Math 313: Computational linear algebra

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1 Preface

Generalities The aim of this course is to give an overview over linear algebra with an emphasize on its arithmetics and applications. For this reason, we restrict ourselves to linear algebra over $\mathbb{R}$ and $\mathbb{C}$. Deeper insights into the theory of linear algebra over arbitrary fields (and eventually algebra over arbitrary rings) are taught in more advanced algebra courses.

By tradition, this course follows the textbook *Introduction to linear algebra* by *Gilbert Strang*. This course is no exception. For convenience, these notes aim to collect the material in a self-contained fashion.

What is this course about? Very broadly speaking, we will study lines. A little more specifically, we will study the geometry of linear systems of equations.

While linear systems of equations look deceptively simple, they underlie a vast amount of (applied) mathematics. Ideas that you will encounter here have some surprising and far-reaching applications. To appreciate this fact, we must understand the mathematics behind linear equations.

An example of such applications derives from an attempt to answer the following question: *Why should we care about lines?* Recall from calculus the idea that, in order to understand a complicated curve, we can attempt to zoom in and see what these objects look *locally*. This leads to the study of the tangent line, which is an example of a linear structure and vector space! So, in extrapolating to higher-dimensional objects, we may conclude that while the world around us may possess complicated geometries, if we look locally, we see linear structures.

Of course, we lose information when we zoom in. The resulting linear structure is in general not identical to the original geometric structure. Still, this linear "approximation" retains a vast amount of information. This is why we can try to understand such complicated structures from studying their linear approximations. At times, it is then possible to extract information about the original geometric structure from these linear structures. For example, (for equidimensional) curves, surfaces etc. the dimension of the tangent space matches the dimension of the original geometric object.

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Typos, mistakes and feedback Please send messages regarding typos, mistakes and general feedback to mbies@sas.upenn.edu. Thank you!

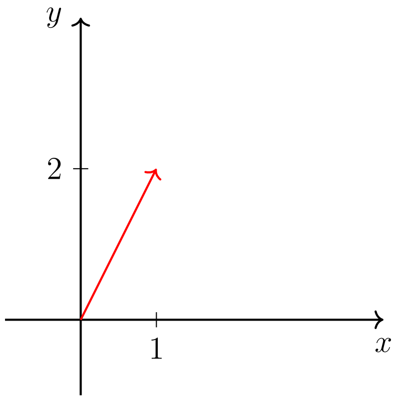
2 Solving Linear Equations

2.1 Revision: Vectors

We begin with a quick revision of vectors, which you should have encountered before, e.g. in Math 240 or Math 260.

Example 2.1.1 (A vector in 2 dimensions)**:**
$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a vector in 2 dimensions. The entries 1 and 2 are the components of $\vec{v}$.

Remark:
We will write our vectors as column vectors. We can picture $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as follows:



(2.1)

Example 2.1.2 (A vector in 3 dimensions)**:**
Similarly, we can work in three dimensions. For instance, the following is a vector in 3-dimensions:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}. \tag{2.2}$$

Exercise:
Draw an image of this vector $\vec{v}$ in three dimensions.

Note:
These images get no easier in higher dimension. However, we may still abstractly think of vectors by merely listing their components. In a sense, this is the first place where we see the benefits of abstraction.

Definition 2.1.1:
A vector $\vec{v} \in \mathbb{R}^n$ is a column with n real components:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \qquad v_i \in \mathbb{R}\,. \tag{2.3}$$

Definition 2.1.2 (Addition)**:**
For $\vec{v}, \vec{u} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we define addition and scalar multiplication:

$$\vec{v} + \vec{u} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} := \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}, \tag{2.4}$$

$$c \cdot \vec{v} = c \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} := \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \\ \vdots \\ c \cdot v_n \end{bmatrix}. \tag{2.5}$$

Note:
A 1-dimensional vector is essentially a real number. The algebraic operations for vectors are lifted from addition and multiplication of real numbers.

Exercise (Addition and scalar multiplication pictorially)**:**
Consider $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$:

- Draw images of $\vec{v} + \vec{u}$ and $2 \cdot (\vec{v} + \vec{u})$.
- Draw the set $\{\lambda \cdot \vec{u} \,|\, \lambda \in \mathbb{R}\}$.

Note:
Later in the course, we will see that addition and scalar multiplication allow us to generate lines, planes and, more generally speaking, any linear object in any dimension.

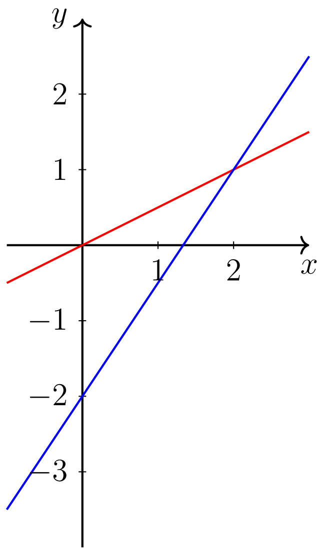
2.2 Approaches to systems of linear equations

For the remainder of chapter 2, we will work with n equations in n unknown, i.e. as many equations as variables.

Note (Row picture)**:**
Let us consider the following set of equations:

$$\begin{aligned} x - 2y &= 0\,, \\ 3x - 2y &= 4\,. \end{aligned} \tag{2.6}$$

The equations $x - 2y = 0$ and $3x - 2y = 4$ define 2 lines in the plane:



(2.7)

We can therefore interpret this set of equations as the task to find the intersection of those two lines. We term this perspective the *row picture*.

Note (Column picture)**:**
Let us again consider the system of equations

$$\begin{aligned} x - 2y &= 0\,, \\ 3x - 2y &= 4\,. \end{aligned} \tag{2.8}$$

It makes sense to consider the coefficients of x and y in both equations simultaneously. We may thus rewrite this system as

$$x \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}\,. \tag{2.9}$$

We are thus trying to find scalar multiples of the vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$, such that their sum matches $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$. We term this perspective the *column picture*.

Exercise:
How can we geometrically determine the right scalars x and y?

Note (Matrix picture)**:**
Yet another way to view the equations

$$\begin{aligned} x - 2y &= 0\,, \\ 3x - 2y &= 4\,. \end{aligned} \tag{2.10}$$

is the matrix picture. Here, we use the coefficients to construct the 2×2 *coefficient matrix*

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{2.11}$$

We record the RHS of eq. (2.10) and collect the unknowns in the vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then we can rewrite eq. (2.10) as

$$\begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}\,. \tag{2.12}$$

Notation:
We use $\mathbb{M}(m \times n, \mathbb{R})$ to denote the set of all $m \times n$ matrices with entries in $\mathbb{R}$. More generally, and whenever applicable, we use $\mathbb{M}(m \times n, \mathbb{F})$ to denote the set of all $m \times n$ matrices with entries in a field $\mathbb{F}$ (or even a ring).

Definition 2.2.1 (Multiplication of matrix and vector)**:**

$$\begin{bmatrix} & \\ n \times n & \\ \text{matrix } A & \\ & \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \begin{bmatrix} \text{Column 1 of } A \end{bmatrix} + x_2 \cdot \begin{bmatrix} \text{Column 2 of } A \end{bmatrix} + \cdots + x_n \cdot \begin{bmatrix} \text{Column } n \text{ of } A \end{bmatrix}\,. \tag{2.13}$$

Consequence:
The following approaches to systems of linear equations are equivalent:

- linear systems,
- equations involving column vectors,
- matrix equations $A\vec{x} = \vec{b}$.

Remark:
For a system of three equations in three variables, the row picture corresponds to finding the intersection of three planes. On the other hand, the column picture concerns combinations of 3-dimensional vectors. As a general theme through the course, we will find that the column picture is more revealing.

2.3 The method of elimination with back substitution

Example 2.3.1:
Let us return once again to the system of linear equations

$$\begin{aligned} x - 2y &= 0\,, \\ 3x - 2y &= 4\,. \end{aligned} \tag{2.14}$$

By multiplying the first equation by 3 and subtracting it from equation 2, we obtain a new system of equations:

$$\begin{aligned} x - 2y &= 0\,, \\ 4y &= 4\,. \end{aligned} \tag{2.15}$$

At this point, we can solve for y and obtain $y = 1$. Once we know that, we can plug this value into the first equation and solve for x. We obtain $x = 2$.

Note:
This procedure is a special instance of the *method of elimination with back substitution.* We aspire to do this in general, i.e. given a system

$$A\vec{x} = \vec{b}\,, \qquad A \in \mathbb{M}(n \times n, \mathbb{R}),\ \vec{b} \in \mathbb{R}^n\,, \tag{2.16}$$

we desire to transform this system into the form

$$U\vec{x} = \vec{c}\,, \qquad U \in \mathbb{M}(n \times n, \mathbb{R}),\ \vec{c} \in \mathbb{R}^n\,, \tag{2.17}$$

where U is upper triangular and, as a consequence, $U\vec{x} = \vec{c}$ is readily solvable.

Example 2.3.2:
To see how this can be achieved, let us consider another example:

$$\begin{aligned} x + 2y + z &= 2\,, \\ 3x + 8y + z &= 12\,, \\ 4y + z &= 2\,. \end{aligned} \tag{2.18}$$

First, notice that the variable names do not play any role. Therefore, we may ditch them. The coefficient matrix A and the column vetor $\vec{b}$ are the only pieces of information that are relevant. We put them in the so-called *augmented matrix*:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]\,. \tag{2.19}$$

Now, let us aim for a triangular form. We can use the blue 1 to eliminate the 3 below. This gives a new matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]\,. \tag{2.20}$$

Thus, we have eliminated the entry in row 2 column 1. Ideally, we can use the blue 1 to eliminate also the entry in row 3 and col 1. In this particular example, this entry is already 0, so no action is required. At this point, we have 0s below the circled 1. This means that the equations corresponding to the 2nd and 3rd row do not involve x.

Now we recurse and use the blue 2 to clear out the entry in row 3 and column 2. This gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]. \tag{2.21}$$

Now, we can solve for z, then (after back substitution) for y and finally for x.

Definition 2.3.1 (Pivot elements)**:**
The **non-zero**, blue entries in eq. (2.21) are called *pivots* – they are pivotal to the execution of this procedure.

Remark:
By definition, we require that a pivot element is *non-zero*.

Consequence:
By performing the back-substitution, we find that the solution to eq. (2.21) is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}. \tag{2.22}$$

Note (Upshot)**:**
We went from $\left[A|\vec{b}\right]$ to $[U|\vec{c}]$ where U is upper triangular. Subsequently, we solved the resulting system by back substitution. Does this procedure always work? What could possibly go wrong?

Remark:
Note that the entries, which we use to clear out columns, should not vanish. Hence, how would we deal with the following augmented matrix?

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \tag{2.23}$$

While the entry at row 1 column 1 vanishes, there is a non-zero element in row 2 column 1. We can swap these two rows. Thereby, we obtain a non-zero entry at row 1 column 1. This new non-zero entry can now be used to clear out the first column, i.e. it can play the role of a pivot.

Note (Two types of failures)**:**
Let us discuss two instances, in which elimination with back-substitution fails:

1. No solution at all:

$$\left[\begin{array}{cc|c} 2 & 3 & 2 \\ 4 & 6 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3 & 2 \\ 0 & 0 & 3 \end{array}\right]. \tag{2.24}$$

This matrix has a *single* pivot 2 and admits no solutions.

2. Infinitely many solutions:

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & 6 & 8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 0 & 0 \end{array}\right]. \tag{2.25}$$

   This matrix has a *single* pivot 2. Still, there are infinitely many solutions to this system of linear equations.

Consequence:
A zero in a pivot position implies either no solution or infinitely many solutions.

Note (Gaussian elimination)**:**
We proceed as follows:

1. Get a pivot in the first row and use it to clear out the column below.

2. Repeat for all other rows.

3. If you find n pivots after starting from an $n \times n$-matrix, then the system has a unique solution.

Definition 2.3.2:
Consider $A \in \mathbb{M}(n \times n, \mathbb{R})$ and $\vec{b} \in \mathbb{R}^n$. The system of linear equations $A\vec{x} = \vec{b}$ is called non-singular if A admits n pivots. Otherwise, we call this system singular.

Consequence:
It follows that:

- A non-singular system $A\vec{x} = \vec{b}$ has a unique solution.

- A singular system $A\vec{x} = \vec{b}$ has either no or infinitely many solutions.

2.4 Matrix multiplication and elementary matrices

2.4.1 Elementary matrices

Observe that when we solve the system $A\vec{x} = \vec{b}$ via row operations, the vector $\vec{b}$ keeps getting transformed. We can understand this transformation by matrix multiplication. Recall that for $A \in \mathbb{M}(n \times n, \mathbb{R})$ we can write

$$A \cdot \vec{x} = \begin{bmatrix} - & \text{row 1 of A} & - \\ - & \text{row 2 of A} & - \\ & \vdots & \\ - & \text{row n of A} & - \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} (-\text{ row 1 of A }-) \cdot \vec{x} \\ (-\text{ row 2 of A }-) \cdot \vec{x} \\ \vdots \\ (-\text{ row n of A }-) \cdot \vec{x} \end{bmatrix}. \tag{2.26}$$

Example 2.4.1 (A first elementary matrix)**:**
We wonder what matrix sends $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ to $\begin{bmatrix} b_1 \\ b_2 - 3b_1 \\ b_3 \end{bmatrix}$? That is, find a 3×3 matrix E such that $E \cdot \vec{b}$ is the vector obtained by subtracting 3 times the first row from the second row. By inspection, the following matrix satisfies this property:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{2.27}$$

The key property of this matrix is the -3 in row 2 column 1.

Example 2.4.2 (Another elementary matrix)**:**
What matrix send $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ to $\begin{bmatrix} b_1 \\ b_2 \\ b_3 - 2b_2 \end{bmatrix}$? Convince yourself, that the following matrix achieves this:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{2.28}$$

The key is the -2 in row 3 column 2.

Note:
This pattern generalizes by considering $E_{ij} \in \mathbb{M}(n \times n, \mathbb{R})$ which has 1's along the diagonal, a real number c in row i and column j and 0's everywhere else. Assume that $i > j$. Then it follows that

$$E_{ij} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{2.29}$$

is the vector obtained by adding c-times row j to row i. E_{ij} is called *elementary matrix*.

Consequence:
Let us go back to our matrix equation $A\vec{x} = \vec{b}$. Then, any elimination move changes the RHS from $\vec{b}$ to $E\vec{b}$ for some elementary matrix E. Let us 'multiply' both sides of the equation $A\vec{x} = \vec{b}$ from the left by E. Then we obtain

$$EA\vec{x} = E\vec{b}\,. \tag{2.30}$$

Thus, we would like matrix multiplication to possess the property that EA is the matrix obtained by performing the row operation corresponding to E. We will now discuss a multitude of ways to compute the product of two matrices A and B. Thereby, we will verify that this expectation is indeed satisfied.

2.4.2 Matrix multiplication

We consider $A \in \mathbb{M}(m \times n, \mathbb{R})$ and $B \in \mathbb{M}(n \times p, \mathbb{R})$. Then the matrix product $A \cdot B$ can be defined in a multitude of ways:

- The matrix $C = AB \in \mathbb{M}(m \times p, \mathbb{R})$ has entry c_{ij} in row i and column j given by

$$c_{ij} = (\text{— row i of A —}) \cdot \begin{pmatrix} | \\ \text{column j of B} \\ | \end{pmatrix} . \tag{2.31}$$

For instance, the entry in row 3 column 4 of c is given by

$$c_{34} = a_{31}b_{14} + a_{32}b_{24} + \cdots = \sum_{k=1}^{n} a_{3k}b_{k4} . \tag{2.32}$$

- We restate the above as follows:

$$B = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} , \tag{2.33}$$

$$C = AB = \begin{bmatrix} A \cdot \begin{pmatrix} | \\ b_1 \\ | \end{pmatrix} & A \cdot \begin{pmatrix} | \\ b_2 \\ | \end{pmatrix} & \cdots & A \cdot \begin{pmatrix} | \\ b_p \\ | \end{pmatrix} \end{bmatrix} . \tag{2.34}$$

Thus, the columns of AB are linear combinations of the columns of A.

- Similarly, the rows of $C = AB$ are linear combination of the rows of B.
- The product of a column of A with a row of B is an $m \times p$ matrix. For instance:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix} . \tag{2.35}$$

Therefore, $C = AB$ is the sum of all matrices obtained by multiplying a column of A by a row of B.

Exercise:
Let us again consider the example

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix} . \tag{2.36}$$

Note that the columns are scalar multiples of each other. Likewise, the three rows are scalar multiples of each other. What does this mean geometrically?

2.4.3 Permutation matrices

Remark:
Recall that if we did not have a pivot in row 1 column 1, there was a possibility of swapping row 1 with some other row (cf. section 2.3). We will now discuss matrices which perform such swaps. For example, we could be looking for a matrix P such that

$$P \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_4 \\ b_3 \\ b_2 \end{bmatrix} . \tag{2.37}$$

This matrix P thus swaps rows 2 and 4. It is readily confirmed that the only solution to this demand is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \mathbb{M}(4 \times 4, \mathbb{R}) . \tag{2.38}$$

P does *not* have 1's at the diagonal positions in rows 2 and 4. Rather, we have 1's in row 2 column 4 and row 4 column 2.

Definition 2.4.1 (Permutation matrix)**:**
Matrices of the above form are called *permutation matrices*.

Consequence:
Each row operation in Gaussian elimination can be seen either as multiplication by elementary matrices or permutation matrices. Of course, we apply a bunch of row operations successively. We can collect these operations into a single operation once we understand the composition of these operations, which – in a sense – is the most fundamental property of matrix multiplication. Let us turn to this next.

2.4.4 Properties of matrix multiplication

Note:
Recall, that we discussed various ways to multiply matrices. Each of them has their benefits in terms of the perspective they provide. One aspect, that we want to remember at all times, is that matrices act on vectors and matrices. Therefore, we can think of matrices as *functions*. Most results about matrices are obtained by interpreting matrices as functions.

Corollary 2.4.1:
Matrix multiplications is

- associative: $A(BC) = (AB)C$.

- non-commutative: $AB \neq BA$.

Remark:
The product BA might not even exist, even when AB does. Even if both products exist, they need not be equal.

Exercise:
Find matrices A, B such that:

- AB exists but BA does not exist.
- AB and BA exist but $AB \neq BA$.

Note:
Take $A \in \mathbb{M}(m \times n, \mathbb{R})$ and $\vec{x} \in \mathbb{R}^n$. Then consider the function

$$f \colon \mathbb{R}^n \to \mathbb{R}^m \, , \, \vec{x} \mapsto f(\vec{x}) = A\vec{x} \, . \tag{2.39}$$

Such a function f, encoded by a matrix A, is termed a *linear transformation*. Importantly, we can compose linear transformations:

$$\mathbb{R}^p = \{ \ p \times 1 \text{ vectors } \} \xrightarrow{B} \mathbb{R}^n = \{ \ n \times 1 \text{ vectors } \} \xrightarrow{A} \mathbb{R}^m = \{ \ m \times 1 \text{ vectors } \} \, . \tag{2.40}$$

The resulting linear transformation is given by the matrix $AB \in \mathbb{M}(m \times p, \mathbb{R})$. We will revisit this idea of matrices as functions in section 3.7.

Definition 2.4.2 (Matrix power)**:**
Consider $A \in \mathbb{M}(n \times n, \mathbb{R})$ and $k \in \mathbb{Z}_{>0}$. Then we define:

$$A^k := \prod_{i=1}^{k} A \, . \tag{2.41}$$

We set $A^0 := I = \mathrm{Diag}\,(1, 1, \ldots, 1)$ the identity matrix, which has 1's along the diagonal and 0's everywhere else.

2.4.5 Matrix inverses

Note:
As mentioned before, matrices act on vectors and matrices and give rise to the notion of linear transformations. This raises the natural question if a matrix can *undo* the action of another matrix. This leads to the notion of the inverse of a matrix.

Example 2.4.3:
Let E_{32} be the 3×3 matrix that subtracts 3 times row 2 from row 3, i.e.

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) \, . \tag{2.42}$$

What is the matrix that undoes this operation? Clearly, we want to add 3 times row 2 to row 3. Consequently, the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}), \tag{2.43}$$

undoes what E_{32} did.

Exercise:
Convince yourself that $A \cdot E_{32} = I = E_{32} \cdot A$.

Example 2.4.4:
Let us repeat this exercise for the 4×4 permutation matrix, which swaps rows 2 and 4, i.e.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \mathbb{M}(4 \times 4, \mathbb{R}). \tag{2.44}$$

In this case, you should see right away, that swapping again would get us back to where we started. Thus P undoes what P did before.

Exercise:
Convince yourself that $P^2 = I$.

Definition 2.4.3 (Inverse of matrix)**:**
A matrix A is said to be invertible if there exists a matrix B such that

$$A \cdot B = I = B \cdot A. \tag{2.45}$$

We then denote the matrix B as A^{-1}.

Exercise:
Convince yourself that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}). \tag{2.46}$$

has no inverse.

Example 2.4.5:
As another example, let us consider

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}). \tag{2.47}$$

Does this matrix have an inverse? In orther words, does there exist a 2×2 matrix B such that $A \cdot B = I = B \cdot A$?

Let us look at the column picture. Then we notice that the columns of A are scalar multiples of each other. Therefore, there is no way to obtain either $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination. Consequently, this matrix A does not have an inverse.

Claim 1:
If $A\vec{x} = \vec{0}$ has a solution $\vec{x} \neq \vec{0}$, then A has no inverse.

Proof
Assume that A was invertible but $A\vec{x} = \vec{0}$ had a solution $\vec{x} \neq \vec{0}$. Then $A\vec{x} = \vec{0}$ would imply $A^{-1}A\vec{x} = A^{-1}\vec{0} = \vec{0}$. Hence, since $A^{-1}A = I$, we would find $\vec{x} = 0$ which is a contradiction to our assumption. ■

Example 2.4.6:
Let us again consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{2.48}$$

Then it holds $A \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \vec{0}$. Hence, by the above result, A is not invertible.

Corollary 2.4.2:
A matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ is invertible if and only if it has n pivots.

Exercise:
Prove this corollary.

Example 2.4.7:
We now wish to compute the inverse of a 2×2 matrix. Let us consider

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{2.49}$$

To find its inverse, we are interested in solving the following two equations:

$$A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\,, \tag{2.50}$$

$$A \cdot \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\,. \tag{2.51}$$

In terms of augmented matrices, we are thus looking at

$$\left[\begin{array}{cc|c} 4 & 5 & 1 \\ 3 & 4 & 0 \end{array}\right]\,, \qquad \left[\begin{array}{cc|c} 4 & 5 & 0 \\ 3 & 4 & 1 \end{array}\right]\,. \tag{2.52}$$

Both systems share the same coefficient matrix. Rather than solving them separately, we solve them together and thus consider

$$\left[\begin{array}{cc|cc} 4 & 5 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]\,. \tag{2.53}$$

By Gaussian elimination we find

$$\left[\begin{array}{cc|cc} 4 & 5 & 1 & 0 \\ 0 & \frac{1}{4} & -\frac{3}{4} & 1 \end{array}\right]. \tag{2.54}$$

At this point, we could use back-substitution and compute x, y, z and w. However, let us do something different instead. Namely, let us use the blue entry to clean out the column above by a row operation. This gives

$$\left[\begin{array}{cc|cc} 4 & 0 & 16 & -20 \\ 0 & \frac{1}{4} & -\frac{3}{4} & 1 \end{array}\right]. \tag{2.55}$$

Let us now rescale both pivots to 1. This gives

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 4 \end{array}\right]. \tag{2.56}$$

The columns $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 4 \end{bmatrix}$ of the right-matrix are solution to the systems eq. (2.52). Thus, the inverse of A is given by

$$A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}). \tag{2.57}$$

Definition 2.4.4:
This procedure of cleaning out columns first from left to right, top to bottom followed by right to left, bottom to top is called *Gauss-Jordan elimination.*

Consequence:
To compute A^{-1} start from the augmented matrix $[A|I]$ and use Gauss-Jordan elimination to reach $[I|B]$. Then $A^{-1} = B$.

Claim 2:
It holds $(AB)^{-1} = B^{-1}A^{-1}$ and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. More generally, it holds

$$\left(\prod_{i=1}^{N} A_i\right)^{-1} = \prod_{i=0}^{N-1} A_{N-i}^{-1}. \tag{2.58}$$

Exercise:
Prove this.

Remark:
It may sound strange on a first encounter, but even if an inverse exists, it is perse not granted that this inverse is unique. This happens when we are looking at non-associative operations.

Example 2.4.8:
For a simple example, define a binary operation $*\colon S \times S \to S$ on the set $S = \{1, 2, 3\}$:

- $x * 1 = 1 * x = x$ for all $x \in S$,
- $x * y = 1$ for all $x, y \in \{2, 3\}$.

Then $2 * 3 = 2 * 2 = 1$, which means that both 2 and 3 are inverse elements to 2. Hence, the inverse in $(S, *)$ is not unique.

Note:
Fortunately, matrix multiplication is associative and we have the following statement.

Claim 3:
If $A \in \mathbb{M}(n \times n, \mathbb{R})$ has an inverse, then this inverse is unique.

Proof
Let us assume that $A \in \mathbb{M}(n \times n, \mathbb{R})$ has two inverses, namely $B_1, B_2 \in \mathbb{M}(n \times n, \mathbb{R})$. Then these matrices satisfy

$$AB_1 = AB_2 = I = B_1 A = B_2 A \,, \tag{2.59}$$

However, by the associativity of matrix multiplication, this implies

$$B_1 = B_1 I = B_1(AB_2) = (B_1 A)B_2 = IB_2 = B_2 \,. \tag{2.60}$$

Hence, $B_1 = B_2$ and the inverse is unique. ■

2.4.6 Transposition

Note:
Consider the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}(4 \times 4, \mathbb{R}) \,. \tag{2.61}$$

Its inverse is given by

$$P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{M}(4 \times 4, \mathbb{R}) \,. \tag{2.62}$$

The key thing is, that P^{-1} is obtained by flipping P across its diagonal. That is, we have turned the columns into rows and the rows into columns.

Definition 2.4.5 (Transposition)**:**
The transpose of $A \in \mathbb{M}(m \times n, \mathbb{R})$ is the matrix $A^T \in \mathbb{M}(n \times m, \mathbb{R})$ obtained by changing the rows to columns and vice versa.

Example 2.4.9:
Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \in \mathbb{M}(3 \times 2, \mathbb{R}) \,. \tag{2.63}$$

Then it holds

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R}) \,. \tag{2.64}$$

Corollary 2.4.3:
For any permutation matrix P it holds $P^{-1} = P^T$.

Exercise:

- Prove this corollary.
- Find other matrices with the property $A^{-1} = A^T$.

Note:
For any two matrices A, B (for which $A \cdot B$ exists) it holds $(AB)^T = B^T A^T$.

Definition 2.4.6:
A matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ with $A = A^T$ is termed a *symmetric* matrix.

Exercise:
Given a matrix $B \in \mathbb{M}(m \times n, \mathbb{R})$, verify or falsify that BB^T is symmetric.

2.5 (P)L(D)U-Factorization

2.5.1 L(D)U-Factorization

Note:
In many applications, one needs to solve equations $A\vec{x} = \vec{b}$ where A is fixed but $\vec{b}$ could be varying. It would thus help to "remember" the elimination moves performed during Gaussian elimination, so that one does not have to repeat this whenever $\vec{b}$ changes. This is precisely what LU-factorization accomplishes. Before we turn to the most general case, let us assume that no row exchanges are needed in the Gauss elimination.

