rank 1.

Note:
For $S \in \mathbb{M}(n \times n, \mathbb{R})$ symmetric and positive-definite, the singular value decomposition of A coincides with what you obtain from the spectral theorem.

Remark:
The idea behind the singular value decomposition is, that for any $A \in \mathbb{M}(m \times n, \mathbb{R})$, the matrix $A^T A \in \mathbb{M}(n \times n, \mathbb{R})$ is symmetric and positive semi-definite. Hence, by the spectral theorem, $A^T A$ has real eigenvalues and we can write $A^T A = VDV^T$ where D is diagonal and V an orthogonal matrix whose columns are the eigenvectors of $A^T A$.

Example 7.1.1:
Let us compute the singular value decomposition of

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R})\,. \tag{7.4}$$

We first notice that

$$A^T A = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{7.5}$$

We find $\mathrm{ch}_{A^T A}(\lambda) = -\lambda \cdot (10 - \lambda) \cdot (12 - \lambda)$. Hence, the eigenvalues are 0, 10 and 12 and we order them in decreasing order:

$$\lambda_1 = 12\,, \qquad \lambda_2 = 10\,, \qquad \lambda_3 = 0\,. \tag{7.6}$$

Since $A^T A$ is guaranteed to be positive semi-definite, these eigenvalues can never be negative. Therefore, we can consider their square roots, which will be important momentarily.

We next compute the eigenspaces of $A^T A$:

$$\begin{aligned} \mathrm{Eig}_{\mathbb{R}}(A^T A, 12) &= \left\{ c \cdot \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, c \in \mathbb{R} \right\}, \\ \mathrm{Eig}_{\mathbb{R}}(A^T A, 10) &= \left\{ c \cdot \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, c \in \mathbb{R} \right\}, \\ \mathrm{Eig}_{\mathbb{R}}(A^T A, 0) &= \left\{ c \cdot \frac{1}{\sqrt{30}} \cdot \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}, c \in \mathbb{R} \right\}, \end{aligned} \tag{7.7}$$

and form the matrix

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{7.8}$$

To find the matrix $U \in \mathbb{M}(2 \times 2, \mathbb{R})$ such that $A = U\Sigma V^T$, we recall that the vectors $\vec{u}_i$ are related to $A\vec{v}_i$ by factors σ_i. The σ_i are the (positive) square roots of the eigenvalues, namely $\sigma_1 = \sqrt{12}$ and $\sigma_2 = \sqrt{10}$. This gives

$$\begin{aligned} \vec{u}_1 &= \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{\sqrt{12}} \cdot \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \vec{u}_2 &= \frac{A\vec{v}_2}{\sigma_2} = \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned} \tag{7.9}$$

In this case, $n = 2$ and we do not have to add any vectors $\vec{u}_i$. Rather, we form

$$U = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}. \tag{7.10}$$

This shows

$$\begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}^T. \tag{7.11}$$

Claim:
Every $A \in \mathbb{M}(m \times n, \mathbb{R})$ admits a singular value decomposition $A = U\Sigma V^T$.

Proof
We consider the symmetric and positive semi-definite matrix $A^T A \in \mathbb{M}(n \times n, \mathbb{R})$. By the spectral theorem, it admits a basis of orthonormal eigenvectors $\vec{v}_i$ with eigenvalues $\lambda_i \in \mathbb{R}$. W.l.o.g. let us assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. We now form the matrix

$$V = \begin{bmatrix} | & \dots & | & | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_r & \vec{v}_{r+1} & \dots & \vec{v}_n \\ | & \dots & | & | & \dots & | \end{bmatrix} \in \mathbb{M}(n \times n, \mathbb{R}). \tag{7.12}$$

