Homework 9 – Coding

Due: Thursday, April 14 – 10:00 EST

Problem 1C: The type of a local extremum [20 Points]

Background information:
In this exercise we study local extrema of maps

$$f\colon \mathbb{R}^n \to \mathbb{R}\,,\ \vec{x} = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T \mapsto f(x_1, x_2, \ldots, x_n)\,. \tag{1}$$

At a local extremum $\vec{a}$ of f, the Jacobian $J(f)(\vec{a}) \in \mathbb{M}(n \times 1, \mathbb{R})$ necessarily vanishes:

$$0 \equiv J(f)(\vec{a}) = \begin{bmatrix} \left(\frac{\partial f}{\partial x_1}\right)(\vec{a}) & \ldots & \left(\frac{\partial f}{\partial x_n}\right)(\vec{a}) \end{bmatrix}^T\,. \tag{2}$$

The type of local extremum is identified by studying the Hessian matrix of f at $\vec{a}$:

$$H(f)(\vec{a}) = \begin{bmatrix} \left(\frac{\partial^2 f}{\partial x_1 \partial x_1}\right)(\vec{a}) & \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)(\vec{a}) & \ldots & \left(\frac{\partial^2 f}{\partial x_1 \partial x_n}\right)(\vec{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\partial^2 f}{\partial x_n \partial x_1}\right)(\vec{a}) & \left(\frac{\partial^2 f}{\partial x_n \partial x_2}\right)(\vec{a}) & \ldots & \left(\frac{\partial^2 f}{\partial x_n \partial x_n}\right)(\vec{a}) \end{bmatrix} \in \mathbb{M}(n \times n, \mathbb{R})\,. \tag{3}$$

Namely, it can be shown that the following holds true:

$$\begin{aligned} \vec{a} \text{ is local maximum} &\quad\Leftrightarrow\quad H(f)(\vec{a}) \text{ negative definite}\,, \\ \vec{a} \text{ is local minimum} &\quad\Leftrightarrow\quad H(f)(\vec{a}) \text{ positive definite}\,, \\ \vec{a} \text{ is saddle point} &\quad\Leftrightarrow\quad H(f)(\vec{a}) \text{ indefinite}\,. \end{aligned} \tag{4}$$

We will eventually prove in the lecture that a *symmetric* matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ (i.e. $A = A^T$) has only real eigenvalues. For such a matrix it holds:

$$\begin{aligned} A \text{ is positive definite} &\quad\Leftrightarrow\quad \text{all eigenvalues of } A \text{ are positive}\,, \\ A \text{ is negative definite} &\quad\Leftrightarrow\quad \text{all eigenvalues of } A \text{ are negative}\,, \\ A \text{ is indefinite} &\quad\Leftrightarrow\quad A \text{ has positive and negative eigenvalues}\,. \end{aligned} \tag{5}$$

Use this information to complete the following tasks.

Tasks:

1. Write a `Python` function `PositiveDefinite`:
   - Input: $A \in \mathbb{M}(n \times n, \mathbb{R})$
   - Output:
     - Check if $A = A^T$. If not, raise an error and exit.
     - Otherwise, return *true* if A is positive definite and *false* otherwise.

2. Similarly, write a `Python` function `NegativeDefinite` and `Indefinite`.

3. We model the profile of a mountainous region via

$$f\colon \mathbb{R}^2 \to \mathbb{R}\,,\ (x, y) \mapsto 2 - 2x^2 + x^4 + xy - 2y^2 + y^4\,. \tag{6}$$

   In the following, we focus on $(x, y) \in R = [-1.5, 1.5] \times [-1.5, 1.5]$.

   a) Make a 3-dimensional plot of f for $(x, y) \in R$.

   b) By analytic means, verify that

$$J(f)\,(x, y) = \begin{bmatrix} 4x(-1 + x^2) + y \\ 4y(-1 + y^2) + x \end{bmatrix}, \tag{7}$$

$$H(f)\,(x, y) = \begin{bmatrix} 12x^2 - 4 & 1 \\ 1 & 12y^2 - 4 \end{bmatrix}, \tag{8}$$

   and that $J(f)(x, y) \equiv 0$ iff

$$y = -4x(-1 + x^2)\,, \tag{9}$$

$$0 = -x\left(-5 + 4x^2\right)\left(-3 + 4x^2\right)\left(1 - 16x^2 + 16x^4\right)\,. \tag{10}$$

   c) Find the 9 points $(x, y) \in R$ at which $J(f)$ vanishes and use your functions from part 1 and 2 to identify the type of extremum.
   Hint: There is 1 local maximum, 4 saddle points and 4 local minima.

   d) **Bonus (for 313 and 513):** Create a contour map of f over the region R. Indicate also the 9 local extrema (maxima in red, minima in blue and saddle points in green color).