Example 2.5.1:
Consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix} \,. \tag{2.65}$$

By Gauss elimination we find $E_{21}A = U$ where

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \tag{2.66}$$

is the elimination matrix and

$$U = \begin{bmatrix} 1 & 4 \\ 0 & -11 \end{bmatrix}. \tag{2.67}$$

Since E_{21} is invertible, we can also write

$$A = E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 0 & -11 \end{bmatrix} \equiv L \cdot U\,. \tag{2.68}$$

This is the lower–upper (LU) factorization of A. Namely, we have represented A as a product of a lower triangular matrix L and an upper triangular matrix U.

Note:
The analogue of this analysis of a 3×3 matrix A is the existence of elementary matrices such that

$$E_{32}E_{31}E_{21}A = U \quad \Leftrightarrow \quad A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U\,. \tag{2.69}$$

Remark:
In staying with a 3×3 matrix A, we can wonder if $E_{32}E_{31}E_{21}$ or $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ does a better job remembering the elimination. In order to answer this question, let us try with an example. For convenience, let us assume that the $(3,1)$-entry of A is 0 and take

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}. \tag{2.70}$$

Note that $E_{32} \cdot E_{21}$ first subtracts 2 times row 1 from row 2. Subsequently, it subtracts 4 times row 2 from row 3. The net result is therefore given by the matrix

$$E_{32} \cdot E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 8 & -4 & 1 \end{bmatrix}. \tag{2.71}$$

The inverse is given by

$$(E_{32} \cdot E_{21})^{-1} = E_{21}^{-1} \cdot E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}. \tag{2.72}$$

Note that E_{21}^{-1} does not alter row 3. In this sense, $E_{21}^{-1} \cdot E_{32}^{-1}$ does a better job at remembering the elimination process.

Claim 4:

- Inverses of triangular matrices are triangular.
- Products of triangular matrices are triangular.

Exercise:
Prove this statement.

Consequence:
When the elimination process does not involve row exchanges, we can write

$$A = LU\,, \tag{2.73}$$

where U is an upper triangular matrix with the pivots of A along the diagonal and L a lower triangular matrix L with 1's along the diagonal and *multipliers below the diagonal.*

Note:
In returning to our opening problem, suppose that we want to solve $A\vec{x} = \vec{b}$, where A is fixed by $\vec{b}$ varies. In this case, write $A = LU$, so that this problem is equivalent to $L\,(U\vec{x}) = \vec{b}$. Next, set $\vec{c} = U\vec{x}$. Thereby, we are left to solve two triangular systems:

$$L \cdot \vec{c} = \vec{b}\,, \qquad U \cdot \vec{x} = \vec{c}\,. \tag{2.74}$$

This is much more efficient for varying $\vec{b}$ than solving $A\vec{x} = \vec{b}$ directly.

Remark:
On a homework assignment, you will quantify the speed of Gaussian elimination. You should find that for an $n \times n$ matrix A, this process requires $\mathcal{O}(n^3)$ operations. This is why, in the computer sciences, Gauss elimination is referred to as an algorithm of $\mathcal{O}(n^3)$.

Example 2.5.2:
Let us now consider the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 6 & 7 \\ 2 & -6 & 9 \end{bmatrix}. \tag{2.75}$$

Convince yourself, that we obtain an LU decomposition of A by application of

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \qquad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.76}$$

and that this LU-decomposition is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -2 \\ 0 & -6 & 15 \\ 0 & 0 & -17 \end{bmatrix}. \tag{2.77}$$

This factorization can also be written as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -17 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.78}$$

Definition 2.5.1:
Such a factorization is termed an LDU-factorization and the D refers to the diagonal middle matrix.

2.5.2 PL(D)U-Factorization

Remark:
Recall that in claiming $A = LU$ we assumed that there are no row exchanges needed in the Gauss elimination. We are now ready to generalize to arbitary matrices.

Corollary 2.5.1:
A square matrix A can be factored as

$$PA = LU\,, \tag{2.79}$$

where P is some permutation matrix (cf. section 2.4.3) and L, U are as above. This yields

$$A = P^T LU\,. \tag{2.80}$$

Recall that $P^{-1} = P^T$ is again a permutation matrix. One terms such a factorization of A a PLU-factorization.

Remark:
Recall that any matrix obtained by permuting the rows of the identity matrix is a *permutation matrix*. Consequently, permutation matrices perform permutations of the rows of a given matrix. For example, the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\,. \tag{2.81}$$

performs the permutation

$$(\text{row } 1) \to (\text{row } 4) \to (\text{row } 3) \to (\text{row } 2) \to (\text{row } 1)\,. \tag{2.82}$$

3 Vector Spaces and Linear Subspaces

3.1 Vector Spaces

Note:
When we say *vector space*, we use the term *space* to emphasize that we are studying a collection of vectors. But not just any collection. There are constraints imposed on this collection.

Example 3.1.1:
An example of a vector space is

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} , x, y \in \mathbb{R} \right\} . \tag{3.1}$$

In other words, the space $\mathbb{R}^2$ consists of all 2-dimensional vectors. Also, recall that we perform two operations with vectors:

- scaling by real numbers,
- component-wise addition.

If we scale $\vec{v} \in \mathbb{R}^2$ by a scalar $c \in \mathbb{R}$, then $c \cdot \vec{v} \in \mathbb{R}^2$. Likewise, if $\vec{v}, \vec{w} \in \mathbb{R}^2$, then $\vec{v} + \vec{w} \in \mathbb{R}^2$. More general, any linear combination of vectors in $\mathbb{R}^2$ is a vector in $\mathbb{R}^2$.

Note:
Clearly, there is nothing special about $\mathbb{R}^2$. We could have said the same thing about $\mathbb{R}^3$, i.e. the collection of vectors with 3 (real) components. This generalizes as follows.

Remark:
The following is the abstract definition of a vector space over a field F. I am presenting it here, because I believe that this level of abstraction emphasizes the important structures of a vector space in the best way. We will exemplify all of this in vector spaces over the real number $\mathbb{R}$, and much later in the course over $F = \mathbb{C}$. Therefore, in the following definition(s), you may think of F as $\mathbb{R}$, $\mathbb{C}$, or for computer implementations as $\mathbb{Q}$ (or the field extension $\mathbb{Q} + i\mathbb{Q}$).

Definition 3.1.1 (Vector space)**:**
A vector space over a field F is a triple $(V, +, \cdot)$ of a set V and operations

$$+\colon V \times V \to V\,,\ (\vec{u}, \vec{v}) \mapsto \vec{u} + \vec{v}\,, \tag{3.2}$$
$$\cdot\colon F \times V \to V\,,\ (c, v) \mapsto c \cdot \vec{v}\,. \tag{3.3}$$

which satisfy the following properties:

- Associativity of addition:
  For all $\vec{u}, \vec{v}, \vec{w} \in V$ it holds $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
- Commutativity of addition:
  For all $\vec{u}, \vec{v} \in V$ it holds $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
- Existence of neutral element of addition:
  There exists $\vec{n} \in V$ such that for all $\vec{u} \in V$ it holds $\vec{u} + \vec{n} = \vec{u}$.
- Existence of an inverse element under addition:
  For every $\vec{u} \in V$ there exists $\tilde{\vec{u}}$ such that $\vec{u} + \tilde{\vec{u}} = \vec{n}$.
- Compatibility of scalar multiplication and vector addition:
  For all $\vec{u} \in V$ and all $c_1, c_2 \in F$ it holds $c_1 \cdot (c_2 \cdot \vec{u}) = (c_1 \cdot c_2) \cdot \vec{u}$
- Neutral element of scalar multiplication:
  There exists $i \in F$ such that for all $\vec{u} \in V$ it holds $i \cdot \vec{u} = \vec{u}$.
- Distributivity laws:
  For all $c_1, c_2 \in F$ and all $\vec{u}, \vec{v} \in V$ the

$$c_1 \cdot (\vec{u} + \vec{v}) = c_1 \cdot \vec{u} + c_1 \cdot \vec{v}\,, \qquad (c_1 + c_2) \cdot \vec{u} = c_1 \cdot \vec{u} + c_2 \cdot \vec{u}\,. \tag{3.4}$$

We term $+\colon V \times V \to V$ the *vector addition* and $\cdot\colon F \times V \to V$ the *scalar multiplication*. Moreover, we term elements of V vectors and element of F scalars.

Note:
The vector space $\mathbb{R}^n$ as vector space over the field $F = \mathbb{R}$ is the triple $(\mathbb{R}^n, +, \cdot)$ with

$$+\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\,, \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) \mapsto \begin{bmatrix} u_1 +_{\mathbb{R}} v_1 \\ u_2 +_{\mathbb{R}} v_2 \\ \vdots \\ u_n +_{\mathbb{R}} v_n \end{bmatrix}, \tag{3.5}$$

$$\cdot\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\,, \left(c, \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) \mapsto \begin{bmatrix} c \cdot_{\mathbb{R}} u_1 \\ c \cdot_{\mathbb{R}} u_2 \\ \vdots \\ c \cdot_{\mathbb{R}} u_n \end{bmatrix}, \tag{3.6}$$

where $+_{\mathbb{R}}$, $\cdot_{\mathbb{R}}$ denotes addition and multiplication of real numbers, respectively.

Exercise:
Verify that $(\mathbb{R}^n, +, \cdot)$ satisfies all properties in the definition of a vector space.

Remark:
For notational simplicity, we denote $(\mathbb{R}^n, +, \cdot)$ simply as $\mathbb{R}^n$ for the rest of this course.

3.2 Linear Subspaces

Example 3.2.1:
The set $(\{0\}, +, \cdot)$ is a vector space over $\mathbb{R}$ via the following operations

$$+\colon \{0\} \times \{0\} \to \{0\}\,,\ (0,0) \mapsto 0\,, \qquad \cdot\colon \mathbb{R} \times \{0\} \to \{0\}\,,\ (c,0) \mapsto 0\,. \tag{3.7}$$

We call this the trivial vector space.

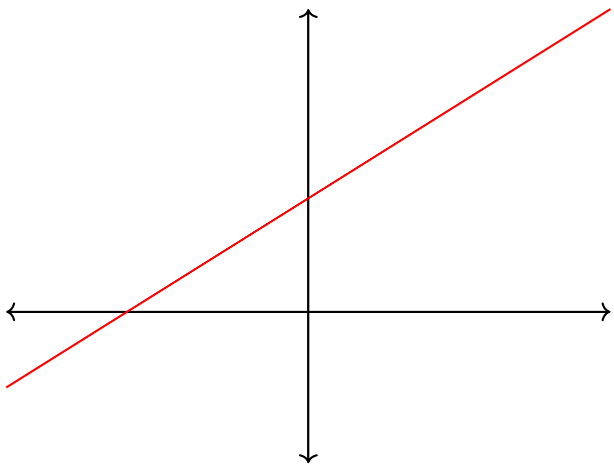
Note:
$\{0\} \subseteq \mathbb{R}^n$. This indicates, that we may want to think of the trivial vector space as a linear subspace of $\mathbb{R}^n$. More generally, we can ask if a vector space V over a field F contains linear subspaces. To this end, let us first define the notation of a linear subspace.

Definition 3.2.1 (Linear subspace)**:**
Be $(V, +, \cdot)$ a vector space over a field F. A linear subspace W of V is a subset $W \subseteq V$ such that $(W, +|_W, \cdot|_W)$ is a vector space over F.

Example 3.2.2:
For any vector space $(V, +, \cdot)$ over a field F, $\{0\}$ and V are linear subspaces.

Note:
At this point we may wonder how we can visualize a subspace W of a vector space V. To this end, we recall that for any two vectors $\vec{u}, \vec{v} \in W$ it must holds that $c_1\vec{u} + c_2\vec{v} \in W$. To fully appreciate this observation, let us exemplify its meaning by looking at examples in $V = \mathbb{R}^2$.

Example 3.2.3:
Consider the collection of points along the following red line:

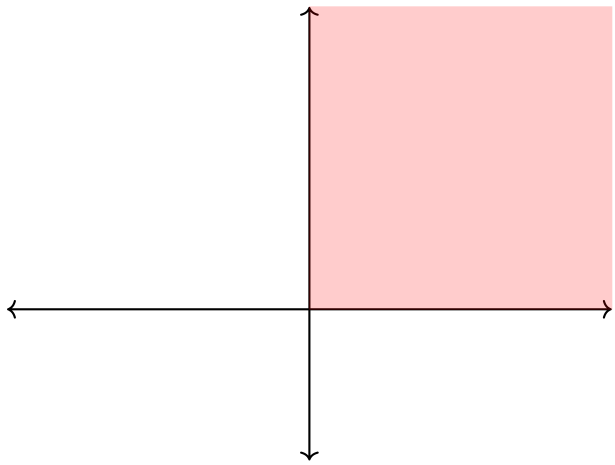


(3.8)

This clearly is a subset of $\mathbb{R}^2$. But is it a linear subspace of $\mathbb{R}^2$? The answer is *no*! Think about what happens if we scale a vector by 0. Hence, any linear subspace of $\mathbb{R}^2$ must contain $\vec{0}$.

Note:
More abstractly, $\vec{0} \in \mathbb{R}^2$ is the neutral element of the vector addition, and this remains true in any linear subspace $W \subseteq \mathbb{R}^2$.

Example 3.2.4:
As another example, let us consider the points W in the first quadrant:



(3.9)

Is W a linear subspace of $\mathbb{R}^2$? Clearly, W contains $\vec{0}$. Also, if $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$. However, we run into the following problem. If $\vec{0} \neq \vec{v} \in W$ then $-\vec{v} \notin W$. So this is not a linear subspace neither.

Example 3.2.5:
Consider a line L in direction $\vec{a} \in \mathbb{R}^2 \setminus \vec{0}$ through the origin of $\mathbb{R}^2$:

$$L(\vec{a}) = \{\lambda \cdot \vec{a} \,|\, \lambda \in \mathbb{R}\} \,. \tag{3.10}$$

Clearly, if $\vec{x} \in L(\vec{a})$, then is $\lambda \cdot \vec{x}$. Also, if $\vec{x}, \vec{y} \in L(\vec{a})$, then

$$\vec{x} = \lambda_1 \cdot \vec{a} \,, \qquad \vec{y} = \lambda_2 \cdot \vec{a} \,. \tag{3.11}$$

Hence $\vec{x} + \vec{y} \in L(\vec{a})$. We conclude from this, that $L(\vec{a})$ is a linear subspace of $\mathbb{R}^2$. Similarly, in $\mathbb{R}^n$, any line through the origin is a linear subspace. These observations lead to the following corollary.

Corollary 3.2.1:
Let $(V, +, \cdot)$ be a vector space over a field F. $W \subseteq V$ is a linear subspace of V if and only if for any two $c_1, c_2 \in F$ and any two $\vec{u}, \vec{v} \in W$ it holds $c_1\vec{u} + c_2\vec{v} \in W$.

Proof
Recall that we consider W equipped with the operations $+|_W$ and $\cdot|_W$. Perse, the sum of two vectors $\vec{u}, \vec{v} \in W$ could be a vector in $V \setminus W$. The same is perse true for $c \cdot \vec{u}$ for

a suitable scalar $c \in F$. However, this is not true if and only if for any two $c_1, c_2 \in F$ and any two $\vec{u}, \vec{v} \in W$ it holds $c_1\vec{u} + c_2\vec{v} \in W$. Put differently, the operations $+|_W$, $\cdot|_W$ are closed in W if and only if for any two $c_1, c_2 \in F$ and any two $\vec{u}, \vec{v} \in W$ it holds $c_1\vec{u} + c_2\vec{v} \in W$. The other vector space axioms now follow immediately from those for $+$ and $\cdot$. This completes the argument. ■

Consequence:
We can now list all linear subspaces of $\mathbb{R}^2$:

- the trivial linear subspace $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$,
- all lines through the origin, i.e. $\{L(\vec{a}) \,|\, \vec{a} \in \mathbb{R}^2\}$,
- $\mathbb{R}^2$.

Note that in the second bullet point we can focus on those $\vec{a}$ with arrow tip on the circle of radius 1. Hence, there are two more linear subspaces of $\mathbb{R}^2$ than there are points on the circle of radius 1. While this seems a lot, it is rather little in comparison to $\mathbb{R}^2$ itself. Hence, being a linear subspace is a *rather non-trivial constraint.*

Exercise:
Find all linear subspaces of $\mathbb{R}^3$.

Example 3.2.6:
Here is a somewhat 'exotic' example of a vector space. Let $M = \mathbb{M}(2 \times 2, \mathbb{R})$ be the set of all 2×2 matrices with real entries. Consider the following operations:

$$+\colon M \times M \to M\,,\; (A, B) \mapsto \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}\,, \tag{3.12}$$

$$\cdot\colon \mathbb{R} \times M \to M\,,\; (c, A) \mapsto \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} \\ c \cdot a_{21} & c \cdot a_{22} \end{bmatrix}\,. \tag{3.13}$$

You will prove in the homeworks that $(M, +, \cdot)$ is a vector space over $\mathbb{R}$.

Exercise:
Let Pol_n denote the set of all polynomials in the variable x, with real coefficients and degree *at most n*. Find operations $+_P$, $\cdot_P$ such that $(\mathrm{Pol}_n, +_P, \cdot_P)$ is a vector space over $\mathbb{R}$.

Note:
Both Pol_n and $\mathbb{R}^m$ are vector spaces over $\mathbb{R}$. In fact, there is a connection between them.

3.3 Vector Space Homomorphisms

Remark:
Relations among vector spaces are encoded by so-called vector space homomorphisms, which is greek for *structure preserving maps.*

Definition 3.3.1:
Consider two vector spaces $(A, +_A, \cdot_A)$ and $(B, +_B, \cdot_B)$ over a field F. A vector space homomorphism from A to B is a map $\varphi\colon A \to B$ which satisfies for all $c \in F$ and $x, y \in A$ that

$$\varphi\left(c \cdot_A (x +_A y)\right) = c \cdot_B \varphi(x) +_B c \cdot_B \varphi(y)\,. \tag{3.14}$$

Remark:
Recall that a map of sets $f\colon S \to T$ is defined to be

- injective if for all $x, y \in S$ the implication $f(x) = f(y) \Rightarrow x = y$ holds true,
- surjective if for all $t \in T$ there exists (at least one) $s \in S$ with $f(s) = t$,
- bijective if it is both injective and surjective.

Example 3.3.1:
Here are a few example of vector space homomorphisms and their properties:

- The canonical embedding

  $$\iota\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}\,,\; \vec{x} \mapsto \begin{bmatrix} \vec{x} \\ 0 \end{bmatrix}\,, \tag{3.15}$$

  is a vector space homomorphism. It is injective but not surjective.

- The canonical projection

  $$\iota\colon \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^n\,,\; \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n\,, \tag{3.16}$$

  is a vector space homomorphism. It is not injective but surjective.

Definition 3.3.2:
A vector space homomorphism $\varphi\colon A \to B$ which at the same time is a bijection of the underlying sets is a vector space isomorphism. We write $A \cong B$ and then consider A and B as essentially the same vector spaces.

Example 3.3.2:
The identity $\mathbb{R}^n \xrightarrow{id} \mathbb{R}^n$ is a vector space homomorphism.

Claim 5:
Be $\vec{a} \in \mathbb{R}^2 \setminus \vec{0}$, then

$$L(\vec{a}) = \{\lambda \cdot \vec{a} \,|\, \lambda \in \mathbb{R}\} \subset \mathbb{R}^2\,, \tag{3.17}$$

understood as linear subspace of $\mathbb{R}^2$, is isomorphic to the vector space $\mathbb{R}$.

Proof
Let us define a map of sets:

$$\varphi\colon \mathbb{R} \to L(\vec{a})\,,\ \lambda \in \mathbb{R} \mapsto \lambda \cdot \vec{a}\,. \tag{3.18}$$

We claim that this is a vector space isomorphism. First, we notice that φ is surjective. To see that φ is also injective, consider $\lambda, \mu \in \mathbb{R}$ and assume that $\varphi(\lambda) = \varphi(\mu)$. This is equivalent to

$$\lambda \cdot \vec{a} = \mu \cdot \vec{a} \qquad \Leftrightarrow \qquad (\lambda - \mu) \cdot \vec{a} = \vec{0}\,. \tag{3.19}$$

By assumption $\vec{a} \neq \vec{0}$, so this is equivalent to $\lambda = \mu$. This shows that indeed φ is injective. Consequently, φ is a bijection of sets. It remains to show that φ is a vector space homomorphism also. This follows since:

$$\varphi\,(\lambda + \mu) = (\lambda + \mu) \cdot \vec{a} = \lambda \cdot \vec{a} + \mu \cdot \vec{a} = \varphi(\lambda) + \varphi(\mu)\,. \tag{3.20}$$

Consequently, φ is a vector space homomorphism from $\mathbb{R}$ to $L(\vec{a})$ and $\mathbb{R} \cong L(\vec{a})$, which completes our argument. ■

Exercise:
Show that $\mathrm{Pol}_n \cong \mathbb{R}^m$ for a suitable $m \in \mathbb{Z}_{\geq 0}$.

3.4 Column space and nullspace

3.4.1 The column space

Example 3.4.1:
Consider the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{b}\,. \tag{3.21}$$

We may ask two questions:

1. Given $\vec{b}$, does this equation have a solution?
2. What are all possible $\vec{b}$ for which the system has a solution?

We can recast these questions in the language of linear combinations:

1. Given $\vec{b}$, is there a linear combination of the columns of A that equals $\vec{b}$?
2. Can we find all the vectors that are linear combinations of the columns of A?

Therefore, we want to understand the set of linear combinations of the columns of A.

Definition 3.4.1:
For $A \in \mathbb{M}(m \times n, \mathbb{R})$, we denote the set of linear combinations of the columns of A as the *column space* $C(A)$. If we label the column of A as $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in \mathbb{R}^m$, then this is equivalent to saying that $C(A)$ is the span of $\vec{a}_1, \ldots, \vec{a}_n$. Explicitly:

$$C(A) := \left\{ \sum_{i=1}^{n} \lambda_i \vec{a}_i \,|\, \lambda_i \in \mathbb{R} \right\} =: \mathrm{Span}_{\mathbb{R}} \left(\vec{a}_1, \ldots, \vec{a}_n \right) \subseteq \mathbb{R}^m . \tag{3.22}$$

Corollary 3.4.1:
$C(A)$ is a real vector space.

Exercise:
Prove this statement.

Example 3.4.2:
Let us return to the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix} . \tag{3.23}$$

$C(A)$ is then a linear subspace of $\mathbb{R}^4$. $C(A)$ is more than a line, since the first two columns of A are not parallel to each other as vectors. Whether $C(A)$ is more than a plane may not be immediate at this point. We will return to this question momentarily. What should be clear at this point is that $C(A)$ is not $\mathbb{R}^4$.

Exercise:
Find $\vec{v} \in \mathbb{R}^4$ with $\vec{v} \notin C(A)$.

3.4.2 The nullspace

Note:
There is another interesting vector space attached to matrices that we have to discuss. Namely, given an $m \times n$ matrix A, we can consider the set of solution to $A\vec{x} = \vec{0}$.

Definition 3.4.2 (Nullspace)**:**
The *nullspace* of a matrix A, denoted by $N(A)$, is the set of solution to $A\vec{x} = \vec{0}$.

Corollary 3.4.2:
$N(A)$ is linear subspace of $\mathbb{R}^n$.

Proof
Any two $\vec{v}, \vec{w} \in N(A)$ satisfy $A\vec{v} = A\vec{w} = \vec{0}$. Consequently, for any $c, d \in \mathbb{R}$ we have

$$A \left(c\vec{v} + d\vec{w} \right) = Ac\vec{v} + Ad\vec{w} = cA\vec{v} + dA\vec{w} = \vec{0} . \tag{3.24}$$

The claim now follows from corollary 3.2.1. ■

Example 3.4.3:
Let us consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix} . \tag{3.25}$$

Then any scalar multiple of $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is contained in $N(A)$. In particular,

$$A \cdot \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} , \quad \text{for } \vec{v} \in \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} , r \in \mathbb{R} \right\} . \tag{3.26}$$

Consequence:
The space $N(A)$ is crucial to find *all* solutions to a linear system. Before we can discuss this important application, we first have to understand the nullspace better. In particular, we have to be able to compute it.

Example 3.4.4:
Let us consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} . \tag{3.27}$$

We are interested in $N(A)$, i.e. the soutions to $A\vec{x} = \vec{0}$. We will essentially execute the same elimination procedure as before. However, since the RHS is $\vec{0}$, we will not carry it along in our computation:

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: U . \tag{3.28}$$

At this point, there are no more eliminations to be performed. We say that the final matrix U is in *(row) echelon form.*

Example 3.4.5:
The following matrices are *not* in (row) echelon form:

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 0 & 0 \\ 3 & 9 & 0 & 0 \end{bmatrix} , \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} . \tag{3.29}$$

But, the following matrices are in (row) echelon form:

$$D = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{bmatrix} , \quad F = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} . \tag{3.30}$$

Definition 3.4.3 (Row rank of a matrix)**:**
Any matrix A can, by use of elementary row operations, be turned into a matrix U which is in echelon form. We call the number of pivots of U the row rank $\mathrm{rk}_R(A)$ of A.

Example 3.4.6:
Since

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: U\,. \tag{3.31}$$

and U has 2 pivots, we conclude that $\mathrm{rk}(A) = 2$.

Note:
Again consider

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: U\,. \tag{3.32}$$

We make two observations:

- The bottom row of 0s in U tells us, that the third row of A is a linear combination of the first and second row of A.
- The columns that do not contain pivots can be expressed in terms of the columns that come before them on the left.

Definition 3.4.4 (Pivot and free columns)**:**
Let U be a matrix in echelon form. Then we distinguish two types of columns:

- Columns which have a pivot are called *pivot columns.*
- All other columns are called *free columns.*

Example 3.4.7 (Continuation)**:**
For the matrix

$$U = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.33}$$

the equations corresponding to $U\vec{x} = \vec{0}$ are given by

$$x_1 + 3x_2 + 3x_3 + 3x_4 = 0\,, \tag{3.34}$$
$$3x_3 + 6x_4 = 0\,. \tag{3.35}$$

x_2, x_4 are refered to as *free* variables.

Comment:
Let us comment on the terminology of *free* variables. Namely, if we randomly assign values to x_2 and x_4, then the values of x_1 and x_3 are uniquely determined. Therefore, we term x_1, x_3 the *pivot* variables.

Example 3.4.8 (Continuation)**:**
For example, let us assign $x_2 = 1$ and $x_4 = 0$. Then $x_3 = 0$ and $x_1 = -3$. Consequently,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)\,. \tag{3.36}$$

In fact, any scalar multiple of this vector is contained in the nullspace. Similarly, we can also try $x_2 = 0$ and $x_4 = 1$. Then, $x_3 = -2$, $x_1 = 3$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in N(A)\,. \tag{3.37}$$

These two *special solutions* enable us to describe all vectors in $N(A)$. Namely, they are all linear combinations of these special solutions! Therefore

$$N(A) = \left\{ c \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \; c, d \in \mathbb{R} \right\}. \tag{3.38}$$

Note:
For $A \in \mathbb{M}(m \times n, \mathbb{R})$ with rank r, there are r pivot variables and $n - r$ free variables.

Remark:
We can reduce the row echelon form further. For example for

$$U = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.39}$$

we can clean the entries above the pivots, just as we did for Gauss-Jordan elimination:

$$U = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.40}$$

This final matrix is called the *reduced (row) echelon form* (RREF) of A. All its pivots are equal to 1 and there are 0s above and below the pivots.