For $1 \leq i \leq r$, we then compute $\sigma_i = \sqrt{\lambda_i}$ and set

$$\vec{u}_i := \frac{1}{\sigma_i} \cdot A\vec{v}_i. \tag{7.13}$$

These vectors are automatically orthonormal:

$$\begin{aligned} \vec{u}_i^T \vec{u}_j &= \left(\frac{A\vec{v}_i}{\sigma_i}\right)^T \cdot \left(\frac{A\vec{v}_j}{\sigma_j}\right) = \frac{\vec{v}_i^T A^T A\vec{v}_j}{\sigma_i \sigma_j} = \frac{\vec{v}_i^T \sigma_j^2 \cdot \vec{v}_j}{\sigma_i \sigma_j} = 0, \\ \vec{u}_i^T \vec{u}_i &= \left(\frac{A\vec{v}_i}{\sigma_i}\right)^T \cdot \left(\frac{A\vec{v}_i}{\sigma_i}\right) = \frac{\vec{v}_i^T A^T A\vec{v}_i}{\sigma_i^2} = \frac{\vec{v}_i^T \sigma_i^2 \cdot \vec{v}_i}{\sigma_i^2} = \vec{v}_i^T \vec{v}_i = 1. \end{aligned} \tag{7.14}$$

We complete $\{\vec{u}_1, \dots, \vec{u}_r\}$ to an orthonormal basis of $\mathbb{R}^m$ and consider the matrices

$$U = \begin{bmatrix} | & \dots & | & | & \dots & | \\ \vec{u}_1 & \dots & \vec{u}_r & \vec{u}_{r+1} & \dots & \vec{u}_m \\ | & \dots & | & | & \dots & | \end{bmatrix} \in \mathbb{M}(m \times m, \mathbb{R}),$$
$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \sigma_r & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R}). \tag{7.15}$$

It follows $A = U\Sigma V^T$. And we can read-off:

- $\vec{v}_1, \dots, \vec{v}_r$ is a basis of $R(A)$,
- $\vec{v}_{r+1}, \dots, \vec{v}_n$ is a basis of $N(A)$,
- $\vec{u}_1, \dots, \vec{u}_r$ is a basis of $C(A)$,
- $\vec{u}_{r+1}, \dots, \vec{u}_m$ is a basis of $N(A^T)$.

This completes the proof. ■

Note:
An important application of the SVD is the so-called principal component analysis. Strang's book has a detailed description of this (cf. the library course resources in Canvas).

Remark:
There is an analogue of the singular value decomposition and the spectral theorem for complex valued matrices. This is what we discuss next.

7.2 Complex Vectors and Matrices

Note:
We will now briefly discuss Hermitian and unitary matrices. They arise for example in quantum mechanics (Hermitian matrices are then so-called observables) or in complex Fourier transform, most notably in the so-called *Fast-Fourier transform.* You can find more information on the latter in Strang's book.

Definition 7.2.1:
For a vector $\vec{z} \in \mathbb{C}^n$ with components z_i, we define $\overline{\vec{z}}^T := \begin{bmatrix} \overline{z_1} & \dots & \overline{z_n} \end{bmatrix}$.

Note:
This conjugate transpose is the appropriate analogue of transposition of real vectors.

Definition 7.2.2:
For $A = [z_{ij}] \in \mathbb{M}(n \times n, \mathbb{C})$ the *Hermitian conjugate* or *adjoint* matrix is $A^H = [\overline{z}_{ji}]$.

Example 7.2.1:
Consider the matrix

$$A = \begin{bmatrix} 1 & 1+i \\ 0 & 1-2i \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{C}) \,. \tag{7.16}$$

Then the Hermitian conjugate matrix is given by

$$A^H = \begin{bmatrix} 1 & 0 \\ 1-i & 1+2i \end{bmatrix} \in \mathbb{M}(3 \times 2, \mathbb{C}) \,. \tag{7.17}$$

Consequence:
For any $A \in \mathbb{M}(n \times n, \mathbb{C})$, it holds $\left(A^H\right)^H = A$.

