Math 313/513, Spring 2022 **Martin Bies**

Homework 9 – Theory

Due: Thursday, April 15 – 10:00 EST

Problem 1: More on diagonalization [20 Points]

1. Diagonalize $A_1, A_2, A_3 \in \mathbb{M}(3 \times 3, \mathbb{R})$ or prove that this is impossible:

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \qquad (1)$$

2. Matrices $A, B \in \mathbb{M}(n \times n, \mathbb{R})$ are similar if there is an invertible $S \in \mathbb{M}(n \times n, \mathbb{R})$ such that $A = SBS^{-1}$. (Hence, A and B describe the same linear map in different bases and the base change is mediated by the matrix S.)
   - If A is diagonalizable and A, B are similar, prove that B is diagonalizable.
   - If A, B are similar, then show that $\det(A) = \det(B)$.
   - Are there A_i in eq. (1) which are similar?
3. Matrices $A, B \in \mathbb{M}(n \times n, \mathbb{R})$ are simultaneously diagonalizable if there is an invertible $S \in \mathbb{M}(n \times n, \mathbb{R})$ such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal.
   - Show that simultaneously diagonalizable matrices commute: $AB = BA$.
   - Are there A_i in eq. (1) which are simultaneously diagonalizable?
   - **Math 513:** Show that if $AB = BA$ and all eigenvalues of A have algebraic multiplicity 1, then A, B are simultaneously diagonalizable.

Problem 2: Markov meets Christmas [20 Points]

Suppose your mood is categorized in exactly three classes (good, even-tempered, bad). We assume the following daily transition probabilities:

	good	even-tempered	bad
good	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{1}{8}$
even-tempered	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$
bad	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{4}$

(2)

Let $\vec{x}^{(k)} = \begin{bmatrix} g_k & e_k & b_k \end{bmatrix}^T \in \mathbb{R}^3$ correspond to your mood in k-days.

1. Find $M \in \mathbb{M}(3 \times 3, \mathbb{R})$ such that $\vec{x}^{(k)} = M\vec{x}^{(k-1)}$ (for $k = 1, 2, \ldots$).
2. Today is April 7. Find the number N of days until Christmas eve.
3. Diagonalize M and use it to find an algebraic expression for $\vec{x}^{(k)}$.
4. For $\vec{x}^{(0)} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \end{bmatrix}^T$, approximate the mood on Christmas eve.
5. **Math 513:** Verify that the components $\vec{x}^{(k)}$ add to one and find $\lim\limits_{k \to \infty} M^k$.