Exercise:
Can you see the special solutions to

$$A\vec{x} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix} \cdot \vec{x} = \vec{0}\,, \tag{3.41}$$

in the RREF of the matrix A?

Example 3.4.9:
Let us now consider the transposed matrix A^T. For this matrix we have

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 3 & 9 & 12 \\ 3 & 12 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.42}$$

This shows $\mathrm{rk}(A^T) = 2$ and

$$N(A^T) = \left\{ c \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \, c \in \mathbb{R} \right\}. \tag{3.43}$$

Exercise:
Find the special solution to $A^T\vec{x} = \vec{0}$ from the RREF of A^T.

Definition 3.4.5 (Column rank of a matrix)**:**
Given a matrix A, then we can bring A by use of elementary column operations into a matrix U which is in column echelon form. We call the number of pivots of U the column rank $\mathrm{rk}_C(A)$ of A.

Claim 6:
For any matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$ it holds $\mathrm{rk}_R(A) = \mathrm{rk}_C(A)$.

Proof
Neither the row nor the column rank are altered by elementary row nor column operations. By use of such elementary row and column operations, we can bring A into the form U of an identity matrix, possibly bordered by rows and columns of zero. It follows that row and column ranks coincide with the number of non-zero entries of U. ■

Remark:
The rank of a matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$ tells us how many "independent" solutions $N(A)$ contains, namely exactly $n - \mathrm{rk}(A)$. We will make precise what we mean by "independent" when we study the precise formulation of this statement, the so-called *rank-nullity theorem*.

3.4.3 All solutions to a linear system

Note:
Let us now return to the question on how to find *all* solutions to $A\vec{x} = \vec{b}$. We want an approach that allows us to tell if there are no solutions, a unique solution or infinitely many solutions. As anticipated before, we will find that the nullspace is crucial in this study.

Example 3.4.10:
We consider the linear system

$$x_1 + 3x_2 + 3x_3 + 3x_4 = b_1\,, \tag{3.44}$$
$$2x_1 + 6x_2 + 9x_3 + 12x_4 = b_2\,, \tag{3.45}$$
$$3x_1 + 9x_2 + 12x_3 + 15x_4 = b_3\,. \tag{3.46}$$

Note that row 1 + row 2 = row 3 for the left hand sides. This already tells us something about the RHS, if we are to solve this system. Namely, for instance, if $b_1 = 1$ and $b_2 = 2$, then b_3 must equal 3 if this system is to have at least one solution.

Of course, we would like elimination to discover this fact for us. To this end, we record this linear system in the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & b_1 \\ 2 & 6 & 9 & 12 & b_2 \\ 3 & 9 & 12 & 15 & b_3 \end{array}\right]. \tag{3.47}$$

Upon elimination we find

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & b_1 \\ 2 & 6 & 9 & 12 & b_2 \\ 3 & 9 & 12 & 15 & b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & b_1 \\ 0 & 0 & 3 & 6 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right]. \tag{3.48}$$

Indeed, this shows us, that this system has no solution unless $b_3 - b_1 - b_2 = 0$.

Exercise:
Find the linear subspace of $\mathbb{R}^3$ which is generated by the 4 column vectors of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 9 & 12 \\ 3 & 9 & 12 & 15 \end{bmatrix}. \tag{3.49}$$

Consequence (Solvability conditions on $\vec{b}$)**:**
Consider the linear system $A\vec{x} = \vec{b}$:

- This system is solvable if and only if $\vec{b} \in C(A)$.
- If a linear combination of rows of A gives a zero row, and the same linear combination of the entries of $\vec{b}$ gives a non-zero value, then the system is not solvable.

Example 3.4.11 (Continuation)**:**
Let us continue to study the linear system eq. (3.47). However, let us proceed by using a specific vector $\vec{b}$, for which the system has a solution. We take $b_1 = 1$, $b_2 = 4$ and $b_3 = 5$. Then the system is represented by

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 3 & 1 \\ 0 & 0 & 3 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] . \tag{3.50}$$

To find all solutions to this system, we execute three steps:

1. Set all free variables to zero and find *one* solution for the resulting system of the pivot variables. For the above system, we set $x_2 = x_4 = 0$ and find

$$x_1 + 3x_3 = 1 \,, \tag{3.51}$$
$$3x_3 = 2 \,. \tag{3.52}$$

   The unique solution to this system is given by $x_3 = \frac{2}{3}$ and $x_1 = -1$. We conclude that a particular solution to the linear system eq. (3.47) is given by

$$\vec{x}_{\text{particular}} = \begin{bmatrix} -1 \\ 0 \\ 2/3 \\ 0 \end{bmatrix} . \tag{3.53}$$

2. Find all vectors in the nullspace of A, i.e. compute $N(A)$.

3. Every solution to eq. (3.47) is then given by

$$\vec{x} = \vec{x}_{\text{particular}} + \vec{x}_{\text{null}} \,, \tag{3.54}$$

   where $\vec{x}_{\text{particular}}$ is the solution in eq. (3.53) and $\vec{x}_{\text{null}}$ is **any** vector in $N(A)$.

Exercise:
Why does that work? Hint: $A(\vec{x}_p + \vec{x}_n) = A\vec{x}_p + A\vec{x}_n$.

Example 3.4.12 (Continuation II)**:**
Every solution to eq. (3.47) is given by

$$\vec{x} \in \left\{ \begin{bmatrix} -1 \\ 0 \\ 2/3 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} . \tag{3.55}$$

Geometrically, we obtain a point and a linear subspace of $\mathbb{R}^4$ isomorphic to $\mathbb{R}^2$. This linear subspace is the nullspace $N(A)$. Thus, the complete set of solutions is an **affine** *plane in* $\mathbb{R}^4$.

Note:
The set of all solutions in eq. (3.55) is *not* a linear subspace of $\mathbb{R}^4$. It is a **translation** of the subspace $N(A) \subseteq \mathbb{R}^4$, i.e. a 'shifted' version of this linear subspace.

Corollary:
The rank r of any $A \in \mathbb{M}(m \times n, \mathbb{R})$ satisfies the inequalities

$$r \leq m\,, \qquad r \leq n\,. \tag{3.56}$$

Exercise:
Prove this statement.

Corollary (Matrices with full column rank)**:**
Consider $A \in \mathbb{M}\,(m \times n, \mathbb{R})$ with full column rank, that is $r = n$. Then the following holds true:

- There are no free variables and $N(A) = \{\vec{0}\}$.
- If there is a solution to $A\vec{x} = \vec{b}$, then this solution is unique.

$\Rightarrow$ $A\vec{x} = \vec{b}$ either has no or a unique solution.

Example 3.4.13:
Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}. \tag{3.57}$$

What is the rank of A? What is the RREF for this matrix? Convince yourself, that the RREF is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.58}$$

Hence, indeed $\mathrm{rk}(A) = 2$. It should also be obvious that this system is not always solvable. However, if it is, then this solution is unique.

Corollary (Matrices with full row rank)**:**
Consider $A \in \mathbb{M}\,(m \times n, \mathbb{R})$ with full row rank, that is $r = m$. Then the following holds:

- There is a pivot in every row.
- $A\vec{x} = \vec{b}$ always has (at least) one solution.

$\Rightarrow$ $A\vec{x} = \vec{b}$ either has one or infinitely many solutions.

Example 3.4.14:
Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 5 & 4 \end{bmatrix} . \tag{3.59}$$

The corresponding RREF is

$$\begin{bmatrix} 1 & 0 & -1 & -5 \\ 0 & 1 & 4 & 3 \end{bmatrix} . \tag{3.60}$$

Clearly, $A\vec{x} = \vec{b}$ is always solvable.

Corollary (Matrices with full rank)**:**
Consider $A \in \mathbb{M}\,(m \times n, \mathbb{R})$ with full rank, that is $r = m = n$. Then the following holds true:

- A is a square matrix,
- there is a pivot in every row and column,
- the RREF is equal to the identity matrix,

$\Rightarrow$ $A\vec{x} = \vec{b}$ always has a *unique* solution.

3.5 Linear (in)dependence, spans, basis and the dimension of vector spaces

3.5.1 Linear (in)dependence

Note:
Suppose $A \in \mathbb{M}(m \times n, \mathbb{R})$ with $m < n$. Then there are non-zero solutions to $A\vec{x} = \vec{0}$. Note that in this case we have more unknowns than equations. Thus, there will be free variables! This fact will become handy, momentarily.

Definition 3.5.1 (Linear independence)**:**
Vectors $\vec{v}_1, \ldots, \vec{v}_n$ are *linearly independent* if no linear combination gives $\vec{0}$, except the zero combination. That is, the vectors are linearly independent if

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}\,, \tag{3.61}$$

implies $c_1 = c_2 = \cdots = c_n = 0$.

Definition 3.5.2 (Linear dependence)**:**
Vectors $\vec{v}_1, \ldots, \vec{v}_n$ which are not *linearly independent* are said to be *linearly dependent*.

Example 3.5.1:
Consider the vectors $\vec{v}_1$ and $\vec{v}_2$ such that $\vec{v}_2 = 2\vec{v}_1$. Let us investigate if these vectors are linearly independent. Thus, consider $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. We note that this is solved by $c_1 = 2$ and $c_2 = -1$. Consequently, $\vec{v}_1$, $\vec{v}_2$ are *not* linearly independent.

Example 3.5.2:
How about $\vec{v}$ and $\vec{0}$, are they linearly independent? No, they are not. Namely, the equation $c_1\vec{v} + c_2\vec{0} = \vec{0}$ can be solved by $c_1 = 0$ and $c_2 = 1$.

Corollary:
Any finite set of vectors which contains $\vec{0}$ is linearly dependent.

Exercise:
Convince yourself that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.62}$$

are linearly independent. Likewise, show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.63}$$

are linearly independent.

Corollary:
Consider $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$ and the matrix $A = \begin{bmatrix} | & \cdots & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & \cdots & | \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R})$. Then:

$$\vec{v}_1, \ldots, \vec{v}_n \text{ linearly independent} \quad \Leftrightarrow \quad N(A) = \left\{\vec{0}\right\}. \tag{3.64}$$

Exercise:
Prove this corollary.

Note:
In particular, $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent iff $N(A) \neq \left\{\vec{0}\right\}$, i.e. there exists a non-zero $\vec{c}$ with $A\vec{c} = \vec{0}$.

Corollary:
Consider $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$ and the matrix $A = \begin{bmatrix} | & \cdots & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & \cdots & | \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R})$. Then the following holds true:

- $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent iff A has full column rank.
- $\vec{v}_1, \ldots, \vec{v}_n$ are linearly dependent iff A does not have full column rank.

3.5.2 Spans, Basis and Dimension

Definition 3.5.3 (Span)**:**
Consider a vector space $(V, +, \cdot)$ over a field $\mathbb{F}$ and vectors $\vec{v}_1, \ldots, \vec{v}_n \in V$. Then $\mathrm{Span}\,(\vec{v}_1, \ldots, \vec{v}_n)$ is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_n$:

$$\mathrm{Span}_{\mathbb{F}}\,(\vec{v}_1, \ldots, \vec{v}_n) = \left\{ \sum_{i=1}^{n} c_i \cdot \vec{v}_i \,,\; c_i \in \mathbb{F} \right\} . \tag{3.65}$$

Corollary:
For the case of $\mathbb{R}^m$ we thus consider $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$. Then

$$\mathrm{Span}_{\mathbb{R}}\,(\vec{v}_1, \ldots, \vec{v}_n) = \left\{ \sum_{i=1}^{n} c_i \cdot \vec{v}_i \,,\; c_i \in \mathbb{R} \right\} . \tag{3.66}$$

In particular, $\mathrm{Span}\,(\vec{v}_1, \ldots, \vec{v}_n)$ is a linear subspace of $\mathbb{R}^m$.

Definition 3.5.4:
Consider a vector space $(V, +, \cdot)$ over $\mathbb{R}$. A collection of vectors $\mathcal{G} \subseteq V$ with $\mathrm{Span}(\mathcal{G}) = V$ is termed a *generating set of* V.

Note:
Every vector space admits a generating set $\mathcal{G}$. In general, $\mathcal{G}$ need not be finite. Convince yourself that this is for example the case for $\mathbb{R}[x]$ – the polynomials in the variable x and coefficients in $\mathbb{R}$. However, the linear subspace Pol_n is 'small' and admits a finite generating set.

Example 3.5.3:
The columns of a matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$ span the column space $C(A)$. Hence, these columns form a generating set $\mathcal{G}$ for the column space $C(A)$ of A. We can wonder if there are smaller generating sets, i.e. if we could span the column space $C(A)$ with fewer columns. In general, the answer depends on the the matrix in question. However, this question leads us to the definition of a very economic generating set.

Definition 3.5.5 (Basis)**:**
Let $(V, +, \cdot)$ be a vector space. A (finite) generating set $\mathcal{G}$ which is linearly independent is termed a *basis of* V.

Example 3.5.4:
Consider the vector space $\mathbb{R}^3$. A generating set is

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} . \tag{3.67}$$

Note that $\mathcal{G}$ is not a basis since these vectors are linearly dependent. This follows for example from the fact that the following matrix does not have full column rank:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{3.68}$$

However, the following set $\mathcal{B}$ is indeed a basis of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} . \tag{3.69}$$

Note:
For the most part of this lecture, we focus on vector space which admit a finite generating set. We then have the following important statement.

Corollary:
Any collection $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is a basis of $\mathbb{R}^n$ iff $A = \begin{bmatrix} | & \cdots & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & \cdots & | \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R})$ is invertible.

Exercise:
Prove this statement.

Corollary:
Be $(V, +, \cdot)$ be a vector space over a field F which admits a finite generating set. Then every basis $\mathcal{B}$ of $(V, +, \cdot)$ is finite and all basis consist of the same number of elements.

Exercise:
Convince yourself that any two basis $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathbb{R}^m$ satisfy $|\mathcal{B}_1| = |\mathcal{B}_2|$.

Definition 3.5.6:
Be $(V, +, \cdot)$ a vector space over a field F which admits a finite generating set. We term the cardinality of a basis $\mathcal{B}$ of $(V, +, \cdot)$ the dimension $\dim_{\mathbf{F}}(V)$ of V, i.e.

$$\dim_{\mathbf{F}}(V) := |\mathcal{B}| . \tag{3.70}$$

Note:
The dimension depends on the field $\mathbb{F}$. For example, as sets we have $\mathbb{R}^2 \cong \mathbb{C}$. However, when we consider $\mathbb{R}^2$ as vector space over $\mathbb{R}$ it follows $\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$. In contrast, $\dim_{\mathbb{C}}(\mathbb{C}) = 1$.

Example 3.5.5:
Convince yourself, that

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\} , \tag{3.71}$$

is not a basis of $\mathbb{R}^3$. We can however wonder if we can find a vector $\vec{v}$ such that $S' = S \cup \{\vec{v}\}$ is a basis of $\mathbb{R}^3$.

Work in this statement!

Remark:
Any vector space over $\mathbb{R}$ is uniquely classified by its dimension. That is, if V is a vector space over $\mathbb{R}$ of dimension $\dim_{\mathbb{R}}(V) = n$. Then $V \cong \mathbb{R}^n$. In particular, $C(A) \cong R(A)$ for any $A \in \mathbb{M}(m \times n, \mathbb{R})$.

Claim 7:
Any vector space over $\mathbb{R}$ is uniquely classified by its dimension. That is, if V is a vector space over $\mathbb{R}$ of dimension $\dim_{\mathbb{R}}(V) = n$. Then $V \cong \mathbb{R}^n$.

Proof
By assumption, V has a basis

$$\mathcal{B}_V = \left\{\vec{b}_1, \ldots, \vec{b}_n\right\} . \tag{3.72}$$

Likewise, consider the standard basis of $\mathbb{R}^n$, that is

$$\mathcal{B}_{\mathbb{R}^n} = \{\vec{e}_1, \ldots, \vec{e}_n\} , \tag{3.73}$$

where $\vec{e}_i \in \mathbb{R}^n$ has 1 in the i-th row and 0s otherwise. We now define the map

$$\varphi\colon V \to \mathbb{R}^n \, , \, \sum_{i=1}^{n} \lambda_i \vec{b}_i \mapsto \sum_{i=1}^{n} \lambda_i \vec{e}_i \, . \tag{3.74}$$

Since $\operatorname{im}(\varphi) = \mathbb{R}^n$, this map is surjective. Take $\vec{x}_1, \vec{x}_2 \in V$ and assume that $\varphi(\vec{x}_1) = \varphi(\vec{x}_2)$. But then we must have $\vec{x}_1 = \vec{x}_2$, so this map is injective as well. Consequently, φ is a bijection of sets. But even more, by construction it obeys the vector space rules in V and $\mathbb{R}^n$. Hence, φ is a vector space homomorphism and the statement follows. ■

Theorem 3.5.1 (Basis Extension Theorem)**:**
Be $(V, +, \cdot)$ a finite dimensional vector space over $\mathbb{F}$. Then, any collection of $\vec{v}_1, \ldots \vec{v}_n \in V$ of linearly independent vectors can be extended to a basis $\mathcal{B}$ of V.

Example 3.5.6:
Note that $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is not a generating set of $\mathbb{R}^3$ since $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin \operatorname{Span}_{\mathbb{R}}(\mathcal{S})$. Let us therefore consider

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} . \tag{3.75}$$

Indeed, $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

3.5.3 Computing dimension and basis of column spaces

Note:
Let us now return to talking about column spaces. Given $A \in \mathbb{M}(m \times n, \mathbb{R})$, how can we find a basis for $C(A)$ as well as its dimension? We will now try to give an answer to this question.

Example 3.5.7:
Consider

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \\ 4 & 8 & -1 & 3 \\ 4 & 8 & -1 & 3 \end{bmatrix}. \tag{3.76}$$

Let us quickly compute the row reduced form:

$$A \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.77}$$

There are two important facts to note at this stage:

- As column 2 does not possess a pivot, but column 1 does, we infer that column 2 can be written in terms of column 1.
- As column 4 does not contain a pivot, we infer that it can be written in terms of pivot columns 1 and 3.

Therefore, to generate $C(A)$, we only need to consider linear combinations of columns 1 and 3. Equivalently, we can say that columns 1 and 3 of A span $C(A)$. Even more, since they are pivot columns, they are linearly independent.

Together, these conclusions shows, that columns 1 and 3 form a basis $\mathcal{B}$ of $C(A)$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix} \right\}. \tag{3.78}$$

Note:
To get the basis elements of $C(A)$, we look at the columns of the original matrix A, not the row echelon form of A.

Exercise:
Explain why we look at the columns of the original matrix A and not its row echelon form, to get a basis of $C(A)$.

Consequence:
For any $A \in \mathbb{M}(m \times n, \mathbb{R})$, the pivot columns of A form a basis for $C(A)$. The dimension of $C(A)$ is equal to the column rank of A.

Exercise:
Pick a finite collection of vectors in $\mathbb{R}^m$ and identify a basis of the space spanned by these vectors.

3.6 Two other important vector spaces of a matrix

Definition 3.6.1 (Row space and left nullspace)**:**
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then we define:

- The row space $R(A)$ of A is the column space $C(A^T)$ of A^T.
- The left null space of A is the null space $N(A^T)$ of A^T.

Corollary:
The row space $R(A)$ and the left null space $N(A^T)$ are vector spaces:

- The row space $R(A)$ of A is a linear subspace of $\mathbb{R}^n$.
- The left null space is a linear subspace of $\mathbb{R}^m$.

Note:
Suppose $\vec{v} \in N(A^T)$. Then, by definition $A^T\vec{y} = \vec{0}$. Upon transposition, this is equivalent to $\vec{y}^T \cdot A = \vec{0}^T$. In this sense, $\vec{y}$ multiplies A from the left to give zero – hence the name *left* nullspace.

Consequence:
Strictly speaking, we may thus term the 'standard' nullspace the *right nullspace*.

Example 3.6.1:
Let us compute bases and dimensions of all these spaces in a single example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix}. \tag{3.79}$$

Let us start with the column and row space. We first recall the following general facts:

- $C(A) \subseteq \mathbb{R}^3$:
  We know that a basis is given by the pivot columns of A. The dimension of $C(A)$ matches the (column) rank r of A.
- $C(A^T) \subseteq \mathbb{R}^4$:
  This should not be too bad, since we already know how to deal with column spaces in general. However, by brute force, we are required to apply elimination to A and A^T. Luckily, it turns out that elimination of A already allows us to find a basis and the dimension of the row space of A^T.

Here is how this works. First, apply elimination to A:

$$A \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} := U\,. \tag{3.80}$$

The rows of U are definitely linear combinations of the rows of A, therefore

$$C(A^T) = R(A) = R(U)\,. \tag{3.81}$$

Furthermore, not that the non-zero rows of U are linearly independent. Therefore, since they span the row space of A, we find:

- A basis of the row space of A is given by the non-zero rows of U.
- The dimension of $R(A)$ is equal to the row rank of A.

Be mindful that row operations preserve row spaces, but not column spaces – $C(A)$ and $C(U)$ are clearly distinct. That said, let us look at the null spaces:

- $N(A) \subseteq \mathbb{R}^3$:
  This is also quite easy, given what we know. Firstly, can we span the nullspace from particular solutions? The answer is yes. Every assignment of numbers to the free variables can be interpreted as a linear combination of assignments where we set one variable to 1 and the rest to 0.

  Are these particular solutions linearly independent? Again, the answer is yes. An informal argument is to realize that the vectors

  $$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{3.82}$$

  are linearly independent. Thus, we conclude the following:

  - A basis for $N(A)$ is given by the particular solutions.
  - The dimension of $N(A)$ is equal to the number of free variables, i.e. $n - r$ where r is the column rank of A.

- The left nullspace $N(A^T) \subseteq \mathbb{R}^4$:
  Note that this corresponds to solving

  $$A^T\vec{y} = \vec{0} \quad \Leftrightarrow \quad \vec{y}^T A = \vec{0}^T\,. \tag{3.83}$$

Since $\vec{y}^T$ is a row vector, $\vec{y}^T A$ is also a row, given by a specific linear combination of the rows of A. As an augmented matrix, we may write this thus as:

$$A = \left[\begin{array}{cccc} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]. \tag{3.84}$$

Let us apply row eliminations. Then we find

$$A \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.85}$$

Thus, we went from

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \to \begin{bmatrix} R_1 \\ R_1 - R_2 \\ R_3 + R_2 - 2R_1 \end{bmatrix}. \tag{3.86}$$

Which matrix does perform this transformation? Well, the following does the job:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}. \tag{3.87}$$

The third row forms an element of $N(A^T)$. Thus, for every row that becomes a zero row during elimination, we get a vector in the null space of A^T.

Remark:
In summary, to compute the four linear subspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix}, \tag{3.88}$$

we first used row eliminations:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.89}$$

This shows $\dim(N(A)) = 2$ and $\mathrm{rk}(A) = 2$. Explicitly, we read-off bases of $N(A)$ and $R(A)$:

$$N(A) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right), \qquad R(A) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right). \tag{3.90}$$

Likewise, by use of row eliminations we can transform A^T as

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{3.91}$$

From this it follows

$$N(A^T) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right) , \qquad C(A) = R(A^T) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) . \tag{3.92}$$

Note:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then consider the map $\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$. Any such map can be factored by the four linear subspaces. By this we mean that there is a diagram

$$\begin{array}{ccccccc} \ker(\varphi_A) \cong N(A) & \xhookrightarrow{\varphi_K} & \mathbb{R}^n & \xrightarrow{\varphi_A} & \mathbb{R}^m & \xrightarrow{\varphi_P} & N(A^T) \cong \mathrm{coker}(\varphi_A) \\ & & \downarrow \varphi_{M_1} & & \uparrow \varphi_{M_2} & & \\ & & \mathrm{coim}(\varphi_A) \cong R(A) & \xrightarrow{\varphi_X} & C(A) \cong \mathrm{im}(\varphi_A) & & \end{array} \tag{3.93}$$

The maps φ_K and φ_{M_2} are injective. They are termed the *kernel embedding* and the *image embedding*, respectively. The maps φ_{M_1} and φ_P are surjective. They are termed the *coimage projection* and the *cokernel projection*, respectively. Crucially, the map φ_X is a vector space isomorphism, that is there exists an invertible matrix X with $A = M_2 \cdot X \cdot M_1$. This factorization is termed the *image-coimage factorization*

Remark:
This factorization exists much more generally, namely for every morphism in an Abelian category. You may encounter this if you every study category theory, which in a pedestrian fashion can be understood as a powerful tool to organize scientific programming.

Example 3.6.2:
For $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix}$ we have

$$\begin{array}{ccccccc} \mathbb{R}^2 \cong N(A) & \xhookrightarrow{\varphi_K} & \mathbb{R}^4 & \xrightarrow{\varphi_A} & \mathbb{R}^3 & \xrightarrow{\varphi_P} & N(A^T) \cong \mathbb{R}^1 \\ & & \downarrow \varphi_{M_1} & & \uparrow \varphi_{M_2} & & \\ & & \mathbb{R}^2 \cong R(A) & \xrightarrow{\varphi_X} & C(A) \cong \mathbb{R}^2 & & \end{array} \tag{3.94}$$

Consequence:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$ and E a matrix such that EA is an echelon form. Then the rows of E corresponding to the zero rows of A are a basis for the left null space of A. Hence

$$\dim \left(N(A^T) \right) = m - r \,, \tag{3.95}$$

where r is the (row) rank of A. Likewise,

$$\dim(N(A)) = n - r\,, \tag{3.96}$$

Thus, just the dimensions of A and its rank tell us a whole lot about the various linear subspaces associated to A. This is a special instance of the following

Theorem 3.6.1 (Rank-nullity theorem)**:**
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then it holds $\dim(N(A)) + \dim(C(A)) = n$.

Proof
$N(A)$ is a linear subspace of $\mathbb{R}^n$. Be $\mathcal{B}_{N(A)}$ a basis of $N(A)$. Then, by the basis-extension-theorem, we may extend $\mathcal{B}_{N(A)}$ to a basis $\mathcal{B}$ of $\mathbb{R}^n$. This amounts to adding a set $\mathcal{I}$ of linearly independent vectors in $\mathbb{R}^n$ to $\mathcal{B}_{N(A)}$. In particular,

$$|\mathcal{I}| = n - |\mathcal{B}_{N(A)}| = n - \dim_{\mathbb{R}}(N(A))\,, \qquad C(A) = \mathrm{Span}_{\mathbb{R}}(I)\,. \tag{3.97}$$

This concludes the proof. ■

Note:
A bit more is true. Let

$$\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m\,,\ \vec{x} \mapsto A\vec{x}\,. \tag{3.98}$$

Then $\mathrm{im}(\varphi_A) \subseteq \mathbb{R}^m$ is a linear subspace. We claim that $\mathcal{I}' = \{A\vec{v}\,,\ \vec{v} \in \mathcal{I}\}$ is a basis of $\mathrm{im}(\varphi_A)$.

Indeed $\mathcal{I}'$ is a generating set. To see that it is linearly independent suppose that

$$\sum_{i \in I} \alpha_i \cdot (A \cdot \vec{v}_i) = 0\,. \tag{3.99}$$

By the linearly of matrix-vector multiplication, this is equivalent to

$$A\left(\sum_{i \in I} \alpha_i \cdot \vec{v}_i\right) = 0\,. \tag{3.100}$$

Hence $\sum_{i \in I} \alpha_i \cdot \vec{v}_i \in N(A)$. This is a contradition to $\mathcal{B} = I \cup \mathcal{B}_{N(A)}$ being a basis of $\mathbb{R}^n$ unless $\alpha_i = 0$ for all $i \in I$.