Construction 7.2.1:
How should one define the dot product of two vectors with complex entries? You probably think we should do it in the same way as we do it for vectors in $\mathbb{R}^n$. Here is one reason why it needs adjusting. Namely, let us consider the vector

$$\vec{z} := \begin{bmatrix} 1 \\ i \end{bmatrix} \,. \tag{7.18}$$

Then we find that $\vec{z}^T \vec{z} = 1^2 + i^2 = 0$. Clearly, we want $\vec{z}^T \vec{z}$ to coincide with the square of the length. Therefore, let us consider $\overline{\vec{z}}^T \vec{z}$ instead:

$$\overline{\vec{z}}^T \vec{z} = \sum_{i=1}^{n} \overline{z}_i z_i = \sum_{i=1}^{n} |z_i|^2 \,. \tag{7.19}$$

In particular, we obtain for $\vec{z} := \begin{bmatrix} 1 & i \end{bmatrix}^T$ that $\overline{\vec{z}}^T \vec{z} = 2$, which is far more reasonable.

Definition 7.2.3 (Inner product)**:**
The *inner product* of two complex vectors $\vec{u}, \vec{z} \in \mathbb{C}^n$ is defined as

$$(\vec{u}, \vec{v}) = \vec{u}^H \cdot \vec{v} = \overline{\vec{u}}^T \cdot \vec{v} = \sum_{i=1}^{n} \overline{u_i} \cdot v_i \,. \tag{7.20}$$

Note:
It holds $\vec{u}^H \vec{v} \neq \vec{v}^H \vec{u}$. Rather $\vec{u}^H \vec{v} = \left(\vec{v}^H \vec{u}\right)^H$.

Example 7.2.2:
We consider the vectors

$$\vec{u} := \begin{bmatrix} 1+2i \\ 2-i \end{bmatrix} , \qquad \vec{v} := \begin{bmatrix} 1 \\ i \end{bmatrix} \,. \tag{7.21}$$

Then it holds $\vec{u}^H \cdot \vec{v} = 0$ and $\vec{v}^H \cdot \vec{u} = 0$.

Definition 7.2.4:
Two vectors $\vec{u}, \vec{v}$ are orthogonal iff $\vec{u}^H \vec{v} = 0$.

Example 7.2.3:
The following two vectors are orthogonal:

$$\vec{u} := \begin{bmatrix} 1+2i \\ 2-i \end{bmatrix}, \qquad \vec{v} := \begin{bmatrix} 1 \\ i \end{bmatrix}. \tag{7.22}$$

Corollary 7.2.1:
Here are two simple consequence from the above:

- For any two $A, B \in \mathbb{M}(n \times n, \mathbb{C})$ it holds $(AB)^H = B^H A^H$.
- For any $A \in \mathbb{M}(n \times n, \mathbb{C})$ and $\vec{u}, \vec{v} \in \mathbb{C}^n$ it holds $(A\vec{u})^H \cdot \vec{v} = \vec{u}^H \cdot \left(A^H \vec{v}\right)$.

Exercise:
Prove these statements.

Definition 7.2.5:
A matrix $H \in \mathbb{M}(n \times n, \mathbb{C})$ with $A = A^H$ is termed a *Hermitian* matrix.

Note:
Hermitian matrices are the complex analogue of real symmetric matrices. Thus, Hermitian matrices have similar properties as their real symmetric counterparts.

Claim:
Every eigenvalue λ of a Hermitian matrix $S \in \mathbb{M}(n \times n, \mathbb{C})$ is real.

