Math 313/513, Spring 2021 **Martin Bies**

Final exam

Due: Friday, May 7 – 23:59 EST

Justify all your answers.

Problem 1: Linear algebra questionary [10 Points]

1. Consider $A \in \mathbb{M}(n \times n, \mathbb{R})$ with $\mathrm{ch}_A(\lambda) = -\lambda^2 (\lambda + 4)^2 (\lambda - 3)$.
   - What is the determinant and trace of A?
   - What values can $\mathrm{rk}(A)$ have? For each value, give one example matrix A.
   - What are the possible numbers of linearly independent eigenvectors of A? For each number, provide an example matrix A.
2. Consider $A \in \mathbb{M}(3 \times 3, \mathbb{R})$ with eigenvalues 1, 2, 3. For which $\mu \in \mathbb{R}$ is the matrix $A - \mu I \in \mathbb{M}(3 \times 3, \mathbb{R})$ invertible?
3. Consider $A \in \mathbb{M}(n \times n, \mathbb{C})$. We perform elementary row operations on A and obtain $B \in \mathbb{M}(n \times n, \mathbb{C})$. Is $\mathrm{ch}_A(\lambda) = \mathrm{ch}_B(\lambda)$?
4. Compute the singular-value-decomposition of
$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 0 & 2 \end{bmatrix} \in \mathbb{M}(2 \times 4, \mathbb{R}) \,. \tag{1}$$
   If $\vec{x} \in \mathbb{R}^4$ is such that $|\vec{x}| = 1$, what is the minimal and maximal length of $A\vec{x}$?

Problem 2: Nilpotents and roots [10 Points]

1. We study a *nilpotent* $A \in \mathbb{M}(n \times n, \mathbb{R})$, i.e., $A^k = 0$ for some $k \in \mathbb{Z}_{>0}$.
   - Find 3 nilpotent matrices which are linearly independent in $\mathbb{M}(2 \times 2, \mathbb{R})$.
   - If A is nilpotent, $S \in \mathbb{M}(n \times n, \mathbb{R})$ invertible, prove that SAS^{-1} is nilpotent.
   - Compute e^A for $A = \begin{bmatrix} 2 & 2 & 2 & -3 \\ 6 & 1 & 1 & -4 \\ 1 & 6 & 1 & -4 \\ 1 & 1 & 6 & -4 \end{bmatrix} \in \mathbb{M}(4 \times 4, \mathbb{R})$.
2. Let $k \in \mathbb{Z}_{\geq 2}$ be arbitrary but fixed. We focus on matrices $A \in \mathbb{M}(n \times n, \mathbb{R})$ with $A^k = I$, i.e., *k-th roots* of the identity matrix I.
   - Find 6 different 2nd roots of $I \in \mathbb{M}(2 \times 2, \mathbb{R})$.
   - **Math 513**: Find *infinitely* many 3rd roots of $I \in \mathbb{M}(3 \times 3, \mathbb{R})$.

Problem 3: Damped spring-mass system [10 Points]

The dynamics of a damped spring-mass system – damping $\delta \in \mathbb{R}_{>0}$ – is described by

$$x''(t) = -2\delta x'(t) - \omega^2 x(t)\,. \tag{2}$$

We will solve this ODE with linear algebra techniques. Recall that for $a \in \mathbb{R}$:

$$e^{ia} = \cos(a) + i\sin(a)\,, \qquad \cosh(a) = \frac{e^a + e^{-a}}{2}\,, \qquad \sinh(a) = \frac{e^a - e^{-a}}{2}\,. \tag{3}$$

Use these three identities without proof to simplify your computations.

1. Find $A \in \mathbb{M}(2 \times 2, \mathbb{R})$ such that $\vec{z}'(t) = A \cdot \vec{z}(t)$ where $\vec{z}(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$.
2. Focus on $\delta^2 < \omega^2$:
   - Diagonalize A with a $\mathbb{C}$-valued base change. Thereby compute e^{At}.
   - Parametrize the solution $x(t)$ to eq. (2) by $x(0) = x_0$, $x'(0) = v_0$.
   - Plot $x(t)$ for $0 \le t \le 250$, $\delta = 0.02$, $\omega = 0.5$, $x_0 = v_0 = 1$ in `Python`.
3. Repeat for $\delta^2 > \omega^2$ and plot $x(t)$ for $0 \le t \le 250$, $\delta = 2$, $\omega = 0.3$, $x_0 = v_0 = 1$.
4. Qualitatively, compare and describe the solutions for $\delta^2 < \omega^2$ and $\delta^2 > \omega^2$.

Problem 4: Quadrics and trajectories of asteroids [10 points]

Consider a symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$, $\vec{b} \in \mathbb{R}^n$ and $R \in \mathbb{R}_{>0}$. We study the quadric

$$Q\left(S, \vec{b}, R\right) := \left\{\vec{x} \in \mathbb{R}^n \,\middle|\, \vec{x}^T S \vec{x} + \vec{b}^T \vec{x} = R\right\} \subseteq \mathbb{R}^n\,. \tag{4}$$

1. Use the spectral theorem to prove that there is a bijection of sets

$$\varphi\colon Q\left(S, \vec{b}, R\right) \to \left\{\vec{y} \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} \lambda_i y_i^2 + \vec{c}^T \vec{y} = R\right\}\,, \quad \lambda_i \in \mathbb{R}, \vec{c} \in \mathbb{R}^n \text{ suitable.} \tag{5}$$

2. Under simplifying assumptions, the trajectory of an asteroid in the gravitational field of a planet is described by

$$T(\epsilon) = \left\{\vec{y} \in \mathbb{R}^2 \,\middle|\, \vec{y}^T \begin{bmatrix} 1-\epsilon^2 & 0 \\ 0 & 1 \end{bmatrix} \vec{y} + \begin{bmatrix} 2\epsilon \\ 0 \end{bmatrix}^T \vec{y} = 1\right\} \subseteq \mathbb{R}^2\,, \tag{6}$$

   where $\epsilon \in \mathbb{R}_{\ge 0}$ is the *eccentricity*. For $\epsilon = 0$, we have

$$T(\epsilon) = \left\{\vec{y} \in \mathbb{R}^2 \,\middle|\, y_1^2 + y_2^2 = 1\right\} \subseteq \mathbb{R}^2\,, \tag{7}$$

   which clearly defines a circle. Similarly, identify the geometric shape of $T(\epsilon)$ for

$$0 < \epsilon < 1\,, \quad \epsilon = 1\,, \quad 1 < \epsilon\,. \tag{8}$$

   Make a *contour plot* in `Python` for $\epsilon \in \{0, 0.5, 1, 1.5\}$.