Example 3.6.3:
For

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix}\,, \tag{3.101}$$

we found

$$N(A) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right), \qquad R(A) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right). \quad (3.102)$$

The basis of $N(A)$ is readily extended to $\mathbb{R}^4$ by adding the basis of $R(A)$. So consider

$$\mathcal{I} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}. \quad (3.103)$$

The images of these vectors under φ_A are

$$\mathcal{I}' = \left\{ \begin{bmatrix} 8 \\ 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix} \right\}. \quad (3.104)$$

Note that

$$\begin{bmatrix} 8 & 7 & 9 \\ 5 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}. \quad (3.105)$$

This is exactly the basis of $C(A)$ that we found above!

Corollary:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$ with $r = \mathrm{rk}(A)$. Then it holds:

- $C(A)$ and $R(A)$ have dimension r. (This already follows from our previous finding that the row and column rank coincide.)
- $N(A)$ has dimension $n - r$.
- $N(A^T)$ has dimension $m - r$.

3.7 Linear transformations

Note:
We have already discussed the idea of a matrix as a function, namely assume that $A \in \mathbb{M}(m \times n, \mathbb{R})$ and $\vec{v} \in \mathbb{R}^n$. Then we defined a function

$$\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m, \ \vec{v} \mapsto A\vec{v}. \quad (3.106)$$

That is, $\vec{v}$ is the input and $A\vec{v}$ is the output. While one can think of $A\vec{v}$ as one vector at a time, the deeper goal is now to see what A does to the whole space!

Remark:
Matrix multiplication is linear. This is equivalent to (for all $c \in \mathbb{R}$ and $\vec{v}, \vec{w} \in \mathbb{R}^n$)

- $\varphi_A(c\vec{v}) = c \cdot A\vec{v}$
- $\varphi_A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$.

Thus, matrix multiplication fits nicely with the operations in a vector space. We now analyse functions which have the same property as matrix multiplication.

Definition 3.7.1 (Linear map)**:**
Let $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ be two vector spaces over $\mathbb{F}$. Then a function $\varphi\colon V \to W$ is called a linear map if for all $c \in \mathbb{F}$ and all $\vec{v}$, $\vec{w} \in V$ it holds:

- $\varphi(c \cdot_V \vec{v}) = c \cdot_W \varphi(\vec{v})$,
- $\varphi(\vec{v} +_V \vec{w}) = \varphi(\vec{v}) +_W \varphi(\vec{w})$.

Claim 8:
Any linear map $\varphi\colon V \to W$ of vector spaces V, W over $\mathbb{F}$ satisfies $\varphi(\vec{0}) = \vec{0}$.

Proof
Since $\vec{0} = \vec{0} + \vec{0}$ it follows from linearity of φ that $\varphi(\vec{0}) = \varphi(\vec{0}) + \varphi(\vec{0}) = 2 \cdot \varphi(\vec{0})$. It follows $\vec{0} = \varphi(\vec{0})$. ■

Example 3.7.1:
Is $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\varphi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_1 + x_2 \end{bmatrix} \tag{3.107}$$

a linear map? No! Because

$$2 \cdot \varphi\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \neq \varphi\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) . \tag{3.108}$$

Essentially, the square in the first component stops this function from being linear.

Example 3.7.2:
Consider the function $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\varphi\left(\vec{v}\right) = v_1 + 2v_2 + 3v_3 . \tag{3.109}$$

This function is linear. In particular, we have

$$\varphi\left(\vec{v}\right) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} . \tag{3.110}$$

Remark:
Note that perse, such a linear map is not defined by a matrix. But that does not mean we cannot find a mapping matrix. For this we focus, unless explicitly stated differently, on *finite*-dimensional vector spaces V and W.

Construction 3.7.1:
Let $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ both be finite-dimensional vector spaces over $\mathbb{F}$ and let $\varphi\colon V \to W$ be a linear map. We denote a basis of V by $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$. Therefore, every vector $\vec{x} \in V$ is expressed as a *unique* linear combination of the vectors in $\mathcal{B}$:

$$\vec{x} = \sum_{i=1}^{n} c_i \cdot \vec{v}_i\,, \qquad c_i \in \mathbb{F}\,. \tag{3.111}$$

Since φ is linear it follows:

$$\varphi(\vec{x}) = \varphi\left(\sum_{i=1}^{n} c_i \cdot \vec{v}_i\right) = \sum_{i=1}^{n} c_i \cdot \varphi\left(\vec{v}_i\right). \tag{3.112}$$

Hence, to compute the image of any vector $\vec{x} \in V$, it suffice to know the images of the basis vectors $\vec{v}_i$ under φ. This we can efficiently encode in the matrix

$$A_{\mathcal{B}} = \begin{bmatrix} | & & | \\ \varphi(\vec{v}_1) & \dots & \varphi(\vec{v}_n) \\ | & & | \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R})\,, \tag{3.113}$$

where $\dim_{\mathbb{F}}(W) = m$. Namely, it then follows

$$\varphi(\vec{x}) = A_{\mathcal{B}} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \tag{3.114}$$

Hence, if we agree to represent eq. (3.111) by the coefficients c_i, which uniquely express this vector in the basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ of V, then φ matches $\varphi_{A_{\mathcal{B}}}\colon V \to W,\ ,\ \vec{x} \mapsto A_{\mathcal{B}} \cdot \vec{x}$ with $A_{\mathcal{B}}$ in eq. (3.113).

Corollary:
Every linear map $\varphi\colon V \to W$ can be expressed as φ_A upon a choice of basis $\mathcal{B}$ of V. The mapping matrix $A = A_{\mathcal{B}}$ depends on the choice of basis!

Remark:
The matrix $A_{\mathcal{B}}$ encodes properties of φ as follows:

- φ is injective iff $A_{\mathcal{B}}$ has full column rank.
- φ is surjective iff $A_{\mathcal{B}}$ has full row rank.
- φ is bijective iff $A_{\mathcal{B}}$ has full rank. In particular, $A_{\mathcal{B}}$ must be a square matrix.

Remark (Index of linear map)**:**
An important quantity of a linear map φ is its index. This is defined as

$$\operatorname{ind}(\varphi) := \dim_{\mathbb{F}}(\ker(\varphi)) - \dim_{\mathbb{F}}(\operatorname{coker}(\varphi)) = \dim_{\mathbb{F}}\left(N(A_{\mathcal{B}})\right) - \dim_{\mathbb{F}}\left(N(A_{\mathcal{B}}^T)\right) . \quad (3.115)$$

Example 3.7.3:
Consider a reflection in $\mathbb{R}^2$ across the line $y = x$. This a linear transformation! To find a mapping matrix, we first pick a basis of $\mathbb{R}^2$. It is particularly convenient to choose

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} . \quad (3.116)$$

Then we have

$$\varphi(\vec{v}_1) = 1 \cdot \vec{v}_1 \,, \qquad \varphi(\vec{v}_2) = (-1) \cdot \vec{v}_2 \,. \quad (3.117)$$

Hence, in this basis, the mapping matrix is given by

$$A_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} . \quad (3.118)$$

This matrix accepts vectors (or actually their coefficients) in the basis $\mathcal{B}$ and returns the coefficient of the image vector in the standard basis $\mathcal{A}$ of $\mathbb{R}^2$:

$$\mathcal{A} = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} . \quad (3.119)$$

This indicates that we should be more careful. We should not only mention in what basis the input is encoded but also in what basis the output is encoded. Therefore, we write

$$A_{\mathcal{AB}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} . \quad (3.120)$$

Of course, we can also consider the matrix $A_{\mathcal{BB}}$. It is not too hard to see that

$$A_{\mathcal{BB}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \quad (3.121)$$

The input to this matrix is the coefficients of $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. And the output is the coefficients of the image, but this time with respect to the basis $\mathcal{B}$. Hence, we learn that $\varphi(1 \cdot v_1 + 0 \cdot v_2)$ is a vector, whose coefficients in the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is given as

$$A_{\mathcal{BB}} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \quad (3.122)$$

Hence, $\varphi(1 \cdot v_1 + 0 \cdot v_2) = 1 \cdot v_1 + 0 \cdot v_2$, just as expected.

Note (Continuation)**:**
We could also find the mapping matrix of the reflection φ of the previous example in the standard basis

$$\mathcal{A} = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} . \tag{3.123}$$

To do this, we need to discuss base change matrices.

Construction 3.7.2:
We are looking at the following two bases of $\mathbb{R}^2$:

- $\mathcal{A} = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$
- $\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$

We are looking for a matrix $T_{\mathcal{BA}} \in \mathbb{M}(2 \times 2, \mathbb{R})$ which mediates from $\mathcal{A}$ to $\mathcal{B}$. To find such a matrix note that

$$\vec{u}_1 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 , \qquad \vec{u}_2 = \frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2 . \tag{3.124}$$

We collect these coefficients in the matrix $T_{\mathcal{BA}}$:

$$T_{\mathcal{BA}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) . \tag{3.125}$$

Hence, $T_{\mathcal{BA}}\vec{u}_i$ is a vector whose components express $\vec{u}_i$ are linear combination of the basis $\mathcal{B}$. The inverse base change matrix is simple the inverse of $T_{\mathcal{BA}}$. Namely:

$$T_{\mathcal{BA}}^{-1} = T_{\mathcal{AB}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{3.126}$$

Remark:
We agree that the indices of a base change matrix $T_{\mathcal{BA}}$ list the "target" basis as first index and the "domain" basis as second argument.

Example 3.7.4 (Continuation)**:**
We just established the base change matrices between $\mathcal{A}$ and $\mathcal{B}$:

$$T_{\mathcal{BA}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} , \qquad T_{\mathcal{BA}}^{-1} = T_{\mathcal{AB}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{3.127}$$

Thereby, we can now compute the mapping matrix of φ discussed in example 3.7.3. The idea is as follows:

1. Convert the input from basis $\mathcal{A}$ to $\mathcal{B}$.

2. Apply $A_{\mathcal{BB}}$.

3. Convert the output back from basis $\mathcal{B}$ to $\mathcal{A}$.

Talking matrices, this means that we are looking at:

$$A_{\mathcal{AA}} = T_{\mathcal{AB}} \cdot A_{\mathcal{BB}} \cdot T_{\mathcal{BA}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{3.128}$$

Exercise:
Compute the mapping matrix $A_{\mathcal{BA}}$.

Exercise:
Consider a function $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ which rotates a vector by 45 degrees. Convince yourself that this is a linear transformation. Find the mapping matrix $A_{\mathcal{AA}}$ in the standard basis $\mathcal{A}$ of $\mathbb{R}^2$. Consider a different basis $\mathcal{B}$ and find the mapping matrix $A_{\mathcal{BB}}$.

4 Orthogonality

Recall the four primary subspaces attached to a matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$:

- the column space $C(A)$,
- the (right) null space $N(A)$,
- the row space $R(A)$,
- the left null space of A, i.e $N(A^T)$.

It so happens, that $R(A)$ and $N(A)$ are orthogonal. Likewise $C(A)$ and $N(A^T)$ are orthogonal. This is hinting at more structure underlying these four fundamental spaces. In this chapter, we wish to investigate this structure. This will allow us to understand Fourier series and least square approximations.

Convention:
Unless stated differently, from now on any vector space V is a vector space over the real numbers $\mathbb{R}$.

4.1 The notion of orthogonality

Remark:
In order to introduce a notion of orthogonality, we start in $\mathbb{R}^n$, which we may access intuitively. This leads to the notion of the "standard inner product" in $\mathbb{R}^n$, which we will generalize momentarily to define inner products also on more general vector spaces such as Pol_n and $\mathbb{M}(m \times n, \mathbb{R})$.

Example 4.1.1 ('Standard' orthogonality in $\mathbb{R}^n$)**:**
Consider the vector space $\mathbb{R}^n$ over $\mathbb{R}$. We consider the map

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\,,\ (\vec{a}, \vec{b}) \mapsto \vec{a}^T \cdot \vec{b} = \sum_{i=1}^{n} a_i \cdot b_i\,. \tag{4.1}$$

This map has the following properties:

- Linearity in the first argument, that is for all $c \in \mathbb{R}$ and $\vec{a}_1, \vec{a}_2, \vec{b} \in \mathbb{R}^n$ it holds

$$\left\langle \vec{a}_1 + c \cdot \vec{a}_2, \vec{b} \right\rangle = \left\langle \vec{a}_1, \vec{b} \right\rangle + c \cdot \left\langle \vec{a}_2, \vec{b} \right\rangle\,. \tag{4.2}$$

- Symmetry, that is for all $\vec{a}, \vec{b} \in \mathbb{R}^n$ it holds

$$\left\langle \vec{a}, \vec{b} \right\rangle = \left\langle \vec{b}, \vec{a} \right\rangle . \tag{4.3}$$

- $\langle \cdot, \cdot \rangle$ is positive-definite, that is for all $\vec{a} \in \mathbb{R}^n \setminus \vec{0}$ it holds

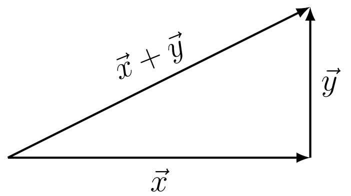
$$\langle \vec{a}, \vec{a} \rangle > 0 . \tag{4.4}$$

We refer to this inner product as the *standard inner product in* $\mathbb{R}^n$. This inner product gives us the following notions of length and orthogonality:

- The length of a vector $\vec{x}$ is given by $|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.
- Two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are orthogonal if and only if $\langle \vec{x}, \vec{y} \rangle = 0$.

Remark:
Why is this the "*right*" criterion? First, there is no right or wrong inner product. Any inner product gives the notion of orthogonality, and it need not be tied to our expectation on our physical surrounding.

However, if we are looking for an inner product which matches the expectation in our physical surroundings, then the above *standard inner product* does a good job. To see this, look at the following triangle with 90 degree angle:



(4.5)

Then, by Pythagoras, it holds

$$|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 , \tag{4.6}$$

where $|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ denotes the length of $\vec{x}$. Hence, this is equivalent to

$$\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle . \tag{4.7}$$

But by linearity and symmetry of the inner product, this in turn is equivalent to

$$\langle \vec{x}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle + 2 \langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle . \tag{4.8}$$

Hence we conclude that

$$|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 \qquad \Leftrightarrow \qquad \langle \vec{x}, \vec{y} \rangle = 0 . \tag{4.9}$$

Definition 4.1.1 (Inner product space)**:**
Be $(V, +_V, \cdot_V)$ a vector space over $\mathbb{R}$. Then a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is termed an inner product on V if and only if it satisfies the following conditions:

1. $\langle \cdot, \cdot \rangle$ is $\mathbb{R}$-linear in its first argument,
2. $\langle \cdot, \cdot \rangle$ is symmetric,
3. $\langle \cdot, \cdot \rangle$ is positive-definite.

The pair $((V, +_V, \cdot_V), \langle \cdot, \cdot \rangle)$ is termed an *inner product space* over $\mathbb{R}$. In this inner product space we define:

- The length of a vector $\vec{x}$ is given by $|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.
- Two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ are orthogonal if and only if $\langle \vec{x}, \vec{y} \rangle = 0$.

Exercise:
Verify that for any $k \in \mathbb{R}_{>0}$

$$\langle \cdot, \cdot \rangle_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\,, \; (\vec{a}, \vec{b}) \mapsto k \cdot a_1 b_1 + \sum_{i=2}^{n} a_i \cdot b_i\,, \tag{4.10}$$

is an inner product. In particular, for $k \neq 1$, this inner product differs from the Standard inner product. This shows, that there are many inner products on a given vector space.

Exercise:
Define inner products on Pol_n and $\mathbb{M}(m \times n, \mathbb{R})$ and explicitly verify that these inner products satisfy the axioms of an inner product. Use these inner products to find 4 orthogonal vectors in Pol_4 and $\mathbb{M}(3 \times 3, \mathbb{R})$.

Claim 9:
Be $((V, +_V, \cdot_V), \langle \cdot, \cdot \rangle)$ an *inner product space* over $\mathbb{R}$. Then $\vec{0} \in V$ is orthogonal to any other vector $\vec{x} \in V$.

Proof
Since $\langle \cdot, \cdot \rangle$ is linear in the first argument, it follows from $\vec{0} + \vec{0} = \vec{0}$ that

$$\left\langle \vec{0}, \vec{x} \right\rangle = \left\langle \vec{0}, \vec{x} \right\rangle + \left\langle \vec{0}, \vec{x} \right\rangle \,. \tag{4.11}$$

Hence $\left\langle \vec{0}, \vec{x} \right\rangle = 0$ and $\vec{0}$ and $\vec{x}$ are orthogonal. ■

Note:
We can extend the notion of orthogonality to vector spaces.

Definition 4.1.2 (Orthogonality of vector spaces)**:**
Be $S, T \subseteq V$ two linear subspaces of an inner product space $(V, \langle \cdot, \cdot \rangle)$ over $\mathbb{R}$. Then we say that S is orthogonal to T – $S \perp T$ – if every vector in S is orthogonal to every vector in T, that is

$$\left\langle \vec{s}, \vec{t} \right\rangle = 0\,, \qquad \forall \vec{s} \in S \text{ and } \forall \vec{t} \in T\,. \tag{4.12}$$

Example 4.1.2:
Consider the real vector space $\mathbb{R}^2$ with the standard inner product. Moreover, let

$$S = \text{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \qquad T = \text{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}. \tag{4.13}$$

Then $S \perp T$.

Corollary:
Every subspace $S \subseteq V$ is orthogonal to the trivial subspace of V.

Claim 10:
Any two distinct 2-dim. linear subspaces $S, T \subseteq \mathbb{R}^3$ are *not* orthogonal.

Proof
Since S and T are distinct, they intersect for dimensional reasons in a line through the origin. Let $\vec{x}$ be a *non-zero* vector that belongs to this line of intersection. Since this vector belongs to S and T, we conclude from $\vec{x}^T\vec{x} > 0$, that S and T are not orthogonal. ■

Note:
A line and a plane in $\mathbb{R}^3$ can be orthogonal subspaces. Namely, the line through the origin normal to the plane gives such a pair of orthogonal subspaces.

Claim 11:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then (w.r.t. the standard inner product in $\mathbb{R}^n$) its row space $R(A)$ and (right) null space $N(A)$ are orthogonal.

Proof
Let $\vec{x} \in N(A)$. Then, by definition $A\vec{x} = \vec{0}$. More explicitly, this means

$$\begin{bmatrix} - & \text{row 1} & - \\ - & \text{row 2} & - \\ \vdots & & \\ - & \text{row m} & - \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.14}$$

Hence, $\text{row}_i \cdot \vec{x} = 0$ for all $1 \leq i \leq m$. Since this is the standard inner product in $\mathbb{R}^n$, we conclude that w.r.t. this inner product $\vec{x}$ is orthogonal to any vector $\vec{r}$ in the row space of A. ■

Consequence:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$ matrix. Then (w.r.t. the standard inner product in $\mathbb{R}^m$) its column space $C(A)$ is orthogonal to the left nullspace $N(A^T)$.

Exercise:
Proof this statement. Hint: Apply the previous result to A^T.

4.2 Orthogonal complements

Note:
In fact, one can say more about the relation between the row space and the null spaces. Not only are they orthogonal to each other, they also fill out the whole space. What does this mean?

Question:
Consider a matrix $A \in \mathbb{M}(n \times 4, \mathbb{R})$ with 4 columns. Is it possible that the row space $R(A)$ is a line in $\mathbb{R}^4$ and the null space $N(A)$ is also just a line in $\mathbb{R}^4$? No! Namely, the rank-nullity theorem tells us that

$$\dim\left(R(A)\right) + \dim\left(N(A)\right) = 4\,. \tag{4.15}$$

Therefore, if the row space $R(A)$ is a line, then the nullspace $N(A)$ is forced to be 3-dimensional. Hence since $N(A) \perp R(A)$ (w.r.t. the standard inner product in $\mathbb{R}^n$), these orthogonal subspaces span $\mathbb{R}^4$. This observation leads to the notion of the orthogonal complement.

Definition 4.2.1 (Orthogonal complement)**:**
Be $(V, \langle\cdot,\cdot\rangle)$ a vector space and $S \subseteq V$ a linear subspace. The orthogonal complement of S in V is the set of all vectors $\vec{v} \in V$ which are perpendicular to S, i.e.

$$S^{\perp} := \{\vec{v} \in V \mid \langle \vec{s}, \vec{v}\rangle = 0 \ \ \forall \vec{s} \in S\}\ . \tag{4.16}$$

Claim 12:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$ matrix. Then the nullspace $N(A)$ is the orthogonal complement of the row space $R(A)$ (w.r.t. the standard inner product in $\mathbb{R}^n$).

Proof
We want to show that $N(A)^{\perp} = R(A)$. To this end we prove that $N(A)^{\perp} \subseteq R(A)$ and that $R(A) \subseteq N(A)^{\perp}$:

- $R(A) \subseteq N(A)^{\perp}$:
  Every vector that is orthogonal to the rows of a matrix A lies in the nullspace $N(A)$ by claim 11.

- $N(A)^{\perp} \subseteq R(A)$:
  We prove the contraposition, that is we assume that there was $\vec{v} \in N(A)^{\perp}$ with $v \notin R(A)$. Then consider the matrix

  $$A' = \left[\begin{array}{c} A \\ \hline \vec{v}^T \end{array}\right] \in \mathbb{M}((m+1) \times n, \mathbb{R})\,. \tag{4.17}$$

  By assumption, it then holds

  $$\mathrm{rk}(A') = \mathrm{rk}_R(A') = \mathrm{rk}_R(A) + 1 = \mathrm{rk}(A) + 1\,. \tag{4.18}$$

Note that

$$N(A') = N(A) \cap \left\{\vec{x} \in \mathbb{R}^n | \vec{v}^T \vec{x} = 0\right\} \subseteq N(A)\,. \tag{4.19}$$

However, by assumption $\vec{v} \in N(A)^\perp$, i.e. $\vec{v}^T \vec{x} = 0$ for all $\vec{x} \in N(A)$. Hence $N(A') = N(A)$ and, by the rank-nullity theorem, we must therefore have

$$n = \dim_{\mathbb{R}}(N(A')) + \mathrm{rk}(A') = \dim_{\mathbb{R}}(N(A)) + \mathrm{rk}(A) + 1 = n + 1\,. \tag{4.20}$$

This is the desired contradiction.

This completes the argument. ■

Exercise:
Prove that $N(A^T)$ is the orthogonal complement of $C(A)$.

Remark:
W.r.t. the standard inner product in $\mathbb{R}^n$, the have:

- $N(A)$ is the orthogonal complement of $R(A)$.
- $N(A^T)$ is the orthogonal complement of $C(A)$.

Claim 13:
Let $S, T \subseteq V$ two linear subspaces of an inner product space $(V, \langle\cdot,\cdot\rangle)$. Suppose that T is the orthogonal complement of S in $(V, \langle\cdot,\cdot\rangle)$. Then every vector $\vec{v} \in V$ can be written *uniquely* in the form

$$\vec{v} = \vec{v}_S + \vec{v}_T\,, \tag{4.21}$$

where $\vec{v}_S \in S$ and $\vec{v}_T \in T$.

Proof
We will discuss the existence of such a decomposition momentarily. It will follow from orthogonal projections onto linear subspaces. Suffice it here to show the uniqueness of this decomposition. We assume the contrary, i.e. assume that

$$\vec{v} = \vec{v}_S + \vec{v}_T = \vec{w}_S + \vec{w}_T\,, \tag{4.22}$$

for distinct $\vec{v}_S, \vec{w}_S \in S$ and distinct $\vec{v}_T, \vec{w}_T \in S$. But this is equivalent to

$$S \ni \vec{v}_S - \vec{w}_S = \vec{v}_T - \vec{w}_T \in T\,, \tag{4.23}$$

Since $S \cap T = \left\{\vec{0}\right\}$ we conclude that $\vec{v}_S = \vec{w}_S$ and $\vec{v}_T = \vec{w}_T$ contrary to our assumption. Hence, the decomposition is unique, as claimed. ■

Consequence:
Let $S \subseteq V$ a linear subspace of an inner product space $(V, \langle \cdot, \cdot \rangle)$. Since every vector $\vec{v} \in V$ has a unique decomposition $\vec{v} = \vec{v}_S + \vec{v}_{S^\perp}$ with $\vec{v}_S \in S$ and $\vec{v}_{S^\perp} \in S^\perp$, we have an isomorphism

$$S \oplus S^\perp := \big\{(s,t) \,|\, s \in S \,,\, t \in S^\perp\big\} \to V \,,\, (s,t) \mapsto s + t \,. \tag{4.24}$$

We term $S \oplus S^\perp$ the direct sum of S and $S^\perp$. Thus, we have found $S \oplus S^\perp \cong V$.

Example 4.2.1:
Consider the line $S = \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^2$ with standard inner product. Then we have $S^\perp \cong \mathbb{R}$ and $\mathbb{R}^2 \cong S \oplus S^\perp \cong \mathbb{R} \oplus \mathbb{R}$. Explicitly

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \end{bmatrix} \,. \tag{4.25}$$

Example 4.2.2:
Consider the vector space $\mathbb{M}(n \times n, \mathbb{R})$. Then the subspaces of symmetric and of anti-symmetric $n \times n$-matrices are orthogonal complements. This follows from

$$A = \left(\frac{A + A^T}{2} \right) + \left(\frac{A - A^T}{2} \right) \,. \tag{4.26}$$

4.3 Orthogonal projections

In this section we will discuss orthogonal projections. As the name indicates, there are projections which are not orthogonal.

Definition 4.3.1:
A linear map $\varphi_P \colon \mathbb{R}^n \to \mathbb{R}^n$ with $P^2 = P$ is called a *projection*.

Example 4.3.1:
Consider the map

$$\varphi_P \colon \mathbb{R}^2 \to \mathbb{R}^2 \,,\, (x,y) \mapsto (x,x) \,. \tag{4.27}$$

It is not too hard to see that

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \,. \tag{4.28}$$

Hence, $P^2 = P$ and φ_P is a projection. Note that

$$C(P) = \mathrm{Span}_{\mathbb{R}} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \,. \tag{4.29}$$

If $\vec{x} \in C(P)$, then $P\vec{x} = \vec{x}$. Hence, φ_P is a projection onto the line $x = y$. Furthermore we have

$$N(P) = \operatorname{Span}_{\mathbb{R}} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) . \tag{4.30}$$

Hence, if $\vec{x}$ is on the y-axis, then it is projected to $\vec{0} \in C(P)$. However, the line $x = y$ and the y-axis are not orthogonal (with respect to the standard inner product). For this reason, this is not an orthogonal projection.

Remark:
We will not discuss such general projections in more detail in this class. Rather, we will now focus on a special type of projections, namely the *orthogonal* projections.

4.3.1 Orthogonal projection onto a line

Note:
We now wish to discuss how we actually write a vector as a sum of vectors coming from orthogonal complements. A first example is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \end{bmatrix} , \tag{4.31}$$

where the first vector is contained in the row space of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and the second in its nullspace. Observe, that we project $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ on the row space and nullspace to obtain the summands. This motivates the discussion of projections.

Remark:
In the following discussion, we focus on the standard inner product in $\mathbb{R}^n$. The results easily generalize to arbitrary inner products.

Construction 4.3.1:
We begin by considering the simplest case, a projection onto a line. Then our challenge is the following: Given a line L through the origin in the direction $\vec{a}$, find the point $\vec{p}$ on the line, which is closest to $\vec{b}$.

Standard Euclidean geometry tells us, that that point $\vec{p}$ is obtained by dropping a line from $\vec{b}$ in the direction perpendicular to the line L. But how do we actually find $\vec{p}$?