Proof
We first note that for any vector $\vec{z}$ it holds:

$$\left(\vec{z}^H S \vec{z}\right)^H = \vec{z}^H S^H \vec{z} = \vec{z}^H S \vec{z}. \tag{7.23}$$

Thus, $\vec{z}^H S \vec{z} \in \mathbb{R}$. Let us now apply this for an eigenvector of S with eigenavlue λ, i.e. $S\vec{z} = \lambda \vec{z}$. Thus

$$\vec{z}^H S \vec{z} = \lambda \cdot \vec{z}^H \vec{z}. \tag{7.24}$$

Since $\vec{z}^H S \vec{z}, \vec{z}^H \vec{z} \in \mathbb{R}$ it follows $\lambda \in \mathbb{R}$. ■

Note:
Hermitian matrices have a basis of orthogonal eigenvectors, which can in turn be normalized to unit length of vectors. This leads to the following

Theorem 7.2.1 (Spectral theorem)**:**
Any Hermitian matrix $S \in \mathbb{M}(n \times n, \mathbb{C})$ can be written as

$$S = U \Lambda U^H \tag{7.25}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with the real eigenvalues $\lambda_i \in \mathbb{R}$ of S and the columns of U are an orthonormal basis of $\mathbb{C}^n$ from eigenvectors of S.

Definition 7.2.6:
$U \in \mathbb{M}(n \times n, \mathbb{C})$ is termed a *unitary matrix* iff $U^H U = I$.

Consequence:
The columns of a unitary matrix U are an orthonormal basis of $\mathbb{C}^n$. They are the analogue of orthogonal matrices.

Example 7.2.4:
Let us consider the Hermitian matrix

$$S = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{C})\,. \tag{7.26}$$

Its characteristic polynomial is $\mathrm{ch}_S(\lambda) = (\lambda - 8) \cdot (\lambda + 1)$. Hence, the eigenvalues are 8 and -1 and the corresponding eigenspaces are found to be

$$\begin{aligned} \mathrm{Eig}_{\mathbb{C}}(S, 8) &= \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \right\}, \\ \mathrm{Eig}_{\mathbb{C}}(S, -1) &= \mathrm{Span}\left\{ \begin{bmatrix} 1-i \\ -1 \end{bmatrix} \right\}. \end{aligned} \tag{7.27}$$

Note that the eigenvectors are orthogonal. We can normalize them to find the following orthonormal basis of $\mathbb{C}^2$:

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1+i \end{bmatrix}, \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1-i \\ -1 \end{bmatrix} \right\}. \tag{7.28}$$

Therefore, we can write

$$S = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \cdot \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}. \tag{7.29}$$

Note that the matrix

$$U = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}, \tag{7.30}$$

is unitary. Even more, it is even Hermitian. This is usually not the case, i.e. the base changes for the spectra theorem of Hermitian matrices are in general only unitary and not Hermitian.

Claim:
For $S \in \mathbb{M}(n \times n, \mathbb{C})$, both unitary and Hermitian, the eigenvalues satisfy $\lambda_i \in \{-1, 1\}$.

Proof
Since S is Hermitian, it holds $\lambda_i \in \mathbb{R}$. Further, by the spectral theorem we can write

$$S = U \Lambda U^H\,. \tag{7.31}$$

Since the matrix U is unitary, it holds $U^{-1} = U^H$. This implies

$$S^{-1} = \left(U\Lambda U^H\right)^{-1} = \left(U^H\right)^{-1} \Lambda^{-1} U^{-1} = U\Lambda^{-1}U^H \,. \tag{7.32}$$

But recall that the matrix S is unitary itself. Hence,

$$S^{-1} = S^H = \left(U\Lambda U^H\right)^H = \left(U^H\right)^H \Lambda^H U^H = U\Lambda U^H \,. \tag{7.33}$$

In the last step we have used that $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$. Consequently, by comparing eq. (7.32) and eq. (7.33) we find $\Lambda^{-1} = \Lambda$. This in turn implies $\lambda_i^{-1} = \lambda_i$. Hence, since $\lambda_i \in \mathbb{R}$, it follows $\lambda_i \in \{-1, 1\}$. This completes the proof. ■

Note:
One of the most important computational application of these theoretical insights is the *Fast Fourier transform.* Strang's book has a detailed exposition on this topic (cf. library course resource in Canvas).