Homework 3 – Theory

Due: Thursday, February 3 – 10:00 am EST

Problem 1T: PLU factorization [10 Points]

1. Consider k invertible matrices $A_i \in \mathbb{M}(n \times n, \mathbb{R})$. Prove that
$$\left(\prod_{i=1}^{k} A_i \right)^{-1} = \prod_{i=0}^{k-1} A_{k-i}^{-1} \,. \tag{1}$$
2. Prove that if a lower triangular matrix has an inverse, it is lower triangular.
3. Find the PLU factorization, pivots and rank of A:
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 11 \\ 3 & 8 & 14 & 20 \\ 4 & 11 & 20 & 30 \end{bmatrix} \,. \tag{2}$$

Problem 2T: Nullspace [10 Points]

1. Compute the row reduced echelon form (RREF) of
$$B = \begin{bmatrix} 0 & 0 & 7 & 2 \\ 0 & 1 & 5 & 2 \\ 1 & 4 & 3 & 2 \end{bmatrix} \,. \tag{3}$$
2. For each free column of the RREF, read off a non-trivial solution to $B\vec{x} = \vec{0}$.
3. Find all solutions to $B\vec{x} = \vec{0}$. Justify your answer.

Problem 3T: "Exotic" vector spaces [20 Points]

1. Consider $M := \mathbb{M}(2 \times 2, \mathbb{R})$ and
$$+_M \colon M \times M \to M \,,\; (A, B) \mapsto \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \,, \tag{4}$$
$$\cdot_M \colon \mathbb{R} \times M \to M \,,\; (c, A) \mapsto \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} \\ c \cdot a_{21} & c \cdot a_{22} \end{bmatrix} \,. \tag{5}$$
Show that $(M, +_M, \cdot_M)$ is a vector space over $\mathbb{R}$.
2. $P := \mathrm{Pol}_n$ is the set of polynomials in the variable x with degree at most n. Find operations $+_P$ and $\cdot_P$ such that $(\mathrm{Pol}_n, +_P, \cdot_P)$ is a vector space over $\mathbb{R}$.
3. **Math 513**:
Argue that $(\mathrm{Pol}_n, +_P, \cdot_P) \cong (\mathbb{R}^m, +, \cdot)$ for a suitable $m \in \mathbb{Z}_{\geq 0}$. $\mathbb{R}^m$ is to be considered with its standard vector space operations.