Clearly, it must hold $\vec{p} = \hat{x} \cdot \vec{a}$ for some $\hat{x} \in \mathbb{R}$. Furthermore, $\vec{b} - \vec{p} = \vec{b} - \hat{x} \cdot \vec{a}$ is orthogonal to $\vec{a}$. Therefore,

$$0 = \vec{a}^T \cdot \left(\vec{b} - \hat{x} \cdot \vec{a} \right) \quad \Leftrightarrow \quad \hat{x} = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} . \tag{4.32}$$

Consequently, it holds

$$\vec{p} = \left(\frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} \right) \cdot \vec{a} . \tag{4.33}$$

Remark:
The vector $\vec{b} - \vec{p}$ is oftentimes referred to as *error vector* and is therefore denoted as $\vec{e}$.

Claim 14:
This projection is a linear transformation.

Proof
We construct a matrix P which sends $\vec{b}$ to $\vec{p}$. From

$$\vec{p} = \vec{a} \cdot \left(\frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} \right) , \tag{4.34}$$

we can see that the matrix

$$P = \frac{\vec{a}\vec{a}^T}{\vec{a}^T \vec{a}} , \tag{4.35}$$

indeed satisfies $P\vec{b} = \vec{p}$. ■

Remark:
For the previous argument it is crucial to notice that $\vec{a}\vec{a}^T \in \mathbb{M}(n \times n, \mathbb{R})$ for $\vec{a} \in \mathbb{R}^n$.

Exercise:
Compute the matrix P for $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then repeat this exercise for $\vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note:
The rank of the matrix P is 1. Its column space is the line we are projection upon.

Example 4.3.2:
Let us compute the projection matrix for projection onto the line

$$L = \{\lambda \cdot \vec{a} | \lambda \in \mathbb{R}\} , \qquad \vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} . \tag{4.36}$$

We plug in the above formula and find

$$P = \frac{\vec{a}\vec{a}^T}{1^2 + 2^2 + 3^2} = \frac{1}{14} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} . \tag{4.37}$$

Next, let us compute the projection of $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Hence, we simply compute

$$\vec{p} = P\vec{b} = \frac{1}{14} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} . \tag{4.38}$$

Note:
Scaling $\vec{a}$ does not change the projection matrix P.

Claim 15:
Projecting twice is the same as projecting once: $P^2 = P$.

Proof

$$P^2 = \left(\frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}\right) \cdot \left(\frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}\right) = \frac{\vec{a}^T \cdot \left(\vec{a}\vec{a}^T\right) \cdot \vec{a}}{\left(\vec{a}\vec{a}^T\right)^2} = \left(\frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}\right) = P\,. \tag{4.39}$$

■

Question:
If P is the projection matrix on the line L through the origin in the direction $\vec{a}$, then what is $I - P$? For any vector $\vec{b}$, it holds

$$(I - P) \cdot \vec{b} = \vec{b} - P\vec{b} = \vec{b} - \vec{p} = \vec{e}\,. \tag{4.40}$$

This is the error vector perpendicular to the line L. But note also that

$$(I - P)^2 = I^2 - 2P + P = I - P\,. \tag{4.41}$$

Consequently, $I - P$ is a projection. It is the projection onto the subspace orthogonal to the line through $\vec{a}$.

Exercise:
Interpret the equation $I = P + (I - P)$. Hint: Orthogonal complements.

4.3.2 Orthogonal projection onto subspaces

Lemma 4.3.1:
Be $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then the following two statements are equivalent:

- A has linearly independent columns.
- $A^T A$ is invertible.

Proof
We prove that A and $A^T A$ have the same nullspaces:

- Be $\vec{x} \in N(A)$. Then $A\vec{x} = \vec{0}$. But then $A^T A\vec{x} = \vec{0}$, i.e. $\vec{x} \in N(A^T A)$.
- Conversely, let $\vec{x} \in N(A^T A)$. Then $A^T A\vec{x} = \vec{0}$. We multiply by $\vec{x}^T$ on the LHS:

  $$0 = \vec{x}^T A^T A\vec{x} \quad \Leftrightarrow \quad (A\vec{x})^T (A\vec{x}) = \vec{0}\,. \tag{4.42}$$

  This means that $\langle A\vec{x}, A\vec{x}\rangle = 0$. But, by the properties of the inner product, this is only possible if $A\vec{x} = \vec{0}$, i.e. $\vec{x} \in N(A)$.

Now we can show the stated equivalence:

- If A has linearly independent columns, then $N(A) = \{\vec{0}\}$. Then, by our previous observation, $N(A^T A) = \{\vec{0}\}$. Consequently, the square matrix $A^T A$ is invertible.
- Conversely, if $A^T A$ is invertible, then $N(A^T A) = \{\vec{0}\}$. But then, also $N(A) = \{\vec{0}\}$ and A has linearly independent columns. ■

Question (Projections onto general subspaces)**:**
Given linearly independent vectors $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^m$, how do we find the linear combination

$$\vec{p} = \widehat{x}_1 \vec{a}_1 + \cdots + \widehat{x}_n \vec{a}_n \tag{4.43}$$

which is closest to a given vector $\vec{b}$?

Note:
The case $n = 1$ is projection onto a line. The case $n = 2$ is projection onto a plane.

Construction 4.3.2:
Let us consider the matrix

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix} \in \mathbb{M}(m \times n, \mathbb{R}) . \tag{4.44}$$

By assumption, A has linearly independent columns. We are now looking for a vector $\vec{p} \in C(A)$ which is closest to $\vec{b}$. Let us set

$$\widehat{\vec{x}} = (\widehat{x}_1, \ldots, \widehat{x}_n) \; . \tag{4.45}$$

The 'right' vector $\widehat{\vec{x}}$ is defined by the property that $\vec{b} - A\widehat{\vec{x}}$ is orthogonal to $C(A)$. This is equivalent to saying that

$$A^T \cdot \left(\vec{b} - A\widehat{\vec{x}}\right) = \vec{0} \, . \tag{4.46}$$

We can rewrite this as

$$A^T A \widehat{\vec{x}} = A^T \vec{b} \, . \tag{4.47}$$

The matrix $S = A^T A \in \mathbb{M}(n \times n, \mathbb{R})$ is, by our previous lemma, invertible since A has linearly independent columns. Therefore, we can write

$$\widehat{\vec{x}} = \left(A^T A\right)^{-1} \cdot A^T \vec{b} \, . \tag{4.48}$$

Consequently, the projection $\vec{p}$ and projection matrix P are given by

$$\vec{p} = P\vec{b} \, , \qquad P = A \cdot \left(A^T A\right)^{-1} \cdot A^T \, . \tag{4.49}$$

Note that A by itself is (in general) not invertible! Hence, you cannot write $(A^T A)^{-1}$ as $A^{-1} \cdot (A^T)^{-1}$.

Example 4.3.3:
Let us find the projection matrix for projection onto the plane in $\mathbb{R}^3$, which is given by $x - 2y + z = 0$. First, we need to find the matrix A. To this end, we pick two linearly independent vectors in $x - 2y + z = 0$, say

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}. \tag{4.50}$$

Then we find

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}, \qquad \left(A^T A\right)^{-1} = \frac{1}{6} \cdot \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}. \tag{4.51}$$

Note that A is *not* invertible! From this, the projection matrix follows as

$$P = \frac{1}{6} \cdot \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}. \tag{4.52}$$

Note:
Properties of projection matrices include the following:

1. P is symmetric:

$$P^T = \left(A(A^T A)^{-1} A^T\right)^T = A\left((A^T A)^{-1}\right)^{-1} A = P. \tag{4.53}$$

2. $P^2 = P$:

$$P^2 = A(A^T A)^{-1} A^T A (A^T A)^{-1} A = A(A^T A)^{-1} A = P. \tag{4.54}$$

3. P does not depend on the choice of vectors that make up A:
   This should be surprising given the expression for P. Yet, it is obvious that projection onto a subspace does not depend on the basis of the subspace.

4. $I - P$ is the matrix for projection onto the orthogonal complement of $C(A)$:

$$\vec{b} = P\vec{b} + (I - P) \cdot \vec{b}. \tag{4.55}$$

5. $P\vec{b} = \vec{b}$ if $\vec{b} \in C(A)$:
   This is clear geometrically. Algebraically, it follows by using $\vec{b} = A\vec{c}$. Then we see:

$$A(A^T A)^{-1} A^T \vec{b} = A(A^T A)^{-1} A^T A \vec{c} = A\vec{c} = \vec{b}. \tag{4.56}$$

6. $P\vec{b} = \vec{0}$ if $\vec{b} \in N(A^T)$:
   Again, this is clear geoemtrically. Algebraically, we note that $b \in N(A^T)$ means $A^T\vec{b} = \vec{0}$. Hence
   $$A(A^TA)^{-1}A^T\vec{b} = \vec{0}\,. \tag{4.57}$$
7. The rank of the projection matrix P matches $\dim(C(A))$, which is the rank of A.

4.4 Application: Least square approximation

Motivated by orthogonal complements, we discussed projections in the previous section. Given a point $\vec{b} \in \mathbb{R}^n$ and a linear subspace $S = \mathrm{Span}_{\mathbb{R}}(\vec{a}_1, \dots, \vec{a}_k) \subseteq \mathbb{R}^n$, we have projected $\vec{b}$ onto the point $\vec{p} \in S$ which is closest to $\vec{b}$. In a sense, this projection $\vec{p} \in S$ which best approximates $\vec{b}$ in terms of S.

In this section, we use this very philosophy and apply it to a real real world problem. Namely, suppose we are given points $\vec{b}_1, \dots, \vec{b}_k \in \mathbb{R}^2$ and we want to fit a line to these points, which best describes/approximates these points. Then we will find momentarily, that projections allow us to achieve this very goal.

4.4.1 A first encounter

Example 4.4.1:
Let us consider three points in the plane $\mathbb{R}^2$:
$$\vec{b}_1 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{4.58}$$
We wonder if we can find $C, D \in \mathbb{R}$ such that the line
$$L(C,D) = \left\{ \begin{bmatrix} t \\ C + D \cdot t \end{bmatrix} \mid t \in \mathbb{R} \right\}, \tag{4.59}$$
is closest to these three points. We notice that the following holds true:

- $\vec{b}_1 \in L(C,D)$ iff $C + D \cdot 0 = 6$,
- $\vec{b}_2 \in L(C,D)$ iff $C + D \cdot 1 = 0$,
- $\vec{b}_3 \in L(C,D)$ iff $C + D \cdot 2 = 0$.

We are thus trying to find $\vec{x} = \begin{bmatrix} C \\ D \end{bmatrix} \in \mathbb{R}^2$ which solves $A\vec{x} = \vec{b}$ with
$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}. \tag{4.60}$$
This equation has a solution iff the points $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are contained in a line. In the case at hand, this is not the case and this equation has no solution.

Remark:
Let us reinterpret the task. We consider $C(A)$ and $\vec{b}$. Of course, we can project $\vec{b}$ onto $C(A)$. As in the previous section, we denote this projection by $A\widehat{\vec{x}}$. This is the best approximation of $\vec{b}$ by $C(A)$, in that it minimizes the error vector $\vec{e} = \vec{b} - A\widehat{\vec{x}}$. Therefore, the vector $\widehat{\vec{x}}$ informs us on the straight line which is closest to the points $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$.

Example 4.4.2 (Continuation)**:**
Recall eq. (4.47). It says that the best $\widehat{\vec{x}}$ satisfies the equation $A^T A\widehat{\vec{x}} = A^T\vec{b}$. In the above example we have

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}. \tag{4.61}$$

It follows

$$A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}, \qquad \left(A^T A\right)^{-1} = \frac{1}{6} \cdot \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}, \qquad A^T\vec{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}. \tag{4.62}$$

The unique solution to this equation is thus $C = 5$ and $D = -3$, i.e. $\widehat{\vec{x}} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$. Based on this, we propose that the line

$$L(5, -3) = \left\{ \begin{bmatrix} t \\ 5 - 3 \cdot t \end{bmatrix} \mid t \in \mathbb{R} \right\}, \tag{4.63}$$

best approximates the points

$$\vec{b}_1 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{4.64}$$

Exercise:
Draw an image of the points $\vec{b}_i$ and the line $L(5, -3)$. Compute the error vector for the orthogonal projection of $\vec{b}$ onto $C(A)$.

4.4.2 Why "least square?" – Jacobians and the Hessian

Remark:
To measure the "distance" of a line parametrized by $\vec{x}$ to the given points, we consider

$$l_e(\vec{x}) = \left\langle \vec{b} - A\vec{x}, \vec{b} - A\vec{x} \right\rangle. \tag{4.65}$$

Note that $l_e(\vec{x})$ is minimal for $\vec{x} = \widehat{\vec{x}}$. By minimizing the length of $l_e(\vec{x})$, i.e. minizing a sum of squares, we perform the so-called *least square approximation* to find the line which best describes the points. This is exactly what we achieve with the orthogonal projection.

Example 4.4.3:
Let us continue with the previous example. There we have

$$l_e(\vec{x}) = (C-6)^2 + (C+D)^2 + (C+2D)^2 = 3C^2 - 12C + 6CD + 5D^2 + 36\,. \tag{4.66}$$

So instead of projecting, we could approach this problem by minimizing the function

$$l_e\colon \mathbb{R}^2 \to \mathbb{R}\,,\ (C,D) \mapsto 3C^2 - 12C + 6CD + 5D^2 + 36\,. \tag{4.67}$$

This is achieve by first looking at the Jacobian matrix

$$J\colon \mathbb{R}^2 \to \mathbb{R}^2\,,\ (C,D) \mapsto \begin{bmatrix} \left(\frac{\partial l_e}{\partial C}\right)(C,D) \\ \left(\frac{\partial l_e}{\partial D}\right)(C,D) \end{bmatrix} = \begin{bmatrix} 6C - 12 + 6D \\ 6C + 10D \end{bmatrix}. \tag{4.68}$$

The Jacobian matrix vanishes at all *local* extrema of the function l_e, i.e. at its local minima, local saddle points and local maxima. We notice that the vanishing of J is equivalent to

$$2 \cdot \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} C \\ D \end{bmatrix} = 2 \cdot \begin{bmatrix} 6 \\ 0 \end{bmatrix}. \tag{4.69}$$

This exactly reproduces eq. (4.62)!

Remark:
The analytics does *not guarantee* that the zeros of the Jacobian matrix are (local) minima. The type of the local extremum is found by analysing the so-called *definiteness* of the Hessian matrix. To argue that a local extremum is a global extremum, a further argument is required, which analyses the behaviour of the function away from the local extremum.

Corollary:
The partial derivatives of $l_e(\vec{x})$ vanish iff $A^T A\vec{x} = A^T\vec{b}$.

Example 4.4.4:
The unique solution to *eq.* (4.69) is $C = 5$ and $D = -3$, i.e. $\widehat{\vec{x}} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$. We then have $l_e(\widehat{\vec{x}}) = 6$. To see that this is a minimum, we consider the Hessian matrix

$$\begin{aligned} &H\colon \mathbb{R}^2 \to \mathbb{M}(2\times 2, \mathbb{R})\,, \\ &(C,D) \mapsto \begin{bmatrix} \left(\frac{\partial^2 l_e}{\partial C^2}\right)(C,D) & \left(\frac{\partial^2 l_e}{\partial C \partial D}\right)(C,D) \\ \left(\frac{\partial^2 l_e}{\partial D \partial C}\right)(C,D) & \left(\frac{\partial^2 l_e}{\partial D^2}\right)(C,D) \end{bmatrix}. \end{aligned} \tag{4.70}$$

For $C = 5$ and $D = -3$ we have

$$H(C,D) = \begin{bmatrix} 6 & 6 \\ 6 & 10 \end{bmatrix}. \tag{4.71}$$

We will learn later in the course that this matrix is indeed positive definite, which proves that $\widehat{\vec{x}}$ is a *local* minimum. Since we found that $l_e(\vec{x})$ in eq. (4.66) has a unique local extremum – namely a minimum – there are no other local minima or maxima. To argue that $\widehat{\vec{x}}$ is the *global* minimum, it remains to compare $l_e(\widehat{\vec{x}}) = 6$ to

$$\lim_{C,D\to\infty} l_e(\vec{x}) = \infty \,. \tag{4.72}$$

It follows that indeed, we have found the global minimum of eq. (4.66). This gives full justification towards calling the $L(5,-3) = \left\{ \begin{bmatrix} t \\ 5 - 3 \cdot t \end{bmatrix} \mid t \in \mathbb{R} \right\}$ the one which best approximates the points

$$\vec{b}_1 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \tag{4.73}$$

in the sense of a *least square approximation.*

4.4.3 Generalization

This situation generalizes. In experiments, we often measure a quantity over and over again. For example, we could have 100 points in $\mathbb{R}^2$, which we intend to explain by one shifted (straight) line. In general, those measured points are not located on a perfectly straight line. It is then our task to find the straight line closest to all these points.

Construction 4.4.1:
To fit $m \geq 2$ points $\{p_i = (t_i, b_i)\}_{1\leq i\leq m} \in \mathbb{R}^2$ to a straight line

$$L(C,D) = \left\{ \begin{bmatrix} t \\ C + D \cdot t \end{bmatrix} \mid t \in \mathbb{R} \right\}, \tag{4.74}$$

we are looking at the equations

$$\begin{aligned} C + Dt_1 &= b_1 \,, \\ C + Dt_2 &= b_2 \,, \\ &\vdots \\ C + Dt_m &= b_m \,. \end{aligned} \tag{4.75}$$

Equivalently, we are looking at

$$A\vec{x} = \vec{b}, \qquad \vec{x} = \begin{bmatrix} C \\ D \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}. \tag{4.76}$$

The closest – in the sense of a least square approximation – line minimizes the length of the error vector, i.e. $l_e(\vec{x}) = \left\langle \vec{b} - A\vec{x}, \vec{b} - A\vec{x} \right\rangle$. Equivalently, it solves $A^T A\widehat{\vec{x}} = A^T\vec{b}$. This we can work out explicitly. Namely

$$A^T A = \begin{bmatrix} m & \sum_{i=1}^m t_i \\ \sum_{i=1}^m t_i & \sum_{i=1}^m t_i^2 \end{bmatrix}, \qquad A^T\vec{b} = \begin{bmatrix} \sum_{i=1}^m b_i \\ \sum_{i=1}^m t_i b_i \end{bmatrix}. \tag{4.77}$$

In a specific problem, i.e. fitting *given* points to a line, the numbers t_i and b_i are known. We can then work out these matrices and find the best solution as

$$\widehat{\vec{x}} = (A^T A)^{-1} \cdot A^T\vec{b}. \tag{4.78}$$

The error vector $\left\langle A\widehat{\vec{x}} - \vec{b}, A\widehat{\vec{x}} - \vec{b} \right\rangle$ gives a measure for how good this fit is. The smaller, the better the line describes the given data. In particular, if all points are on a line, then $\left\langle A\widehat{\vec{x}} - \vec{b}, A\widehat{\vec{x}} - \vec{b} \right\rangle$ vanishes.

Remark:
Note that A is not invertible, so $(A^T A)^{-1}$ cannot be written as $A^{-1} \cdot (A^T)^{-1}$ because in general neither A nor A^T are invertible.

Construction 4.4.2:
This strategy is by no means limited to fitting straight lines. For example, suppose that we are given $m \geq 3$ points $\vec{b}_i = (t_i, b_i)$. Then we can fit a parabola

$$P(C, D, E) = \left\{ \begin{bmatrix} t \\ C + Dt + Et^2 \end{bmatrix} | \, t \in \mathbb{R} \right\}, \tag{4.79}$$

to these points. We are then looking at

$$C + Dt_1 + Et_1^2 = b_1\,, \tag{4.80}$$
$$C + Dt_2 + Et_2^2 = b_2\,, \tag{4.81}$$
$$\vdots \tag{4.82}$$
$$C + Dt_m + Et_m^2 = b_m\,. \tag{4.83}$$

Equivalently, we are looking at

$$A\vec{x} = \vec{b}, \qquad \vec{x} = \begin{bmatrix} C \\ D \\ E \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \\ 1 & t_m & t_m^2 \end{bmatrix}. \tag{4.84}$$

4.5 Orthonormal bases and Gram-Schmidt

4.5.1 Orthonormal bases

Note:
So far, when we discussed projections P onto a linear subspace $S \subseteq \mathbb{R}^n$ we have considered a basis of S, i.e. $S = \mathrm{Span}_{\mathbb{R}}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k)$ and $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$ is a linearly independent family of vectors. The key difficulty in finding the projection matrix was to invert $A^T A$. We will now discuss basis in which this task is trivial. Hence, in these basis the computation of the projection matrix simplifies a lot.

Remark:
Recall that the projection $\vec{p}$ of $\vec{b} \in \mathbb{R}^n$ onto $S = \mathrm{Span}_{\mathbb{R}}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k) \subseteq \mathbb{R}^n$ is given by

$$\vec{p} = P\vec{b}\,, \qquad P = A \cdot \left(A^T A\right)^{-1} \cdot A^T \in \mathbb{M}(n \times n, \mathbb{R})\,. \tag{4.85}$$

The expression for P simplifies provided that $A^T A = I$. We may thus wonder for which basis of the subspace S this condition is satisfied. Recall that

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_k \\ | & | & & | \end{bmatrix} \in \mathbb{M}(n \times k, \mathbb{R})\,. \tag{4.86}$$

Hence, the entries of $A^T A$ are the inner products of the basis vectors $\vec{a}_i$ (w.r.t. to $\langle \cdot, \cdot \rangle_{\mathrm{Std}}$). We conclude that $A^T A = I$ if and only if for all $1 \leq i, j \leq k$ with $i \neq j$ it holds

$$\langle \vec{a}_i, \vec{a}_i \rangle_{\mathrm{std}} = \vec{a}_i^T \vec{a}_i = 1\,, \qquad \langle \vec{a}_i, \vec{a}_j \rangle_{\mathrm{std}} = \vec{a}_i^T \vec{a}_j = 0\,. \tag{4.87}$$

This gives rise to the following notions.

Definition 4.5.1 (Orthogonal basis)**:**
Be V an inner product space and $S \subseteq V$ a linear subspace with basis $\mathcal{B}_S = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$. The basis $\mathcal{B}_S$ is said to be orthogonal if for all $1 \leq i, j \leq k$ with $i \neq j$ it holds

$$\langle \vec{a}_i, \vec{a}_j \rangle = 0\,. \tag{4.88}$$

Definition 4.5.2 (Orthonormal basis)**:**
Be V an inner product space and $S \subseteq V$ a linear subspace with basis $\mathcal{B}_S = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$. The basis $\mathcal{B}_S$ is said to be orthonormal if for all $1 \leq i, j \leq k$ with $i \neq j$ it holds

$$\langle \vec{a}_i, \vec{a}_i \rangle = 1\,, \qquad \langle \vec{a}_i, \vec{a}_j \rangle = 0\,. \tag{4.89}$$

4.5.2 Orthogonal matrices

Corollary:
Be $Q \in \mathbb{M}(m \times n, \mathbb{R})$. Then the following holds true:

- The columns of Q are orthonormal iff $Q^TQ = I$.
- If Q is square, then the columns of Q are orthonormal iff $Q^T = Q^{-1}$.

Definition 4.5.3 (Orthogonal matrix)**:**
A matrix $Q \in \mathbb{M}(n \times n, \mathbb{R})$ with $Q^TQ = I$ is termed an *orthogonal matrix.*

Corollary:
The rows of an orthogonal matrix are orthonormal.

Example 4.5.1:
All rotation and permutation matrices are orthogonal. For example:

- Rotation matrix: $R = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ $(\alpha \in \mathbb{R})$.
- Permutation matrix: $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Exercise:
Verify that these matrices are orthogonal.

Note:
Another important class of orthogonal matrices are reflections.

Construction 4.5.1:
Consider a **unit** vector $\vec{u} \in \mathbb{R}^n$. The reflection matrix about $\vec{u}$ is given by

$$Q = I - 2\vec{u}\vec{u}^T \in \mathbb{M}(n \times n, \mathbb{R})\,. \tag{4.90}$$

Example 4.5.2:
Consider $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\,. \tag{4.91}$$

This is indeed the expected matrix for reflection at the x-axis.

Corollary:
Consider a **unit** vector $\vec{u} \in \mathbb{R}^n$ and the reflection matrix

$$Q = I - 2\vec{u}\vec{u}^T \in \mathbb{M}(n \times n, \mathbb{R})\,. \tag{4.92}$$

Then Q has the following properties:

1. Q is symmetric.
2. Q is orthogonal.

Proof

1. Symmetry: $Q^T = \left(I - 2\vec{u}\vec{u}^T\right)^T = I - 2\vec{u}\vec{u}^T = Q$.
2. Orthogonality: $Q^TQ = \left(I - 2\vec{u}\vec{u}^T\right) \cdot \left(I - 2\vec{u}\vec{u}^T\right) = I - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T = I = QQ^T$.

This completes the argument. ■

4.5.3 Applications of orthogonal matrices

Note:
A base change by an orthogonal matrix has a very important property – it preserves the (standard) inner product, and thereby lengths and angles! Here is the proof.

Claim 16:
Consider the inner product space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\text{Std}})$ and orthogonal $Q \in \mathbb{M}(n \times n, \mathbb{R})$. Then for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ it holds

$$\langle Q\vec{x}, Q\vec{y}\rangle_{\text{Std}} = \langle \vec{x}, \vec{y}\rangle_{\text{Std}} \ . \tag{4.93}$$

Proof
By definition of the standard inner product in $\mathbb{R}^n$ it holds

$$\langle Q\vec{x}, Q\vec{y}\rangle_{\text{Std}} = (Q\vec{x})^T \cdot (Q\vec{y}) = \vec{x}^T(Q^TQ)\vec{y} = \vec{x}^T\vec{y} = \langle \vec{x}, \vec{y}\rangle_{\text{Std}} \ . \tag{4.94}$$

In the second equality we have used the defining property of the orthogonal matrix Q, namely $Q^TQ = I$. ∎

Note:
We will have more to say about base changes with orthogonal matrices later in the course when we discuss the spectral theorem.

Construction 4.5.2 (Projections from orthonormal basis)**:**
Let us look at the projection onto a linear subspace $S = \text{Span}_{\mathbb{R}}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k) \subseteq \mathbb{R}^n$. However, instead of considering an arbitrary basis of S, we wish to work with an orthonormal basis. This is always possible by the Gram-Schmidt procedure, which we will discuss momentarily. For the time being, suffice it to assume the existence of an orthonormal basis of S, namely

$$S = \text{Span}_{\mathbb{R}}(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k)\,, \qquad \{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k\} \text{ orthonormal basis of } S \ . \tag{4.95}$$

This replaces the matrix A by

$$Q = \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \ldots & \vec{q}_k \\ | & | & & | \end{bmatrix} \in \mathbb{M}(n \times k, \mathbb{R})\,. \tag{4.96}$$

In particular, A^TA becomes $Q^TQ = I$ and the projection formula simplies enormously:

$$\vec{p} = Q\left(Q^TQ\right)^{-1} Q^T\vec{b} = QQ^T\vec{b}\,. \tag{4.97}$$

In particular, $P = QQ^T$ and there is no matrix left to invert. This is the key simplification that we achieve with an orthonormal basis.

Example 4.5.3:
Let us exemplify this with the projection matrix onto the plane in $\mathbb{R}^3$ given by $x-2y+z=0$. We already discussed this in example 4.3.3. This time, we will pick an orthonormal basis of this plane. Namely

$$\vec{q}_1 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{4.98}$$

Then we find

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.99}$$

As before, the projection matrix follows as

$$P = \frac{1}{6} \cdot \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}. \tag{4.100}$$

4.5.4 The Gram-Schmidt procedure and QR decompositions

Remark:
Given the significance of orthonormal basis and orthogonal matrices, we can ask two important questions:

- Does every linear subspace S of an inner product space have an orthonormal basis?
- If yes, how do we find such a basis?

