Math 313/513, Spring 2021 **Martin Bies**

Midterm 1

Due: Thursday, February 25 – 10:00 am EST

Problem 1: The four linear subspaces [10 Points]

Every map $\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$, $A \in \mathbb{M}(m \times n, \mathbb{R})$ can be factored as

$$\begin{array}{ccccccc} \ker(\varphi_A) \cong N(A) & \xhookrightarrow{\varphi_K} & \mathbb{R}^n & \xrightarrow{\varphi_A} & \mathbb{R}^m & \xrightarrow{\varphi_P} & N(A^T) \cong \operatorname{coker}(\varphi_A) \\ & & \downarrow \varphi_{M_1} & & \uparrow \varphi_{M_2} & & \\ & \operatorname{coim}(\varphi_A) \cong R(A) & & \xrightarrow{\varphi_X} & & C(A) \cong \operatorname{im}(\varphi_A) & \end{array} \tag{1}$$

We compute this factorization for $A = \begin{bmatrix} 1 & 4 & 7 & 10 & 5 \\ 2 & 5 & 8 & 11 & 10 \\ 3 & 6 & 9 & 12 & 15 \end{bmatrix} \in \mathbb{M}(3 \times 5, \mathbb{R})$.

1. *Kernel embedding* of φ_A:
   - Verify $N(A) = \operatorname{Span}_{\mathbb{R}}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Form $K = [\ \vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3\ ] \in \mathbb{M}(5 \times 3, \mathbb{R})$.
   - Argue that $\operatorname{im}(\varphi_K) = N(A)$ and that φ_K is injective.
2. *Coimage projection* of φ_A:
   - Verify $R(A) = \operatorname{Span}_{\mathbb{R}}\{\vec{b}_1, \vec{b}_2\}$ and form $M_1 = [\ \vec{b}_1 \quad \vec{b}_2\ ]^T \in \mathbb{M}(2 \times 5, \mathbb{R})$.
   - Argue that φ_{M_1} is surjective.
3. *Image embedding* of φ_A:
   - Verify $C(A) = \operatorname{Span}_{\mathbb{R}}\{\vec{c}_1, \vec{c}_2\}$ and form $M_2 = [\ \vec{c}_1 \quad \vec{c}_2\ ] \in \mathbb{M}(3 \times 2, \mathbb{R})$.
   - Argue that $\operatorname{im}(\varphi_{M_2}) = C(A)$ and that φ_{M_2} is injective.
4. *Cokernel projection* of φ_A:
   - Verify $N(A^T) = \operatorname{Span}_{\mathbb{R}}\{\vec{d}_1\}$. Form $P = [\ \vec{d}_1\ ]^T \in \mathbb{M}(1 \times 3, \mathbb{R})$.
   - Argue that φ_P is surjective.
5. Draw eq. (1) for A. Indicate the dimension of all vector spaces.
6. **Math 513**: Isomorphism φ_X:
   Find an invertible $X \in \mathbb{M}(2 \times 2, \mathbb{R})$ with $A = M_2 \cdot X \cdot M_1$.
7. **Bonus**: Write a `Python` function *image-coimage-factorization*:
   Input: Arbitrary matrix $B \in \mathbb{M}(m \times n, \mathbb{R})$
   Output:
   - The matrices K, M_1, M_2, P in the image-coimage factorization of φ_B.
   - **Nice** image of eq. (1). Indicate the vector spaces by their dimension.

   Apply it to the matrix $B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 8 & 10 \\ 9 & 10 & 11 & 12 & 15 \end{bmatrix} \in \mathbb{M}(3 \times 5, \mathbb{R})$.

Problem 2: Basis extension theorem [10 Points]

1. Which subsets $S' \subseteq \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ extend to a basis of $\mathbb{R}^4$?

2. Extend one such subset S' to a basis $\mathcal{B}$ of $\mathbb{R}^4$. Verify that $\mathcal{B}$ is a basis of $\mathbb{R}^4$.

3. Write a `Python` function `lin_independent` and justify why it operates correctly:
   - Input: k vectors in $\mathbb{R}^n$.
   - Output: `True` if the vectors are linearly independent and `false` otherwise.

4. Write a `Python` function `basis_check` and justify why it operates correctly:
   - Input: k vectors in $\mathbb{R}^n$.
   - Output: `True` if the vectors are a basis of $\mathbb{R}^n$ and `false` otherwise.

5. Use these functions to verify your answers to (1) and (2).

Problem 3: Rank-nullity theorem [10 Points]

The rank-nullity theorem states that for $A \in \mathbb{M}(m \times n, \mathbb{R})$ it holds

$$\dim_{\mathbb{R}}(N(A)) + \dim_{\mathbb{R}}(C(A)) = \dim_{\mathbb{R}}\left(\text{Source}(\varphi_A)\right) . \qquad (2)$$

1. Argue that this is equivalent to $\dim_{\mathbb{R}}(N(A)) + \text{rk}(A) = n$.

2. Test the rank-nullity theorem for $A = \begin{bmatrix} 1 & 3 & 2 & 6 \\ -1 & 1 & 2 & 2 \\ 3 & 7 & 4 & 14 \end{bmatrix} \in \mathbb{M}(3 \times 4, \mathbb{R})$.

3. In `Python`, generate a family $\mathcal{A}$ of 1000 random matrices in $\mathbb{M}(5 \times 5, \mathbb{R})$. The entries of these matrices are required to be the integers 0 or 1. Compute the dimension of the null space of all matrices $\mathcal{A}$. What dimensions occur?

4. Write a `Python` function:

   Input: $A \in \mathbb{M}(m \times n, \mathbb{R})$. Output: $\dim_{\mathbb{R}}(N(A)) + \text{rk}(A) - n$.

   Use this function to test the rank-nullity theorem for all matrices in $\mathcal{A}$.

5. Formulate a hypothesis how $\dim_{\mathbb{R}}(N(A^T))$, $\dim_{\mathbb{R}}(C(A^T))$ and $\dim_{\mathbb{R}}\left(\text{Range}(\varphi_A)\right)$ are related. With `Python`, verify your hypothesis for all matrices in $\mathcal{A}$.

6. **Bonus**: Use eq. (2) to prove your hypothesis.

Problem 4: A matrix questionary [10 Points]

Suppose A is a matrix with a two-dimensional null space and a one-dimensional column space. Suppose $\vec{b}$ belongs to the column space of A. Answer each of the following questions as true, false, or not enough information. Justify your answer.

1. The system $A\vec{x} = \vec{b}$ is solvable.
2. A has three columns.
3. A has three rows.
4. $\mathrm{rk}(A)$ is divisible by three.
5. The solutions to $A\vec{x} = \vec{b}$ form a plane $P \subset \mathbb{R}^3$.