Math 313/513, Spring 2022 **Martin Bies**

Homework 7 – Theory

Due: Thursday, March 31 – 10:00 EST

Problem 1T: Determinants – Generalities [10 Points]

1. You were given $A \in \mathbb{M}(3 \times 3, \mathbb{R})$ and performed the following row operations:
   - Added the first row to the third row.
   - Swapped rows 2 and 3.
   - Scaled row 2 by the number 4.

   You arrive at $E = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$. What is $\det(A)$? Explain your answer.

2. Let $A \in \mathbb{M}(4 \times 4, \mathbb{R})$ with $\det(A) = 7$ and $\vec{b} \in \mathbb{R}^4$. Argue that $A\vec{x} = \vec{b}$ has exactly one solution.

3. Solve $A\vec{x} = \vec{b}$ with Cramer's rule for

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,, \qquad \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}\,. \tag{1}$$

   Hint: Verify $\det(A) \neq 0$, as otherwise Cramer's rule does not apply.

Problem 2T: Determinants and applications [10 Points]

1. Show that the Vandermonde determinant satisfies ($a_i \in \mathbb{R}$)

$$\det\left(\begin{bmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{bmatrix}\right) = \prod_{1 \leq i < j \leq n}^{n} (a_j - a_i)\,. \tag{2}$$

2. You are given points $\{(x_i, y_i) \in \mathbb{R}^2 | 1 \leq i \leq n \text{ and } x_i \neq x_j \text{ whenever } i \neq j\}$. We are looking for a polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}\,, \tag{3}$$

   with $P(x_i) = y_i$ for all $1 \leq i \leq n$. Under what condition does a unique P exist?

Problem 3T: A first encounter with diagonalization [20 Points]

In this problem, we find a basis in which a linear transformation is diagonal.

1. Compute the polynomial $\mathrm{ch}_A(\lambda) = \det(A - \lambda I) \in \mathbb{R}[\lambda]$ for

$$A = \begin{bmatrix} -2 & -2 & -2 \\ -2 & 1 & -5 \\ -2 & -5 & 1 \end{bmatrix} . \tag{4}$$

2. Find the three zeros $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{R}$ of this polynomial.

3. Find linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$ with

$$A\vec{v}_1 = \lambda_1 \cdot \vec{v}_1 , \qquad A\vec{v}_2 = \lambda_2 \cdot \vec{v}_2 , \qquad A\vec{v}_3 = \lambda_3 \cdot \vec{v}_3 . \tag{5}$$

4. Find the base change matrix $T_{\mathcal{B}_2\mathcal{B}_1}$ where $\mathcal{B}_2 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\mathcal{B}_1 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

5. For the linear transformation $\varphi_A \colon \mathbb{R}^3 \to \mathbb{R}^3$ with $A = A_{\mathcal{B}_2\mathcal{B}_2}$, compute $A_{\mathcal{B}_1\mathcal{B}_1}$.
   Hint: You should find that $A_{\mathcal{B}_1\mathcal{B}_1}$ is diagonal.