This is answered by the Gram-Schmidt procedure. It provides an algorithmic procedure to find an orthonormal basis. In particular, indeed every linear subspace S of an inner product space admits an orthonormal basis.

Construction 4.5.3:
Let us exemplify the Gram-Schmidt procedure by looking at a 3-dimensional linear subspace S, i.e. $S = \text{Span}_{\mathbb{R}}(\vec{a}, \vec{b}, \vec{c})$. We first wish to construct three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ which are an orthogonal basis of S. Subsequently, we will normalize them to form an orthonormal basis of S. For the first step, we perform the following tasks:

1. Take $\vec{A} = \vec{a}$.
2. Next consider $\vec{b}$ and subtract its projection along $\vec{A}$. This gives

$$\vec{B} = \vec{b} - \frac{\left\langle \vec{A}, \vec{b} \right\rangle}{\left\langle \vec{A}, \vec{A} \right\rangle} \cdot \vec{A}. \tag{4.101}$$

In particular $\left\langle \vec{A}, \vec{B} \right\rangle = 0$.

3. Now consider $\vec{c}$. To construct $\vec{C}$ which is orthogonal to both $\vec{A}$ and $\vec{B}$ we subtract the projections of $\vec{c}$ along $\vec{A}$ and along $\vec{B}$:
$$\vec{C} = \vec{c} - \frac{\langle \vec{A}, \vec{c} \rangle}{\langle \vec{A}, \vec{A} \rangle} \cdot \vec{A} - \frac{\langle \vec{B}, \vec{c} \rangle}{\langle \vec{B}, \vec{B} \rangle} \cdot \vec{B}\,. \tag{4.102}$$
Indeed, $\langle \vec{A}, \vec{C} \rangle = \langle \vec{B}, \vec{C} \rangle = 0$.

Finally, all that is left is to normalize these vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, i.e. we devide $\vec{A}$, $\vec{B}$ and $\vec{C}$ by their lengths.

Note:
This generalizes to any finite family of vectors. For example, if there was also a vector $\vec{d}$ above, then we would form $\vec{D}$ by subtracting from $\vec{d}$ the projections along $\vec{A}$, $\vec{B}$ and $\vec{C}$.

Example 4.5.4:
In going back to example 4.3.3 once more, i.e. the projection matrix onto the plane in $\mathbb{R}^3$ given by $x - 2y + z = 0$. We have taken a basis
$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \tag{4.103}$$
previously. We construct an orthonormal basis by first computing:
$$\vec{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{B} = \vec{a}_2 - \frac{\langle \vec{A}, \vec{a}_2 \rangle}{\langle \vec{A}, \vec{A} \rangle} \cdot \vec{A} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{4.104}$$
Now normalize these vectors, then we find
$$\vec{q}_1 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \tag{4.105}$$
This is exactly the basis used in example 4.5.3.

Example 4.5.5:
Here is another example. Consider the following basis of $\mathbb{R}^3$:
$$\vec{a} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \qquad \vec{c} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}. \tag{4.106}$$
By applying the Gram-Schmidt procedure, we find
$$\vec{q}_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \vec{q}_2 = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \qquad \vec{q}_3 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{4.107}$$

Recall that the original basis A and the new basis Q are related by a base change. Explicitly, it holds here

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{18} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix} = QR\,. \tag{4.108}$$

This pattern holds more generally.

Corollary:
Consider the inner product space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$. Then, starting with linearly independent vectors $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$, the Gram-Schmidt procedure constructs orthonormal vectors $\vec{q}_1, \ldots, \vec{q}_n \in \mathbb{R}^n$. Let A be the matrix with columns $\vec{a}_i$ and Q the matrix with columns $\vec{q}_i$. Then $R = Q^T A$ is an upper triangular matrix.

Corollary (QR decomposition)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then $A = QR$ where $Q \in \mathbb{M}(n \times n, \mathbb{R})$ is an orthogonal matrix and R an upper triangular matrix. This is a QR decomposition/factorization of A.

Note:
Similarly to the LU-factorization, the QR-factorization is key to efficiently perform a computation in linear algebra, in particular projections. Let us illustrate the use of the QR-factorization for the least square approximation. Recall that this amounts to solving

$$A^T A\widehat{\vec{x}} = A^T\vec{b}\,. \tag{4.109}$$

Now use $A = QR$. Then we find

$$R^T R\widehat{\vec{x}} = R^T Q^T QR\widehat{\vec{x}} = (QR)^T QR\widehat{\vec{x}} = (QR)^T\vec{b} = R^T Q^T\vec{b}\,. \tag{4.110}$$

Since R^T is invertible (it is a base change matrix), we conclude that this is equivalent to

$$R\widehat{\vec{x}} = Q^T\vec{b}\,. \tag{4.111}$$

Since R is upper triangular, we can use back-substitution to solve this equation efficiently and fast. The real cost are the operations in the Gram-Schmidt procedure, respectively the computation of the QR-decomposition.

4.6 Application: Fourier series

4.6.1 An infinite dimensional vector space

Note:
We will now, for the only time in this course, leave the terrain of finite-dimensional vector spaces. Namely, we will use our knowledge about orthogonality to perform linear algebra in two infinite dimensional vector spaces. Generally speaking, we have to be careful which results generalize from finite-dimensional linear algebra to infinite-dimensional linear algebra. For example, we might not be able to write matrices and vectors, but the ideas about orthogonality still do apply.

Definition 4.6.1:
We consider the vector space

$$V = \{f\colon [0,2\pi] \to \mathbb{R} \,|\; f \text{ measurable and square integrable}\}\ , \tag{4.112}$$

that is $f\colon [0,2\pi] \to \mathbb{R}$ belongs to V if and only if the following exists and is finite

$$(f,f) = \int_0^{2\pi} |f(x)|^2 \, dx\,. \tag{4.113}$$

Remark:
There is nothing special about our integrals ranging from 0 to 2π. We can study square integrable spaces by integrating over $[0,1]$ or $(-\infty,\infty)$ just as well.

Note:
The function $f\colon [0,2\pi] \to \mathbb{R}\,,\ x \mapsto f(x) = \sin(x)$ satisfies $(f,f) = \pi$, i.e. $f \in V$. This follows as follows. We first use $\sin^2(x) + \cos^2(x) = 1$ to see

$$(f,f) = \int_0^{2\pi} \sin(x)^2 dx = \int_0^{2\pi} \left(1 - \cos(x)^2\right) dx = 2\pi - \int_0^{2\pi} \cos(x)^2 dx\,. \tag{4.114}$$

Hence

$$\int_0^{2\pi} \sin(x)^2 dx + \int_0^{2\pi} \cos(x)^2 dx = 2\pi\,. \tag{4.115}$$

By periodicity of $\sin(x)$ and $\cos(x)$, we also have

$$\int_0^{2\pi} \sin(x)^2 dx = \int_0^{2\pi} \cos(x)^2 dx\,. \tag{4.116}$$

Hence

$$\int_0^{2\pi} \sin(x)^2 dx + \int_0^{2\pi} \sin(x)^2 dx = 2\pi\,. \tag{4.117}$$

This implies $(f,f) = \pi$.

Definition 4.6.2:
We define an inner product for $f,g \in V$ by

$$(f,g) = \int_0^{2\pi} f(x)g(x)dx\,. \tag{4.118}$$

Consequence:
Any two $f, g \in V$ have finite length. But then, we also have

$$\begin{aligned}(f+g, f+g) &= (f,f) + 2\,(f,g) + (g,g) \\ &\leq (f,f) + 2\sqrt{(f,f)}\cdot\sqrt{(g,g)} + (g,g) < \infty\,.\end{aligned} \tag{4.119}$$

This is the famous *Schwarz inequality*:

$$|\,(f,g)\,|^2 \leq (f,f)\cdot(g,g)\ . \tag{4.120}$$

Hence, if $f, g \in V$, then also $f + g \in V$.

4.6.2 Interlude: Hilbert spaces in a nutshell

In honour of the German mathematician David Hilbert, inner product spaces are also termed a *Prähilbertraum* – pre-Hilbert space. Hence, $(V, (\cdot,\cdot))$ is a pre-Hilbert space. It actually has more structure. Namely, the inner product induces a length for all vectors – also called a *norm*. A pre-Hilbert space in which every Cauchy sequence converges w.r.t. to this norm is called a Hibert space. The above space $(V, (\cdot,\cdot))$ satisfies this "completeness relation" and is therefore a Hilbert space.

Hilbert spaces are of fundamental interest in quantum mechanics and quantum field theory. For example, in quantum mechanics, we have the following dictionary between mathematics and physics:

Physics	Mathematics
State of a quantum system	(Equivalence class of) vector in Hilbert space $\mathcal{H}$
Measurement on quantum system	(Special) operator ($\sim$ linear map) $\mathcal{H} \to \mathcal{H}$
Possible measurement values	Eigenvalues of these (special) operators

We will discuss eigenvalues at length in chapter 6.

4.6.3 The Fourier series

Lemma 4.6.1:
For any two $x, y \in \mathbb{R}$ it holds $\sin(x)\cdot\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$.

Proof
We use a central property of the exponential function, namely

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}\,, \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}\,. \tag{4.121}$$

Then it follows

$$\begin{aligned}\sin(x)\cdot\sin(y) &= \frac{1}{4i^2}\cdot\left[e^{i(x+y)} + e^{-i(x+y)} - e^{i(x-y)} - e^{-i(x-y)}\right] \\ &= \frac{1}{2}\cdot\left(\cos(x-y) - \cos(x+y)\right)\ .\end{aligned} \tag{4.122}$$

This completes the argument. ■

Consequence:
The function $\sin(m \cdot x)$ and $\sin(n \cdot x)$ are orthogonal in V iff $m \neq n$.

Proof
We consider

$$(\sin(m \cdot x), \sin(n \cdot x)) = \frac{1}{2} \cdot \int_0^{2\pi} \cos\left((m-n)x\right) - \cos\left((m+n)x\right) dx\,. \tag{4.123}$$

This vanishes iff $m \neq n$. ■

Note:
Similarly, one can show that for any two $x, y \in \mathbb{R}$ it holds

$$\cos(x) \cdot \cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)\,. \tag{4.124}$$

As a consequence, it follows that $\cos(m \cdot x)$ and $\cos(n \cdot x)$ are orthogonal in V iff $m \neq n$.

Consequence:
Consider the set

$$\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots\}\,. \tag{4.125}$$

Any two distinct functions in this list are orthogonal. It would be nice if we could construct a nice basis of the vector space

$$V = \{f\colon [0, 2\pi] \to \mathbb{R} \mid f \text{ measurable and square integrable}\}\,, \tag{4.126}$$

from them. This is essentially the idea behind the Fourier series.

Definition 4.6.3:
The *Fourier series* of (a function) $f \in V$ is its expansion

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx)\right). \tag{4.127}$$

Note:
As $\sin(x)$ and $\cos(x)$ are 2π-periodic, our function f must be 2π-periodic as well.

4.6.4 The Fourier inversion theorem

Remark:
Let us turn the tables around. Given a choice of coefficients a_i, we may wonder if the resulting series is the Fourier series of a function $f \in V$. As preparation for this, let us introduce the vector space of these coefficients.

Definition 4.6.4:
We consider the vector space

$$W = \left\{ \vec{v} = (v_1, v_2, v_3, \dots) \mid v_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} v_i^2 < \infty \right\} . \tag{4.128}$$

We define an inner product for $\vec{v}, \vec{w} \in W$ by $\vec{v} \cdot \vec{w} = \sum_i v_i w_i$.

Example 4.6.1:
The vector $\vec{v} = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \dots\right)$ has

$$\vec{v} \cdot \vec{v} = 1 + \frac{1}{2} + \frac{1}{4} + \dots = \sum_{k=0}^{\infty} \frac{1}{2^k} . \tag{4.129}$$

This is a famous geometric series and it is a famous result that

$$\vec{v} \cdot \vec{v} = 1 + \frac{1}{2} + \frac{1}{4} + \dots = \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1 - \frac{1}{2}} = 2 . \tag{4.130}$$

Hence, $\vec{v} \in W$.

Note:
W is another example of a *Hilbert spaces*. If $\vec{v}, \vec{w} \in W$, then it again follows from the *Schwarz inequality* that also $v + w \in W$.

Remark:
We now have to answer the question, which Fourier series are actually honest functions. To this end, let us compute the length of a function $f \in V$ from its expansion. For this, let us use the orthonormal basis

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \dots \right\} . \tag{4.131}$$

Then we have

$$\begin{aligned} (f, f) &= \int_0^{2\pi} \left(\frac{a_0}{\sqrt{2\pi}} + \frac{a_1}{\sqrt{\pi}} \cos(x) + \frac{b_1}{\sqrt{\pi}} \sin(x) + \dots \right)^2 dx \\ &= \int_0^{2\pi} \left(\frac{a_0^2}{2\pi} + \frac{a_1^2}{\pi} \cos^2(x) + \frac{b_1^2}{\pi} \sin^2(x) + \frac{a_2^2}{\pi} \cos^2(2x) + \dots \right)^2 dx \\ &= a_0^2 + a_1^2 + b_1^2 + a_2^2 + b_2^2 + \dots . \end{aligned} \tag{4.132}$$

This implies $f \in V$ if and only if its vector of coefficients belongs to W.

Note:
Given $f \in V$, we term the vector

$$\vec{v}(f) = (a_0, a_1, b_1, a_2, b_2, \dots) \in W\,, \tag{4.133}$$

formed from the coefficients of the Fourier series of f, the *Fourier transform* of f. In this sense, we have just found a 1-to-1 correspondance between function $f \in V$ and their Fourier transforms. Put differently, for all $f \in V$ it is possible to recover the function f uniquely from its Fourier transform. This is the so-called *Fourier inversion theorem.* Let us mention again, that there is nothing special about our integrals ranging over $[0, 2\pi]$. It is for example possible to generalize to $(-\infty, \infty)$, which is common to formulate the Fourier inversion theorem.

4.6.5 Computing the Fourier series – orthogonality to the rescue

Example 4.6.2:
Let us consider the function

$$f\colon [0, 2\pi] \to \mathbb{R}\,,\; x \mapsto f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ -1 & \pi \leq x \leq 2\pi \end{cases}\,. \tag{4.134}$$

and compute its Fourier series:

$$f(x) = a_0 + \sum_{k\geq 1} a_k \cos(kx) + \sum_{k\geq 1} b_k \sin(kx)\,. \tag{4.135}$$

We can identity the coefficients by apply our knowledge of inner products. Namely, in order to find a_k, we simply compute the inner product with $\cos(kx)$:

$$\int_0^{2\pi} f(x)\cos(kx)dx = \int_0^{2\pi} \left(a_0 + \sum_{k\geq 1} a_k \cos(kx) + \sum_{k\geq 1} b_k \sin(kx) \right) dx\,. \tag{4.136}$$

By orthogonality, we then find

$$\begin{aligned} \int_0^{2\pi} f(x)\cos(kx)dx = a_k \int_0^{2\pi} \cos^2(kx)dx = \frac{1}{2}\cdot a_k \cdot 2\pi\,, \\ \Leftrightarrow \qquad a_k = \frac{1}{\pi}\cdot \int_0^{2\pi} f(x)\cos(kx)dx\,. \end{aligned} \tag{4.137}$$

Similarly, we can find

$$b_k = \frac{1}{\pi}\cdot \int_0^{2\pi} f(x)\sin(kx)dx\,, \qquad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx\,. \tag{4.138}$$

So in particular, a_0 is the average value of f on $[0, 2\pi]$. Let us apply this to the function in eq. (4.134). Since this function is odd, the cosine terms in the Fourier series vanish. Moreover, we readily confirm $b_k = \frac{4}{\pi k}$. Thus

$$a_0 + \sum_{k\geq 1} a_k \cos(kx) + \sum_{k\geq 1} b_k \sin(kx) = \frac{4}{\pi} \cdot \left[\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \ldots \right] . \tag{4.139}$$

Note that this series vanishes at $x = 0$, which is different from $f(0) = 1$! This is because $f(x)^2$ is not continous at $x = 0$!

Note:
In general, you want to be careful to compare a function to a series expansion. Another example of this sort is the Taylor expansion. For "well-behaved" functions, these series expansions coincide with the original functions. But this is not true in general.

Remark:
The above Fourier series for the function f in eq. (4.134) is reliable at $x = \frac{\pi}{2}$. Therefore, we find

$$f\left(\frac{\pi}{2}\right) = 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) . \tag{4.140}$$

This is equivalent to the famous *Leibniz formula for* π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots . \tag{4.141}$$

Remark:
We have come a long way in terms of computing π using series. Here is an example:

$$\frac{4}{\pi} = \frac{1}{882} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(4^n n!)^4} \cdot \frac{(4n)!}{882^{2n}} \cdot (1123 + 21460 \cdot n) . \tag{4.142}$$

Note:
Let us conclude this discussion, by analysing the Fourier coefficient computation with our knowledge about inner products. Given an orthonormal basis $\vec{v}_1, \ldots, \vec{v}_n$ of an inner product space $(V, \langle\cdot,\cdot\rangle)$, we can express any $\vec{v} \in V$ as

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n . \tag{4.143}$$

The coefficients c_i are simply given by

$$\langle \vec{v}_i, \vec{v} \rangle = \sum_{j=1}^{n} c_j \langle \vec{v}_i, \vec{v}_j \rangle = c_i . \tag{4.144}$$

In our computation of Fourier series, we did exactly the same thing. Namely, for

$$f(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{k\geq 1} \frac{a_k \cos(kx)}{\sqrt{\pi}} + \sum_{k\geq 1} \frac{b_k \sin(kx)}{\sqrt{\pi}} , \tag{4.145}$$

we computed for $k \geq 1$ the inner products

$$a_k = (f(x), \cos(kx)) , \qquad b_k = (f(x), \sin(kx)) . \tag{4.146}$$

4.7 Interlude: The Hesse normal form

4.7.1 Generalities

The Hesse normal form is a particulary convenient way to describe an affine plane in $\mathbb{R}^3$. It has many applications in analytic geometry, as it easily allows to determine the distance of points from this affine plane, project onto that plane etc.

Definition 4.7.1 (Affine plane)**:**
To $\vec{x}_0 \in \mathbb{R}^3$ and $\vec{a}, \vec{b} \in \mathbb{R}^3 \setminus \{\vec{0}\}$ we associate the affine plane

$$S(\vec{x}_0, \vec{a}, \vec{b}) = \left\{ \vec{x}_0 + \mu\vec{a} + \nu\vec{b} \,|\, \mu, \nu \in \mathbb{R} \right\} . \tag{4.147}$$

Note:
The interpretation of the quantites in this expression is as follows:

- $\vec{x}_0$ is a vector that points from the origin $\vec{0}$ to a point in $S(\vec{x}_0)$.
- $\vec{a}, \vec{b}$ are two vectors that indicate the orientation of the affine plane.

Example 4.7.1:
Take the following:

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \qquad \vec{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{4.148}$$

Then

$$S(\vec{x}_0, \vec{a}, \vec{b}) = \left\{ \vec{x} \in \mathbb{R}^3 \,\middle|\, x_1 = 1 \right\} . \tag{4.149}$$

In particular, $\vec{a}, \vec{b} \notin S(\vec{x}_0, \vec{a}, \vec{b})$! Those two vectors indicate the orientation of the affine plane, but we must not forget the offset from the origin given by $\vec{x}_0$. This is exactly what $\vec{a}, \vec{b}$ are missing. However,

$$\{\vec{x}_0 + \vec{a}, \vec{x}_0 + \vec{b}\} \subset S(\vec{x}_0, \vec{a}, \vec{b}) . \tag{4.150}$$

Consequence:
A vector $\vec{n} \in \mathbb{R}^3$ is orthogonal to $S(\vec{x}_0, \vec{a}, \vec{b})$ iff

$$\langle \vec{a}, \vec{n} \rangle_{\text{std}} = \left\langle \vec{b}, \vec{n} \right\rangle_{\text{std}} = 0 . \tag{4.151}$$

If we demand that $\vec{n}$ has length 1, then there are exactly two vectors which satisfy this condition ("up/down" or "left/right"or "front/back").

Example 4.7.2:
Again take

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.152}$$

Then $S(\vec{x}_0, \vec{a}, \vec{b}) = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = 1\}$. It follows that $\vec{n} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Consequence:
Let $\vec{n}$ be one of the two normalized normal vectors to $S(\vec{x}_0, \vec{a}, \vec{b})$ and $\vec{x} \in S(\vec{x}_0, \vec{a}, \vec{b})$. Then we can write

$$\vec{x} = \vec{x}_0 + \mu\vec{a} + \nu\vec{b}\,, \tag{4.153}$$

for suitable $\mu, \nu \in \mathbb{R}$. In particular, it follows

$$\langle \vec{n}, \vec{x} \rangle_{\text{std}} = \langle \vec{n}, \vec{x}_0 \rangle_{\text{std}} := d\,. \tag{4.154}$$

By slightly extending this argument, we obtain the following.

Corollary:
$\vec{x} \in S(\vec{x}_0, \vec{a}, \vec{b})$ iff $\langle \vec{n}, \vec{x} \rangle_{\text{std}} - d = 0$ where

- $\vec{n}$ is one of the normalized normal vectors to $S(\vec{x}_0, \vec{a}, \vec{b})$,
- $d = \langle \vec{n}, \vec{x}_0 \rangle_{\text{std}}$.

The representation

$$S(\vec{x}_0, \vec{a}, \vec{b}) = \left\{ \vec{x} \in \mathbb{R}^3 \middle| \, \langle \vec{n}, \vec{x} \rangle_{\text{std}} - d = 0 \right\}\,, \tag{4.155}$$

is known as the *Hesse normal form*.

Example 4.7.3:
Again take

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.156}$$

Then $S(\vec{x}_0, \vec{a}, \vec{b}) = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = 1\}$. It follows that $\vec{n} = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Consequently,

$$d = \langle \vec{x}_0, \vec{n} \rangle_{\text{std}} = \pm 1\,. \tag{4.157}$$

It follows that

$$S(\vec{x}_0, \vec{a}, \vec{b}) = \left\{ \vec{x} \in \mathbb{R}^3 \middle| \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x} \right\rangle_{\text{std}} - 1 = 0 \right\} \tag{4.158}$$

$$= \left\{ \vec{x} \in \mathbb{R}^3 \middle| \left\langle \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \vec{x} \right\rangle_{\text{std}} + 1 = 0 \right\}. \tag{4.159}$$

Of course, those two expressions just differ by a negative sign. Still, it would be desirable to find a unique representation.

Convention:
If $\vec{x}_0 \neq \vec{0}$, then $d \neq 0$. As we want to interpret d as the distance from the origin, we impose the condition $d > 0$. This then picks one the normalized normal vectors $\vec{n}$ and leads to a unique expression.

Note:
If $\vec{x}_0 = \vec{0}$, then $d = 0$ and no preferred choice of normalized normal vector $\vec{n}$ exists.

Construction 4.7.1:
Before we discuss a possible application of the Hesse normal form towards computer vision, let us briefly discuss orthogonal projections. To this end, consider $\vec{x} \in \mathbb{R}^3$. We want to find the orthogonal projection $\vec{p} \in S(\vec{x}_0, \vec{a}, \vec{b})$ of $\vec{x}$. This amounts to finding $\lambda \in \mathbb{R}$ such that

$$\vec{x} = \vec{p} + \lambda \vec{n}. \tag{4.160}$$

Since $\vec{p} \in S(\vec{x}_0, \vec{a}, \vec{b})$, it obeys the Hesse normal form. Hence:

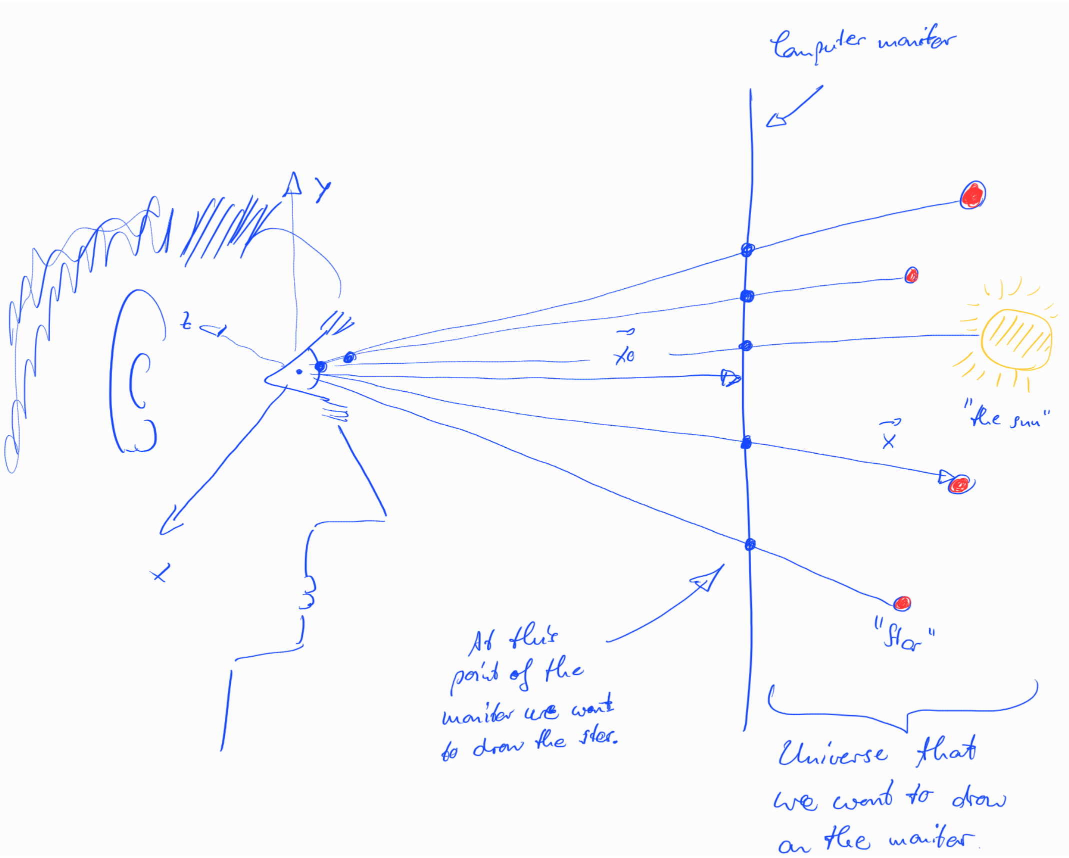
$$0 = \langle \vec{n}, \vec{p} \rangle_{\text{std}} - d = \langle \vec{n}, \vec{x} - \lambda \vec{n} \rangle_{\text{std}} - d = \langle \vec{n}, \vec{x} \rangle_{\text{std}} - \lambda - d. \tag{4.161}$$

This implies $\lambda = \langle \vec{n}, \vec{x} \rangle_{\text{std}} - d$. Or equivalently, we have

$$\vec{p} = \vec{x} - \langle \vec{n}, \vec{x} \rangle_{\text{std}} \cdot \vec{n} + d\vec{n}. \tag{4.162}$$

4.7.2 Hesse normal form meets computer vision

Suppose our task is to draw an image of the universe on a computer monitor. Then, in order to properly represent the positions of all stars and planets, we can sketch the task as follows:



Hence, we take one eye (or the center of both eyes) of the user, who sits in front of the computer, as the center of our coordinate system. The monitor, assumed flat for this simple approximation, plays the role of the affine plane $S(\vec{x}_0, \vec{a}, \vec{b})$. Its orientation is specified by the vectors $\vec{a}$, $\vec{b}$ and $\vec{x}_0$ parametrizes the relative position of the user and the monitor. In particular, is specifies the distance d of the user from the monitor.

