**Math 313/513, Spring 2022** **Martin Bies**

# Homework 4

Due: Thursday, February 17 – 10:00 am EST

## Problem 1T: More about ranks [10 Points]

Consider $A \in \mathbb{M}(m \times n, \mathbb{R})$ and $\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m\,,\ \vec{x} \mapsto A\vec{x}$.

1. Argue that:
   - $\varphi_A$ is injective iff $N(A) = \{0\}$,
   - $\varphi_A$ is surjective iff $N(A^T) = \{0\}$.

   Use this and the rank-nullity theorem to compare $\mathrm{rk}(A)$ with $n$, $m$ for the following cases:

   (a) $\varphi_A$ is injective, (b) $\varphi_A$ is surjective, (c) $\varphi_A$ is bijective.

2. Is it always true that $\mathrm{rk}(B) \leq \mathrm{rk}(AB)$ for $B \in \mathbb{M}(n \times l, \mathbb{R})$?

3. **Math 513**: Is it always true that $\mathrm{rk}(AB) \leq \mathrm{rk}(A)$ for $B \in \mathbb{M}(n \times l, \mathbb{R})$?

4. The rank-nullity theorem states that for any $A \in \mathbb{M}(m \times n, \mathbb{R})$ it holds

$$\dim_{\mathbb{R}}(N(A)) + \dim_{\mathbb{R}}(C(A)) = \dim_{\mathbb{R}}\left(\mathrm{Source}(\varphi_A)\right)\,. \tag{1}$$

   Argue that this is equivalent to $\dim_{\mathbb{R}}(N(A)) + \mathrm{rk}(A) = n$.

## Problem 2T: The four linear subspaces [20 Points]

Every map $\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m\,,\ \vec{x} \mapsto A\vec{x}$, $A \in \mathbb{M}(m \times n, \mathbb{R})$ can be factored as

$$\begin{array}{ccccccc} \ker(\varphi_A) \cong N(A) & \xhookrightarrow{\varphi_K} & \mathbb{R}^n & \xrightarrow{\varphi_A} & \mathbb{R}^m & \xrightarrow{\varphi_P} & N(A^T) \cong \mathrm{coker}(\varphi_A) \\ & & \downarrow{\scriptstyle \varphi_{M_1}} & & \uparrow{\scriptstyle \varphi_{M_2}} & & \\ & \mathrm{coim}(\varphi_A) \cong R(A) & & \xrightarrow{\varphi_X} & C(A) \cong \mathrm{im}(\varphi_A) & & \end{array} \tag{2}$$

We compute this factorization for $A = \begin{bmatrix} 1 & 4 & 7 & 10 & 5 \\ 2 & 5 & 8 & 11 & 10 \\ 3 & 6 & 9 & 12 & 15 \end{bmatrix} \in \mathbb{M}(3 \times 5, \mathbb{R})$.

1. *Kernel embedding* of $\varphi_A$:
   - Verify $N(A) = \mathrm{Span}_{\mathbb{R}}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Form $K = [\ \vec{a}_1\ \ \vec{a}_2\ \ \vec{a}_3\ ] \in \mathbb{M}(5 \times 3, \mathbb{R})$.
   - Argue that $\mathrm{im}(\varphi_K) = N(A)$ and that $\varphi_K$ is injective.

2. *Coimage projection* of $\varphi_A$:
   - Verify $R(A) = \mathrm{Span}_{\mathbb{R}}\{\vec{b}_1, \vec{b}_2\}$ and form $M_1 = [\ \vec{b}_1\ \ \vec{b}_2\ ]^T \in \mathbb{M}(2 \times 5, \mathbb{R})$.
   - Argue that $\varphi_{M_1}$ is surjective.

3. *Image embedding* of $\varphi_A$:
   - Verify $C(A) = \operatorname{Span}_{\mathbb{R}}\{\vec{c}_1, \vec{c}_2\}$ and form $M_2 = [\ \vec{c}_1 \quad \vec{c}_2\ ] \in \mathbb{M}(3 \times 2, \mathbb{R})$.
   - Argue that $\operatorname{im}(\varphi_{M_2}) = C(A)$ and that $\varphi_{M_2}$ is injective.
4. *Cokernel projection* of $\varphi_A$:
   - Verify $N(A^T) = \operatorname{Span}_{\mathbb{R}}\{\vec{d}_1\}$. Form $P = [\ \vec{d}_1\ ]^T \in \mathbb{M}(1 \times 3, \mathbb{R})$.
   - Argue that $\varphi_P$ is surjective.
5. Compare your findings with the rank-nullity theorem for $A$ and $A^T$.
6. **Math 513**: Isomorphism $\varphi_X$:
   Find an invertible $X \in \mathbb{M}(2 \times 2, \mathbb{R})$ with $A = M_2 \cdot X \cdot M_1$.

## Problem 3T: Bases of vector spaces [10 Points]

1. Which subsets $S' \subseteq \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$ extend to a basis of $\mathbb{R}^4$?
2. Find $n \times m$ linearly independent vectors in $\mathbb{M}(m \times n, \mathbb{R})$.
3. Prove $\dim_{\mathbb{R}}(\mathbb{M}(m \times n, \mathbb{R})) = m \times n$.
4. Find $n + 1$ linearly independent vectors in $\operatorname{Pol}_n$.
5. Prove $\dim_{\mathbb{R}}(\operatorname{Pol}_n) = n + 1$.