If we are given the 3-dimensional coordinates $\vec{x}$ of a star, we want to find coordinates (μ, ν) on the monitor, at which we should draw this star to convey a 3-dimensional impression to the user.[1]

1. We intersect the line

$$L(\vec{x}) = \{\lambda \vec{x} | \lambda \in \mathbb{R}\} , \tag{4.163}$$

with the plane $S(\vec{x}_0, \vec{a}, \vec{b})$. This is achieved most easily by use of the Hesse normal form. Then, the intersection point is quantified by

$$\langle \lambda \vec{x}, \vec{n} \rangle_{\text{std}} - d = 0 \,. \tag{4.164}$$

This is equivalent to $\lambda = \frac{d}{\langle \vec{n}, \vec{x} \rangle_{\text{std}}}$.This expression diverges if $\langle \vec{n}, \vec{x} \rangle = 0$, i.e. if $\vec{x}$ belongs to the orthogonal complement of $\vec{n}$. This orthogonal complement is parallel

[1] Another important aspect is to determine the size of the objects accordingly. Objects far away should be smaller than objects at near range. We will ignore this here.

to the monitor plane and contains the origin. Hence, in order to avoid this, one want to focus on points "right/behind" the monitor plane, as already indicated in the picture above.

2. Consequently, it holds

$$\vec{x}_S = \frac{d}{\langle \vec{n}, \vec{x} \rangle_{\text{std}}} \cdot \vec{x} \in S(\vec{x}_0, \vec{a}, \vec{b})\,. \tag{4.165}$$

Hence we can write

$$\vec{x}_S = \vec{x}_0 + \mu\vec{a} + \nu\vec{b}\,. \tag{4.166}$$

Let us assume that $\vec{x}_0$ points from the eye to the center of the monitor. Then $(\mu, \nu) \in \mathbb{R}^2$ are exactly the coordinates that we are looking for. To compute them, we can for example note that

$$\vec{x}_S - \vec{x}_0 = \mu\vec{a} + \nu\vec{b} \in C\left(\begin{bmatrix} | & | \\ \vec{a} & \vec{b} \\ | & | \end{bmatrix}\right) = C(A)\,. \tag{4.167}$$

Corollary 4.7.1:
Given the 3d coordinates $\vec{x}$ of a star, we should draw this star at the coordinates $(\mu, \nu) \in \mathbb{R}^2$ given as the unique solution to

$$\begin{bmatrix} | & | \\ \vec{a} & \vec{b} \\ | & | \end{bmatrix} \cdot \begin{bmatrix} \mu \\ \nu \end{bmatrix} = \frac{d \cdot \vec{x}}{\langle \vec{n}, \vec{x} \rangle_{\text{std}}} - \vec{x}_0\,. \tag{4.168}$$

Remark:
Recall that the computation of a matrix inverse is not an easy problem. In the case at hand, the matrix is not even square! Still, we can find a left inverse, that is another matrix B such that

$$B \cdot \begin{bmatrix} | & | \\ \vec{a} & \vec{b} \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.169}$$

As an example consider

$$\vec{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.170}$$

Then

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.171}$$

Consequence:

If $B \in \mathbb{M}(2 \times 3, \mathbb{R})$ is a left-inverse of $\begin{bmatrix} | & | \\ \vec{a} & \vec{b} \\ | & | \end{bmatrix}$ and we are given 3-dimensional coordinates $\vec{x}$ of a star. Then we draw this star at position $(\mu, \nu) \in \mathbb{R}^2$ of the monitor:

$$\begin{bmatrix} \mu \\ \nu \end{bmatrix} = B \cdot \left(\frac{d \cdot \vec{x}}{\langle \vec{n}, \vec{x} \rangle_{\text{std}}} - \vec{x}_0 \right) . \tag{4.172}$$

Example 4.7.4:
If we return to an earlier example and make the choice

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{4.173}$$

then $d = 1$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Also,

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{4.174}$$

Hence,

$$\begin{bmatrix} \mu \\ \nu \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left(\frac{\vec{x}}{x_1} - \vec{x}_0 \right) = \begin{bmatrix} \frac{x_2}{x_1} \\ \frac{x_3}{x_1} \end{bmatrix} . \tag{4.175}$$

Recall that we focus on objects that are more than $d = 1$ away from the user, i.e. are located "behind" the monitor. This means that $x_1 > 1$. Therefore, the above expression is well-defined for such objects. In the limit $x_1 \to \infty$, i.e. the limit in which the object moves very far away, the above coordinates tend to $\vec{0}$ matching our expectation.

5 Determinants

In this section we go back to the question of inverses of matrices. We already found that we can find the inverse of $A \in \mathbb{M}(n \times n, \mathbb{R})$ by Gauss-Jordan elimination. Also, this process fails exactly when A has no inverse.

We will now extend this analysis by studying determinants. For $A \in \mathbb{M}(n \times n, \mathbb{R})$, the determinant is a real number i.e. $\det(A) \in \mathbb{R}$. This number tells us immediately if a matrix is invertible or not. Namely, we will find that A is invertible iff $\det(A) \neq 0$. In extending, we can even find formuli for A^{-1}.

On a more theoretical level, it must be noted that the determinant is by itself a remarkable function with interesting properties. In fact, these properties uniquely fix the determinant. This is a feature addressed formally as *universal properties* in category theory. While we will not touch upon this point of view in much detail here, we will argue that the properties of the determinant uniquely fix it. As a consequence, knowing these properties is as good as knowing an explicit mapping rule for the determinant. This in turn is beneficial in practical computations, for which abstract arguments can replace or at least simplify brute force computations.

5.1 The Properties of Determinants

Note:
In this section we consider fixed but arbitary $n \in \mathbb{Z}_{>0}$. Then the determinant is a map $\det\colon \mathbb{M}(n \times n, \mathbb{R}) \to \mathbb{R}$. We define it by its properties.

Remark:
More generally, one can consider any field k and a map

$$\det\colon \mathbb{M}(n \times n, k) \to k\,. \tag{5.1}$$

The most prominent case in the literature is probably $k = \mathbb{C}$. This will be of relevance once we discuss diagonalization in section 6.2.

Definition 5.1.1 (Determinant)**:**
The determinant $\det\colon \mathbb{M}(n \times n, \mathbb{R}) \to \mathbb{R}$ is a map with the following properties:

1. The determinant is linear in all rows of A (one says it is multi-linear):

$$\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_{k-1}^T \\ \vec{a}_k^T + \vec{b}_k^T \\ \vec{a}_{k+1}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_{k-1}^T \\ \vec{a}_k^T \\ \vec{a}_{k+1}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) + \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_{k-1}^T \\ \vec{b}_k^T \\ \vec{a}_{k+1}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right). \tag{5.2}$$

2. The determinant is alternating in the rows of A:

$$\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_i^T \\ \vdots \\ \vec{a}_j^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_j^T \\ \vdots \\ \vec{a}_i^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right). \tag{5.3}$$

3. The determinant of the identity matrix is 1, i.e. $\det(I) = 1$.

Note:
It is not automatic, that a map with these properties does even exist. Neither does it follow immediately that this map is unique. To see that it exists and is unique, we will use the above properties to derive rules for how to compute the determinant of a given matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$. This will estabish both the existence and the uniqueness of this map.

Corollary 5.1.1:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ with two identical rows. Then $\det(A) = 0$.

Proof
We denote the two identical rows of A as $\vec{a}^T$. When we exchange those rows, the matrix A remains unchanged. However, since the determinant is alternating in the rows of A, the sign of the determinant changes:

$$\det(A) = \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}^T \\ \vdots \\ \vec{a}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}^T \\ \vdots \\ \vec{a}^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = -\det(A)\,. \tag{5.4}$$

The only real number with this property is 0. Hence $\det(A) = 0$. ■

Corollary 5.1.2:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. Subtracting a multiple of one row of A from another row of A leaves $\det(A)$ unchanged.

Proof
W.l.o.g. let us assume that the relevant rows are the first two of A. Since the determinant is linear in the rows of A, it follows

$$\det\left(\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T - \lambda \cdot \vec{a}_1^T \\ \vec{a}_3^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vec{a}_3^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) - \lambda \cdot \det\left(\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_1^T \\ \vec{a}_3^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}\right) \tag{5.5}$$

The last matrix has two identical rows. Hence, by corollary 5.1.1, its determinant vanishes which proves our claim. ■

Note:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ with (at least) one zero row. Then $\det(A) = 0$. Namely, we add any other row of A to this zero row. By corollary 5.1.2, this leaves the determinant unchanged. The resulting matrix has now two identical rows and by corollary 5.1.1, the determinant vanishes.

Corollary 5.1.3:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. We rescale its rows by $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \setminus 0$ to obtain A'. Then

$$\det(A') = \prod_{i=1}^{n} \lambda_i \cdot \det(A)\,. \tag{5.6}$$

Proof
This follows from the multi-linearity of the determinant in the rows of A. ■

Corollary 5.1.4:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ be an upper (or lower) triangular matrix. Then

$$\det(A) = \prod_{i=1}^{n} a_{ii}\,. \tag{5.7}$$

Proof
We distinguish two cases:

- All diagonal entries of A are non-zero:
  By elementary row operations, we can bring A into diagonal form. By corollary 5.1.2, the determinant remains unchanged. By corollary 5.1.3, it follows then

$$\det(A) = \prod_{i=1}^{n} a_{ii} \cdot \det(I)\,. \tag{5.8}$$

  The normalization of the determinant states $\det(I) = 1$. Hence, the claim follows.

- At least one diagonal entry of A vanishes:
  Then A is singular. We can again perform elementary row operations, and the determinant remains unchanged by corollary 5.1.2. However, since A is singular, this process leads to a *zero row*. The determinant vanishes then as consequence of corollary 5.1.1. This matches the product of the diagonal entries, since at least one diagonal entry was assumed to vanish.

This completes the proof. ■

Convention:
By using corollary 5.1.3, corollary 5.1.4 we can compute the determinant for any matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ as follows:

- Employ elementary row transformations to bring A into upper triangular form U. This process may involve row exchanges. Let their number by N. Then, by corollary 5.1.3 and the alternatingness of the determinant, if follows

$$\det(A) = (-1)^N \cdot \det(U)\,. \tag{5.9}$$

- Use corollary 5.1.4 to infer $\det(U)$ and thereby $\det(A)$.

Note:
This proves existence and uniqueness of the determinant. Even more, it shows that if we consider a function $f\colon \mathbb{M}(n \times n, \mathbb{R}) \to \mathbb{R}$ which is multi-linear and alternating in the rows of A, but not normalized as the determinant, then $f(A) = c \cdot \det(A)$ for a suitable constant $c \in \mathbb{R}$, which is given by $c = f(I)$. This observation allows us to prove the following.

Corollary 5.1.5:
Let $A, B \in \mathbb{M}(n \times n, \mathbb{R})$. Then $\det(AB) = \det(A) \cdot \det(B)$.

Proof
We distinguish two cases:

- B is singular:
  Then also AB is singular and $\det(AB) = 0 = \det(B)$.

- B non-singular:
  Then $\det(B) \neq 0$. We can therefore consider the map

$$f\colon \mathbb{M}(n \times n, \mathbb{R}) \to \mathbb{R}\,,\; A \mapsto \frac{\det(AB)}{\det(B)}\,. \tag{5.10}$$

  This map f is multilinear and alternating in the rows of A. Furthermore, for $A = I$ we have $f(I) = 1$. Hence, this map has the defining properties of the determinant of A and it follows $f(A) = \det(A)$.

This completes the proof. ■

Note:
This generalizes to $\det\left(\prod_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \det(A_i)$.

Corollary 5.1.6:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then the following holds true:

- If A is singular, then $\det(A) = 0$.
- If A is invertible, then $\det(A) \neq 0$.

Proof

- If A is singular, then at least one of its pivots vanishes. Consequently, we can bring A into an upper triangular form, but at least one diagonal entry vanishes. It follows from corollary 5.1.4 that $\det(A) = 0$.
- If A is invertible, then none of its pivots vanish. Hence, we bring A into an upper tringular form U for which all diagonal entries are non-zero. It follows from corollary 5.1.4 that $\det(A) \neq 0$. ■

Corollary 5.1.7:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then $\det(A) = \det(A^T)$.

Proof
We first note the following:

- Consider a permutation matrix $P \in \mathbb{M}(n \times n, \mathbb{R})$. All pivots of P are 1. Hence $\det(P) = \pm 1$. Furthermore, $P \cdot P^T = I$ implies $\det(P) \cdot \det(P^T) = 1$. It follows $\det(P) = \det(P^T)$.
- Consider an upper triangular matrix $U \in \mathbb{M}(n \times n, \mathbb{R})$. Then, since U and U^T have the same pivots, it follows $\det(U) = \det(U^T)$. Similarly, for a lower triangular matrix $L \in \mathbb{M}(n \times n, \mathbb{R})$ it holds $\det(L) = \det(L^T)$.

With this we can now prove the general statement. To this end, consider $A \in \mathbb{M}(n \times n, \mathbb{R})$. We express A by a PLU-decomposition as $A = PLU$. Hence

$$\det(A) = \det(P) \cdot \det(L) \cdot \det(U)\,. \tag{5.11}$$

Upon transposition, we then find $A^T = U^T L^T P^T$. Hence

$$\det(A^T) = \det(U^T) \cdot \det(L^T) \cdot \det(P^T) = \det(P) \cdot \det(L) \cdot \det(U) = \det(A)\,. \tag{5.12}$$

This completes the proof. ■

Note:
As a consequence of this result, we notice that the determinant is also multi-linear and alternating in the *columns* of A.

5.2 Applications of determinants

5.2.1 Symbolic computation of matrix inverses

In this section we wish to solve $A\vec{x} = \vec{b}$ algebraically, and not by elimination. This will involve quotients of certain determinants.

Note:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ with $\det(A) \neq 0$, $\vec{b} \in \mathbb{R}^n$ and $\vec{x} \in \mathbb{R}^n$ with $A\vec{x} = \vec{b}$. Let us write

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix} \in \mathbb{M}(n \times n, \mathbb{R}) \,. \tag{5.13}$$

Then we notice that

$$A \cdot \left[\begin{array}{c|cccc} | & 0 & 0 & \dots & 0 \\ | & 1 & 0 & \dots & 0 \\ \vec{x} & 0 & 1 & \ddots & \vdots \\ | & \vdots & \ddots & \ddots & 0 \\ | & 0 & \dots & 0 & 1 \end{array}\right] = \begin{bmatrix} | & | & & | \\ \vec{b} & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix} . \tag{5.14}$$

Consequently,

$$\det(A) \cdot \underbrace{\det\left(\left[\begin{array}{c|cccc} | & 0 & 0 & \dots & 0 \\ | & 1 & 0 & \dots & 0 \\ \vec{x} & 0 & 1 & \ddots & \vdots \\ | & \vdots & \ddots & \ddots & 0 \\ | & 0 & \dots & 0 & 1 \end{array}\right]\right)}_{=x_1} = \underbrace{\det\left(\begin{bmatrix} | & | & & | \\ \vec{b} & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix}\right)}_{:=\det(B_1)} . \tag{5.15}$$

Hence, since we assumed $\det(A) \neq 0$, the component x_1 of $\vec{x}$ necessarily satisfies

$$x_1 = \frac{\det(B_1)}{\det(A)} \,, \tag{5.16}$$

where B_1 is the matrix obtained by replacing the first columns of A by $\vec{b}$. The converse is also true and leads to the following result.

Corollary (Cramer's rule)**:**
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ with $\det(A) \neq 0$ and $\vec{b} \in \mathbb{R}^n$. The unique vector $\vec{x} \in \mathbb{R}^n$ with

$$A\vec{x} = \vec{b} \,, \tag{5.17}$$

satisfies

$$x_i = \frac{\det(B_i)}{\det(A)} \,, \tag{5.18}$$

where $B_i \in \mathbb{M}(n \times n, \mathbb{R})$ is obtained by replacing the i-th column of A by $\vec{b}$.

Note:
For matrices with numbers as entries, Cramer's rule is inefficient. But for symbolic operations, Cramer's rule can be useful.

Example 5.2.1:
Consider the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}), \tag{5.19}$$

where $a, b, c, d \in \mathbb{R}$ are arbitrary but fixed real numbers. We assume that A is invertible, i.e. $\det(A) \neq 0$. Then we wish to find the inverse of this matrix from Cramer's rule. To this end, we definitely need to compute a number of determinants, in particular $\det(A)$. Without loss of generality, we may assume that a is a pivot of A, and thus $a \neq 0$. Then, by subtracting $\left(\frac{c}{a}\right)$-times the first row from the second, we find

$$A \to \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix}. \tag{5.20}$$

Since this is an upper triangular matrix, it follows that $\det(A) = a \cdot \left(d - \frac{bc}{a}\right) = ad - bc$.

Next we compute the entries of $A^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. Hence $A \cdot \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. By Cramer's rule, this is equivalent to

- $\alpha = \frac{\det\left(\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}\right)}{\det(A)} = \frac{d}{\det(A)}$,
- $\gamma = \frac{\det\left(\begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix}\right)}{\det(A)} = -\frac{\det\left(\begin{bmatrix} c & 0 \\ a & 1 \end{bmatrix}\right)}{\det(A)} = -\frac{c}{\det(A)}$.

Similarly, for β and δ we have $A \cdot \begin{bmatrix} \beta \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Cramer's rule now gives

- $\beta = \frac{\det\left(\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}\right)}{\det(A)} = -\frac{\det\left(\begin{bmatrix} 1 & d \\ 0 & b \end{bmatrix}\right)}{\det(A)} = -\frac{b}{\det(A)}$,
- $\delta = \frac{\det\left(\begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix}\right)}{\det(A)} = \frac{a}{\det(A)}$.

So overall,

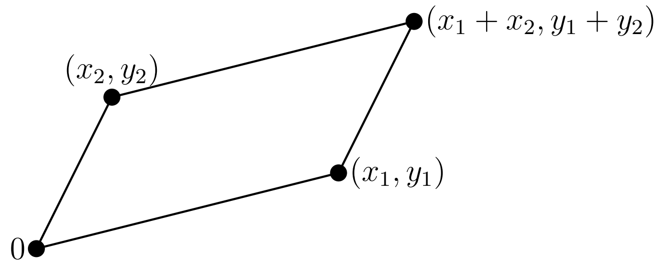
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{5.21}$$

Note:
While inverses of matrices may or may not exist, we can in general say the following:

- In general, an analytic expression for the inverse of $A \in \mathbb{M}(n \times n, \mathbb{R})$ is hard to remember. The above (2×2)-case may the be sole exception.
- A closed analytic expression for A^{-1} does exist in terms of the so-called cofactors of A. We will discuss this momentarily.

5.2.2 Areas and volumes

Note:
Let us consider a parallelogram with corners $(0,0)$, (x_1, y_1), (x_2, y_2) and (x_1+x_2, y_1+y_2):



(5.22)

I claim that its area is given by

$$A = \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} . \tag{5.23}$$

We establish this fact by considering the area A as a function

$$A \colon \mathbb{M}(2 \times 2, \mathbb{R}) \to \mathbb{R}\,, \tag{5.24}$$

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \mapsto \text{area of parallelogram eq. (5.22)} . \tag{5.25}$$

To verify that A is the determinant, it suffices to verify that A satisfies the three defining properties of determinants:

- Property 3:
  If $\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = I$, then the parallelogram eq. (5.22) is a square with area 1.
- Property 2:
  If we exchange the rows, then the parallelogram remains the same as collection of points. On the other hand, the determinant changes sign, indicating whether the edges form a right-handed $(\det(A) > 0)$ or a left-handed coordinate system $(\det(A) < 0)$. We include this information as sign in the area of the parallelogram.
- Property 1:
  The area of the parallelogram associated to

$$\begin{bmatrix} x_1 & y_1 \\ \lambda x_2 + x_2' & \lambda y_2 + y_2' \end{bmatrix} , \tag{5.26}$$

is the sum of the areas of the parallelograms $\begin{bmatrix} x_1 & y_1 \\ \lambda x_2 & \lambda y_2 \end{bmatrix}$ and $\begin{bmatrix} x_1 & y_1 \\ x_2' & y_2' \end{bmatrix}$.

Remark:
Whilst this proof may seem exotic – we could have simply done with basic geometry – it will allow us to extend this result to arbitrary dimension. Before we get to this, let us point out the following result.

Claim 17:
The area of the triangle T in $\mathbb{R}^3$ with corners (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by

$$A(T) = \frac{1}{2} \cdot \det\left(\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}\right). \tag{5.27}$$

Proof
Let us consider

$$\Delta_1 = (x_1 - x_3, y_1 - y_3), \qquad \Delta_2 = (x_2 - x_3, y_2 - y_3). \tag{5.28}$$

These are two of the three sides of the triangle in a shifted coordinate system, in which (x_3, y_3) is considered the origin. Hence, by applying the above results, we conclude that

$$A(T) = \frac{1}{2} \cdot \det\left(\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}\right). \tag{5.29}$$

This completes the proof. ■

Exercise:
By an explicit computation one can show that

$$\det\left(\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}\right) = \det\left(\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right). \tag{5.30}$$

Note:
To see how these results generalize to $\mathbb{R}^n$, we first introduce the *convex hull.*

Definition 5.2.1 (Convex hull)**:**
Be $\{\vec{v}_1, \ldots, \vec{v}_k\} \subset \mathbb{R}^n$. We consider these vectors points in $\mathbb{R}^n$ and define their convex hull as

$$\mathrm{Conv}\{\vec{v}_1, \ldots, \vec{v}_k\} = \left\{ \sum_{i=1}^{k} a_i \vec{v}_i \in \mathbb{R}^n \,\middle|\, a_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} a_i = 1 \right\}. \tag{5.31}$$

Remark:
Sometimes, the *convex hull* is also referred to as the *convex envelope* or *convex closure.*

Consequence:
Suppose $\vec{v}_1 = \vec{0}$. Then it follows that

$$\mathrm{Conv}\left\{\vec{v}_1, \ldots, \vec{v}_k\right\} = \left\{ \sum_{i=2}^{k} a_i \vec{v}_i \in \mathbb{R}^n \,\middle|\, a_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=2}^{n} a_i \leq 1 \right\}. \tag{5.32}$$

In particular, the triangle with corners $(0,0)$, (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ can be described as the convex hull

$$\begin{aligned} T &= \mathrm{Conv}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\} \\ &= \left\{ a \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, a, b \in \mathbb{R}_{\geq 0} \text{ and } a + b \leq 1 \right\}. \end{aligned} \tag{5.33}$$

Similarly, the parallelogram in eq. (5.22) can be described as

$$\begin{aligned} P &= \mathrm{Conv}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \right\} \\ &= \left\{ a \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + c \cdot \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, a, b, c \in \mathbb{R}_{\geq 0} \text{ and } a + b + c \leq 1 \right\}. \end{aligned} \tag{5.34}$$

Corollary:
For any two $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^2$, the area of the convex hull $T = \mathrm{Conv}\left\{\vec{0}, \vec{a}_1, \vec{a}_2\right\}$ is given by

$$A(T) = \frac{1}{2} \cdot \det\left(\begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \right). \tag{5.35}$$

and for $P = \mathrm{Conv}\left\{\vec{0}, \vec{a}_1, \vec{a}_2, \vec{a}_1 + \vec{a}_2\right\}$ it holds

$$A(P) = \det\left(\begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \right). \tag{5.36}$$

Note:
In two dimensions, we talk about the area of a triangle, parallelogram etc. The established wording for the equivalent quantity in 3 and higher dimensions is the volume. For example, we talk about the volume of a bottle, whereas the area of a bottle is not clearly defined. It requires to make reference to the surface, bottom, . . . of the bottle. With this terminology in mind, let me generalization our results to $\mathbb{R}^n$ by replacing *area* by *volume*.

Consequence:
Let $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$. Then the *volume* of the "hyper-parallelogram"

$$P = \text{Conv}\left\{\vec{0}, \vec{a}_1, \ldots, \vec{a}_n, \sum_{i=1}^{n} \vec{a}_i\right\} \tag{5.37}$$

is given by

$$V(P) = \det\left(\begin{bmatrix} | & \cdots & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & \cdots & | \end{bmatrix}\right). \tag{5.38}$$

Cross and triple product

These notions are special to three dimensions. I will not disuss them here in detail, but additional information is for example available in Strang's book on pages 279ff.

5.3 Three ways to compute determinants

In the previous section we have convinced ourselves that the determinant exists and is unique. Moreover, we already derived quite a few useful properties for computing determinants. We will now extend this study in order to find rules by which we can easily compute the determinant.

5.3.1 The Pivot formula

Remark:
This pivot formula follows directly from section 5.1. Namely, given $A \in \mathbb{M}(n \times n, \mathbb{R})$, we can write

$$A = P \cdot L \cdot U\,. \tag{5.39}$$

L is lower triangular and has 1s along the diagonal, so its determinant is 1. $\det(P) = (-1)^N$ with N the number of row exchanges required to bring A into the form LU. Finally, on the diagonal of U we list the pivots of A. Hence

$$\det(A) = \det(P) \cdot \det(U) = (-1)^N \cdot \prod_{i=1}^{n} U_{ii}\,. \tag{5.40}$$

Example 5.3.1:
Let us compute the determinant of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{5.41}$$

We readily find a PLU decomposition of A:

$$A = I \cdot \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ \frac{1}{3} & \frac{-2}{3} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = PLU\,. \tag{5.42}$$

In this case, we have $P = I$ and hence $\det(P) = 1$. This corresponds to $N = 0$ above. Furthermore, $\det(L) = 1$. Consequently

$$\det(A) = \det(U) = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} = 4\,. \tag{5.43}$$

5.3.2 The Big Formula for Determinants

Example 5.3.2:
Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{5.44}$$

Then one can compute the determinant as follows:

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}\,. \tag{5.45}$$

This is an instance of the big formula for the determinant. It generalizes by use of the symmetric group.

Remark:
The symmetric group S_n is the following set of bijections and group operation being concatenation of functions:

$$S_n = \{\varphi\colon\ \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \,|\, \varphi \text{ is bijective}\}\,. \tag{5.46}$$

This group thus coincides with the permutations of the numbers $1, 2, \ldots, n$, i.e. has $n!$ elements and one says that S_n has order $n!$.

A transposition is a permutation which exchanges two elements and keeps all other elements fixed. It is a fact, but maybe not entirely obvious, that any permutation can be written as concatenation of transpositions. The representation of a permutation as a product of transpositions is not unique. However, the number of transpositions needed to represent a given permutation is either always even or always odd. Based on this parity, one defines the sign of $\sigma \in S_n$:

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \sigma \text{ even} \\ -1 & \sigma \text{ odd} \end{cases}\,. \tag{5.47}$$

Claim 18:
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. We consider the symmetric group S_n. Then

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \,. \tag{5.48}$$

Proof
We have to show that eq. (5.48) has the three defining properties of the determinant. Linearity in the rows is clear. It is alternating in the rows as a consequence of $\operatorname{sgn}(\sigma)$. Finally, for the identity matrix we have

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} = \det(I) \cdot 1 \cdot \dots \cdot 1 = 1. \tag{5.49}$$

This completes the proof. ∎

Note:
For $A \in \mathbb{M}(n \times n, \mathbb{R})$, this big formula consists of $n!$ terms. Half of them have positive and the remaining ones negative sign. In particular, the total number of terms increases sharply with n:

- $n = 1$: $n! = 1$,
- $n = 5$: $n! = 120$,
- $n = 10$: $n! = 3628800$,
- $n = 15$: $n! \sim 10^{12}$.

Example 5.3.3:
Let us again compute the determinant of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) \,. \tag{5.50}$$

With the big formula we then find

$$\det(A) = 8 + 0 + 0 - 0 - 2 - 2 = 4 \,. \tag{5.51}$$

5.3.3 Determinant by cofactors

Note:
Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \,. \tag{5.52}$$

Then one can compute the determinant as follows:

$$\begin{aligned}\det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \\ &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) .\end{aligned} \tag{5.53}$$

The three quantities in parentheses are called *cofactors*. We can understand them clearer by writing this finding as

$$\begin{aligned}\det(A) = \det\left(\begin{bmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33}\end{bmatrix}\right) - \det\left(\begin{bmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33}\end{bmatrix}\right) \\ + \det\left(\begin{bmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{bmatrix}\right) .\end{aligned} \tag{5.54}$$

We thus understand the cofactors as determinants of 2×2 "submatrices" of A. These submatrices M_{1j} are obtained by crossing out row 1 and column j from A. This leads to the following observation.

Corollary 5.3.1 (Cofactor expansion)**:**
Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then

$$\det(A) = \sum_{j=1}^{n} a_{1j} \cdot C_{1j} \qquad C_{1j} = (-1)^{1+j} \cdot \det(M_{1j}) . \tag{5.55}$$

Note:
In fact, we can expand the determinant about any row and any column of A in this spirit. For example, expanding about the i-th row is given by

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot C_{ij} \qquad C_{ij} = (-1)^{i+j} \cdot \det(M_{ij}) . \tag{5.56}$$

M_{ij} is obtained by crossing out the i-th row and the j-th column.

6 Eigenvalues and Eigenvectors

The primary topic of this chapter are eigenvalues and eigenvectors. The former are certain special numbers, wheres the latter are certain special vectors.

To motivate this topic, let us look at a reflection about a 2-dimensional linear subspace $S \subset \mathbb{R}^3$ with mapping matrix A. Observe that if $\vec{x}$ belongs to S, then $A\vec{x} = \vec{x}$. Furthermore, observe that any non-zero vector $\vec{x}$ orthogonal to S satisfies $A\vec{x} = -1\vec{x}$. So in a sense, looking for vector $\vec{x}$ with

$$A\vec{x} = \lambda\vec{x}\,, \tag{6.1}$$

for suitable $\lambda \in \mathbb{R}$ seems to tell us a lot about the reflection. In particular, the solution to this equation for $\lambda = +1$ span S and those with $\lambda = -1$ span $S^{\perp}$. In other words, by using these vectors as a basis of $\mathbb{R}^3$, the mapping matrix A becomes diagonal!

The upshot of this chapter is that this observation holds true more generally. The vectors $\vec{x}$ with $A\vec{x} = \lambda\vec{x}$ are the eigenvectors of A and the corresponding $\lambda \in \mathbb{R}$ are the eigenvalues.

Note:
In this chapter we work with square matrices. Given $A \in \mathbb{M}(n \times n, \mathbb{R})$, we will often think of this matrix as linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, so that A is a function which accepts $\vec{x} \in \mathbb{R}^n$ as input and outputs $A\vec{x} \in \mathbb{R}^n$.

6.1 Basic properties of eigenvalues and eigenvectors

Remark:
A first attempt towards defining eigenvalues and eigenvectors could be as follows:

Consider $A \in \mathbb{M}(n \times n, \mathbb{R})$. A *non-zero* vector $\vec{x} \in \mathbb{R}^n$ with the property $A\vec{x} = \lambda\vec{x}$ (with $\lambda \in \mathbb{R}$) is called an *eigenvector* of A. The number $\lambda \in \mathbb{R}$ is termed the *eigenvalue* of $\vec{x}$.

This is flawed!

Example 6.1.1:
The eigenvalues of real matrices do not have to exist/be real. To this end consider the following matrix, which corresponds to a rotation by 90 degrees in the plane $\mathbb{R}^2$:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{6.2}$$

Then it holds

$$A \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = (-i) \cdot \begin{bmatrix} 1 \\ i \end{bmatrix}, \qquad A \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \cdot \begin{bmatrix} i \\ 1 \end{bmatrix}. \tag{6.3}$$

Hence, this matrix has *complex eigenvalues* $\pm i$ and *complex valued eigenvectors*!

We will uncover the reasons behind this momentarily. Therefore, let us define eigenvectors and eigenvalues as follows.

Definition 6.1.1 (Eigenvectors and eigenvalues)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R}) \subset \mathbb{M}(n \times n, \mathbb{C})$. A *non-zero* vector $\vec{x} \in \mathbb{C}^n$ with the property $A\vec{x} = \lambda\vec{x}$ ($\lambda \in \mathbb{C}$!) is called an *eigenvector* of A. λ is termed the *eigenvalue* of $\vec{x}$.

Note:
Since any $A \in \mathbb{M}(n \times n, \mathbb{R})$ can naturally be understood as element of $\mathbb{M}(n \times n, \mathbb{C})$, it makes sense to think of the eigenvalues and eigenvectors as complex valued. However, there are interesting cases, in which all eigenvalues of a matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ are real. Most importantly, we will eventually show, that this is true for symmetric matrices, i.e. matrices with $A = A^T$. In such cases, it makes sense to think of the eigenvalues and eigenvectors are real valued.

Note:
One immediate goal is to compute eigenvalues and eigenvectors in an efficient manner. We begin with the notion of the eigenspace.

Definition 6.1.2 (Eigenspace)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$ and $\lambda \in \mathbb{C}$ an eigenvalue of A. Then we term

$$\operatorname{Eig}(A, \lambda) \equiv \operatorname{Eig}_{\mathbb{C}}(A, \lambda) := \{\vec{x} \in \mathbb{C}^n \mid A\vec{x} = \lambda\vec{x}\}, \tag{6.4}$$

the *complex eigenspace* of A to the eigenvalue λ (or for short the λ-eigenspace of A). In case $\lambda \in \mathbb{R}$, we define

$$\operatorname{Eig}_{\mathbb{R}}(A, \lambda) := \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}, \tag{6.5}$$

and term it the *real eigenspace* of A to the eigenvalue λ.

Example 6.1.2:
For a reflection $A \colon \mathbb{R}^3 \to \mathbb{R}^3$ about a plane $S \subseteq \mathbb{R}^3$ it holds:

- $\operatorname{Eig}_{\mathbb{R}}(A, 1) = \{\vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{x}\} = S$,
- $\operatorname{Eig}_{\mathbb{R}}(A, -1) = \{\vec{x} \in \mathbb{R}^3 \mid A\vec{x} = -\vec{x}\} = S^{\perp}$.

Consequently, for reflections, we can identify the eigenvectors, eigenvalues and eigenspaces from simple geometric intuition. Is this true more generally?

Note:
In the previous example, the 1-eigenspace $\operatorname{Eig}_{\mathbb{R}}(A, 1)$ has dimension 2 while the 0-eigenspace $\operatorname{Eig}_{\mathbb{R}}(A, 0)$ has dimension 1. The dimension of $\mathbb{R}^3$ is 3. We will return to this aspect very soon.

Example 6.1.3:
Consider the permutation matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.6}$$

Thus, A takes a vector in $\mathbb{R}^2$ as input and spits out the vectors obtained by swapping the components of the input. Geometrically, A performs a reflection at the line $y = x$. Again, we can use this to find the eigenvalues and eigenvectors. Namely, all vector $\vec{x}$ on the line $x = y$ are of the form

$$\vec{x} = \begin{bmatrix} a \\ a \end{bmatrix} \in \mathbb{R}^2\,, \tag{6.7}$$

and satisfy $A\vec{x} = \vec{x}$. Consequently:

$$\mathrm{Eig}_{\mathbb{R}}(A, 1) = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} \in \mathbb{R}^2 \,|\, a \in \mathbb{R} \right\}\,. \tag{6.8}$$

Are there any other eigenvalues? Indeed, namely observe the line normal to the line of reflection. It forms the eigenspace of A to the eigenvalue -1:

$$\mathrm{Eig}_{\mathbb{R}}(A, -1) = \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} \in \mathbb{R}^2 \,|\, a \in \mathbb{R} \right\}\,. \tag{6.9}$$

Note that both eigenspaces are of dimension 1 and that $1 + 1 = 2 = \dim(\mathbb{R}^2)$.

Question:
How do we compute eigenvalues and eigenvectors in general? We are trying to solve $A\vec{x} = \lambda\vec{x}$, except that we know neither λ nor $\vec{x}$.

Suppose that $\vec{x}$ is an eigenvector. Then $A\vec{x} = \lambda\vec{x}$. Equivalently, $A - \lambda I$ has non-zero nullspace, i.e. $A - \lambda I$ is singular. It then follows that $\det(A - \lambda I) = 0$.

Definition 6.1.3 (Characteristic polynomial)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then

$$\mathrm{ch}_A(\lambda) := \det(A - \lambda I) \in \mathbb{R}[\lambda] \subset \mathbb{C}[\lambda]\,, \tag{6.10}$$

is called the characteristic polynomial of A.

Note:
If $A \in \mathbb{M}(n \times n, \mathbb{R})$, then $\mathrm{ch}_A(\lambda)$ is a polynomial of degree n. At this point, you want to recall the fundamental theorem of algebra.

Theorem 6.1.1 (Fundamental theorem of algebra)**:**
Let $p \in \mathbb{C}[\lambda]$ be a polynomial of degree n. Then p has, counted with multiplicity, exactly n zeros. That is, there are $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $a \in \mathbb{C}$ such that

$$p = a \cdot \prod_{i=1}^{n} (\lambda - \lambda_i)\,. \tag{6.11}$$

Remark:
It is crucial to note that the λ_i in the fundamental theorem of algebra need not be pairwise distinct! For example for $p = \lambda^2$, we have $\lambda_1 = \lambda_2 = 0$ and $a = 1$.

Exercise:
What does this theorem tell you about the eigenvalues of a matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$?

Consequence:
Once we know all zeros of $\mathrm{ch}_A(\lambda)$, i.e. all eigenvalues of A, then we find the corresponding eigenspace by $\mathrm{Eig}(A, \lambda) = N(A - \lambda I)$.

Example 6.1.4:
Consider the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.12}$$

Then it is readily verified that $\mathrm{ch}_A(\lambda) = \lambda^2 - 8\lambda + 15$. Note that

- 8 is the sum of the entries along the diagonal of A – the so-called *trace* tr(A),
- 15 is the determinant of A.

We note that

$$\mathrm{ch}_A(\lambda) = \lambda^2 - \mathrm{tr(A)} \cdot \lambda + 15 = (\lambda - 5) \cdot (\lambda - 3)\ . \tag{6.13}$$

Hence, the eigenvalues of A are 3 and 5. It is then readily verified that

$$\begin{aligned} \mathrm{Eig}_{\mathbb{R}}(A, 3) &= \left\{ c \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ c \in \mathbb{R} \right\}, \\ \mathrm{Eig}_{\mathbb{R}}(A, 5) &= \left\{ c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ c \in \mathbb{R} \right\}, \end{aligned} \tag{6.14}$$

Exercise:
Convince yourself that the matrices $A, B \in \mathbb{M}(2 \times 2, \mathbb{R})$ with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \tag{6.15}$$

have the same eigenspaces. Their eigenvalues are $-1, 1$ and $3, 5$, respectively. Moreover $B = A + 4 \cdot I$.

Show that if $B = A + c \cdot I$, then the eigenvalues of B are obtained by adding c to the eigenvalues of A. The eigenspaces remain unchanged.

Example 6.1.5:
Let us again come back again to note that real matrices need not have real eigenvalues nor eigenvectors. We already discussed the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.16}$$

Convince yourself that $\mathrm{ch}_A(\lambda) = \lambda^2 + 1$. This polynomial has no real zeros, but complex zeros ± 1. Geometrically, we could have forseen this as rotations do not scale any non-zero vector in $\mathbb{R}^2$. By computing $N(A \pm i \cdot I)$, it is readily verified that $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} i \\ 1 \end{bmatrix}$ are eigenvectors of A.

Example 6.1.6:
Let us compute the eigenvalues and eigenspaces of

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.17}$$

Observe that A is triangular and its eigenvalues are simply the entries on the diagonal. Thus, in this case, 3 is the only eigenvalue. We readily verify that

$$\mathrm{Eig}_{\mathbb{R}}(A, 3) = \left\{ c \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \, c \in \mathbb{R} \right\}. \tag{6.18}$$

Thus, the 3-eigenspace of this matrix A is 1-dimensional.

Definition 6.1.4 (Algebraic and geometric multiplicity of an eigenvalue)**:**
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$, $\mathrm{ch}_A(\lambda)$ its characterstic polynomial and λ and eigenvalue. Then we define:

- The algebraic multiplicity $\mu_{\mathrm{alg}}(A, \lambda)$ of λ is the order of vanishing of ch_A at λ.
- The geometric multiplicity $\mu_{\mathrm{geo}}(A, \lambda)$ of λ is $\dim_{\mathbb{R}}(\mathrm{Eig}_{\mathbb{R}}(A, \lambda))$.

Example 6.1.7:
Let us exemplify these notions:

- In the previous example, we thus have $\mu_{\mathrm{alg}}(A, 3) = 2$ and $\mu_{\mathrm{geo}}(A, 3) = 1$.
- As another example consider the $n \times n$ identity matrix I. Then $\mu_{\mathrm{alg}}(I, 1) = n$ and $\mu_{\mathrm{geo}}(I, 1) = n$.

We will come back to these observations when we discuss diagonalizations.

Definition 6.1.5:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. The sum of the entries along the diagonal of A is termed the trace $\mathrm{tr}(A)$ of A.

Claim 19:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. Then the following hold true:

- The sum of the eigenvalues of A equals $\mathrm{tr}(A)$.
- The product of all eigenvalues of A equals $\det(A)$.

- Be $k \geq 0$. Then, the eigenvalues of A^k are obtained by raising the eigenvalues of A to the k-th power.
- If A is invertible, then λ^{-1} is an eigenvalue of A^{-1} iff λ is an eigenvalue of A.

Exercise:
Prove these statements.

Claim 20:
Eigenvectors of distinct eigenvalues are linearly independent.

Proof
We suffice it to give the proof for two vectors. Let us thus consider vectors $\vec{v}_1$ and $\vec{v}_2$ which are eigenvectors to A with distinct eigenvalues λ_1 and λ_2. Then we consider

$$0 = c_1\vec{v}_1 + c_2\vec{v}_2\,. \tag{6.19}$$

Let us now solve for c_1 and c_2. To this end, we perform two distinct steps:

- Multiply *eq.* (6.19) from the left with A. This gives
$$0 = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2\,. \tag{6.20}$$
- Multiply *eq.* (6.19) with λ_1. This gives
$$0 = c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2\,. \tag{6.21}$$

Now consider the difference of eq. (6.20) and eq. (6.21):

$$0 = c_2(\lambda_1 - \lambda_2)\vec{v}_2\,. \tag{6.22}$$

Since λ_1 and λ_2 are distinct and $\vec{v}_2$ non-zero (defining property of eigenvectors!), we conclude $c_2 = 0$. Consequently, eq. (6.19) implies $c_1 = 0$ and it follows that $\vec{v}_1$, $\vec{v}_2$ are linearly independent. ■

6.2 Diagonalizing matrices

6.2.1 The notation of diagonalizability

Note (Motivation)**:**
Why is it nice to have a basis of eigenvectors of $A \in \mathbb{M}(n \times n, \mathbb{R})$? Here is one reason. Say we want to compute $A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$. If we have a basis of eigenvectors $\vec{v_1}, \ldots, \vec{v_n}$ we may express $\vec{x}$ as

$$\vec{x} = c_1\vec{v}_1 + \ldots c_n\vec{v}_n\,. \tag{6.23}$$

Then we find

$$A\vec{x} = c_1A\vec{v}_1 + \ldots c_nA\vec{v}_n = c_1\left(\lambda_1\vec{v}_1\right) + \cdots + c_n\left(\lambda_n\vec{v}_n\right)\,. \tag{6.24}$$

The last expression involves ordinary multiplication and not matrix multiplication! So it is less cumbersome than computing $A\vec{x}$.

Definition 6.2.1:
If a matrix possesses a basis of eigenvectors, it is said to be *diagonalizable*.

Remark:
Diagonalizability is a very important property of matrices.

Note:
Suppose $A \in \mathbb{M}(n \times n, \mathbb{R})$ has a basis of eigenvectors $\vec{x}_1, \ldots \vec{x}_n \in \mathbb{R}^n$. Put these vectors into the columns of the eigenvector matrix X:

$$X = \begin{bmatrix} | & | & \cdots & | \\ \vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n \\ | & | & \cdots & | \end{bmatrix}. \tag{6.25}$$

Let us then compute AX:

$$\begin{aligned} AX &= \begin{bmatrix} | & | & \cdots & | \\ \vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n \\ | & | & \cdots & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & \cdots & | \\ A\vec{x}_1 & A\vec{x}_2 & \ldots & A\vec{x}_n \\ | & | & \cdots & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & \cdots & | \\ \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \ldots & \lambda_n\vec{x}_n \\ | & | & \cdots & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & \cdots & | \\ \vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n \\ | & | & \cdots & | \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}. \end{aligned} \tag{6.26}$$

By assumption, the columns of X are a basis of $\mathbb{R}^n$. Thus X is invertible. So in particular, we find

$$X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}. \tag{6.27}$$

Definition 6.2.2:
The matrix Λ is called the *eigenvalue matrix*.

Remark:
The equality $X^{-1}AX$ has a geometric meaning via a base change. We will come back to this momentarily. First however, we want to look at a few examples and develop a criterion which tells if a matrix is diagonalizable or not.

Example 6.2.1:
Consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.28}$$

Let us find X and Λ such that $X^{-1}AX = \Lambda$. To this end, we first find the characteristic polynomial of A:

$$\mathrm{ch}_A(\lambda) = (1-\lambda)(3-\lambda) - 8 = (\lambda+1)(\lambda-5)\,. \tag{6.29}$$

Thus, the eigenvalues are -1 and 5. It is readily verified that the eigenspaces are

$$\begin{aligned} \mathrm{Eig}_{\mathbb{R}}\,(A,-1) &= \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}, \\ \mathrm{Eig}_{\mathbb{R}}\,(A,5) &= \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}. \end{aligned} \tag{6.30}$$

Consequently, we conclude

$$X = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}. \tag{6.31}$$

Corollary 6.2.1:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$ such that its characteristic polynomial $\mathrm{ch}_A(\lambda)$ has n distinct real zeros, then A is diagonalizable.

Proof
Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be the distinct eigenvalues and $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ corresponding eigenvectors. Then $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly independent. Hence, these eigenvectors form a basis of $\mathbb{R}^n$. ■

Consequence:
Under these assumptions, all (real) eigenspaces of A are 1-dimensional linear subspaces of $\mathbb{R}^n$.

Note (Exponentiation of diagonalizable matrices)**:**
Consider a diagonalizable matrix A, i.e. $A = X\Lambda X^{-1}$ for a diagonal matrix Λ. Then

$$A^k = \left(X\Lambda X^{-1}\right) \cdot \left(X\Lambda X^{-1}\right) \cdot \cdots \cdot \left(X\Lambda X^{-1}\right)\,. \tag{6.32}$$

Since matrix multiplication is associative, we ignore those brackets and then, there is a massive cancellation leading to

$$A^k = X\Lambda^k X^{-1}\,. \tag{6.33}$$

Even more, since Λ is a diagonal matrix, computing Λ^k is very easy!

Example 6.2.2:
Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} . \tag{6.34}$$

We are interested in computing A^k. To this end, let us diagonalize A. We first find

$$\mathrm{ch}_A(\lambda) = \lambda^2 - \lambda - 1 . \tag{6.35}$$

There are thus two distinct real eigenvalues

$$\lambda_1 = \frac{1+\sqrt{5}}{2} , \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} . \tag{6.36}$$

Exercise:
Use $\lambda_1 + \lambda_2 = 1$ and $\lambda_1\lambda_2 = -1$ to conclude that

$$\begin{aligned} \mathrm{Eig}\,(A, \lambda_1) &= \mathrm{Span}\left\{ \begin{bmatrix} -\lambda_2 \\ 1 \end{bmatrix} \right\} , \\ \mathrm{Eig}\,(A, \lambda_2) &= \mathrm{Span}\left\{ \begin{bmatrix} -\lambda_1 \\ 1 \end{bmatrix} \right\} . \end{aligned} \tag{6.37}$$

Example 6.2.3 (Continuation)**:**
We thus have

$$X = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} , \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} , \qquad \Lambda^{-1} = \frac{1}{\lambda_2 - \lambda_1} \cdot \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} . \tag{6.38}$$

Thus we have

$$\begin{aligned} A^k &= \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \cdot \frac{1}{\lambda_2 - \lambda_1} \cdot \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} , \\ &= \frac{1}{\lambda_2 - \lambda_1} \cdot \begin{bmatrix} \lambda_1^{k-1} - \lambda_2^{k-1} & \lambda_2^k - \lambda_1^k \\ \lambda_1^k - \lambda_2^k & \lambda_2^{k+1} - \lambda_1^{k+1} \end{bmatrix} . \end{aligned} \tag{6.39}$$

We have used that $\lambda_1 \cdot \lambda_2 = \det(A) = -1$, to simplify the expression.

Note:
The above matrix A allows us to compute the Fibonacci numbers:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \xrightarrow{A} \begin{bmatrix} 5 \\ 8 \end{bmatrix} \xrightarrow{A} \dots . \tag{6.40}$$

Hence, we identify the k-th Fibonacci number F_k as $F_k = \frac{\lambda_2^k - \lambda_1^k}{\lambda_2 - \lambda_1}$ from

$$A^k \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_2^k - \lambda_1^k}{\lambda_2 - \lambda_1} \\ \frac{\lambda_2^{k+1} - \lambda_1^{k+1}}{\lambda_2 - \lambda_1} \end{bmatrix} . \tag{6.41}$$

Definition 6.2.3:
Two matrices $A, B \in \mathbb{M}(n \times n, \mathbb{R})$ are said to be *similar* if there exists an invertible $P \in \mathbb{M}(n \times n, \mathbb{R})$ satisfying $A = PBP^{-1}$.

Claim 21:
If A and B are similar, then they have the same eigenvalues.

Proof
We compute the characteristic polynomial:

$$\begin{aligned} \mathrm{ch}_A(\lambda) &= \det\left(A - \lambda I\right) = \det\left(PBP^{-1} - \lambda I\right) \\ &= \det\left(P(B - \lambda I)P^{-1}\right) \\ &= \det\left(B - \lambda I\right) . \end{aligned} \tag{6.42}$$

This completes the proof. ■

Consequence:
If A and B are similar and A is diagonalizable, then also B is diagonalizable.

Exercise:
Prove this statement.

6.2.2 Failure of matrices to be diagonalizable

Remark:
Given $A \in \mathbb{M}(n \times n, \mathbb{R})$, we associated to an eigenvalue $\lambda \in \mathbb{R}$ two integers:

- The algebraic multiplicity $\mu_{\text{alg}}(A, \lambda)$ is the order of vanishing of $\mathrm{ch}_A(\lambda)$ at λ.
- The geometric multiplicity $\mu_{\text{geo}}(A, \lambda)$ is the dimension $\mathrm{Eig}_{\mathbb{R}}(A, \lambda)$.

Note:
It of utmost importance that

$$\mu_{\text{geo}}(A, \lambda) \leq \mu_{\text{alg}}(A, \lambda) . \tag{6.43}$$

The consequences of this result are profund! Namely, by the fundamental theorem of algebra, the sum of the algebraic multiplicities is n, i.e. the dimension of $\mathbb{R}^n$. Hence, the sum of the algebraic multiplicity matches the length of any basis of $\mathbb{R}^n$. But since $\mu_{\text{geo}}(A, \lambda) \leq \mu_{\text{alg}}(A, \lambda)$, we can only find a basis of $\mathbb{R}^n$ which consists of eigenvectors of the matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ if and only if the algebraic and geometric multiplicities agree. This leads to the following very important theorem.

Theorem 6.2.1:
Be $A \in \mathbb{M}(n \times n, \mathbb{R})$ with eigenvalues $\lambda_i \in \mathbb{R}$. A is diagonalizable if and only if

$$\mu_{\text{geo}}(A, \lambda_i) = \mu_{\text{alg}}(A, \lambda_i) , \tag{6.44}$$

holds true for all eigenvalues. Thus, if and only if this condition is satisfied, there exists an invertible $S \in \mathbb{M}(n \times n, \mathbb{R})$ and a diagonal matrix $D \in \mathbb{M}(n \times n, \mathbb{R})$ with $A = SDS^{-1}$.

Remark:
Similarly, a matrix $A \in \mathbb{M}(n \times n, \mathbb{C})$ is diagonalizable if and only if

$$\mu_{\text{geo}}(A, \lambda_i) = \mu_{\text{alg}}(A, \lambda_i)\,, \tag{6.45}$$

for all its eigenvalues. In constrast to the above theorem, the matrices S and D are then in general complex valued.

Example 6.2.4:
Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.46}$$

It has an eigenvalue 3 with algebraic multiplicity 2 and geometric multiplicity 1. Hence, A is not diagonalizable.

6.3 Diagonalizability and linear transformations

Question:
In the previous section, we discussed factorizations $A = X\Lambda X^{-1}$ where Λ is a diagonal matrix. We now want to understand what this factorization means by connecting it to linear transformations. A natural question to ask is as follows: "How would one have forseen the fact that $A = X\Lambda X^{-1}$ whenever A has a basis of eigenvectors?" The natural perspective is that of linear transformations.

Note:
Say we want to compute $A\vec{v}$ for some vector $\vec{v} \in \mathbb{R}^n$. Let X be a matrix formed by our basis of eigenvectors. One may think of the eigenvectors as giving a new coordinate system. What do the components of $X^{-1}\vec{v}$ represent?

They give the precise linear combination of the eigenvectors that equals $\vec{v}$. One refers to these components as the *coordinates in the basis of eigenvectors*. Now we apply our linear transform corresponding to A. On the basis of eigenvectors, this linear transform simply scales the axes. Hence, in this new coordinate frame, the linear transform is given by a diagonal matrix of eigenvalues, which we called Λ. Thus, $\Lambda X^{-1}\vec{v}$ gives the coordinates of the linear transform applied to $\vec{v}$. When we want to return to our original coordinate system, we multiply by X on the left to undo what X^{-1} did. Thus, the linear transform we are studying sends $\vec{v}$ to $X\Lambda X^{-1}\vec{v}$. Thus, the corresponding matrix of transformation must be $X\Lambda X^{-1}$!

A more condensed and alternative way to see it is to recall the construction of the matrix X. Namely, we formed it from a basis of eigenvectors $\vec{x}_1, \ldots \vec{x}_n \in \mathbb{R}^n$ of A:

$$X = \begin{bmatrix} | & | & \cdots & | \\ \vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n \\ | & | & \cdots & | \end{bmatrix}\,. \tag{6.47}$$

Then we see that X is a base change matrix. Specifically, $X = T_{\mathcal{B}_1\mathcal{B}_2}$ where

- $\mathcal{B}_2$ is the eigenbasis $\{\vec{x}_1, \ldots \vec{x}_n\}$,
- $\mathcal{B}_1$ is the standard basis $\{\vec{e}_1, \ldots \vec{e}_n\}$ of $\mathbb{R}^n$.

Consequence:
"Every" linear transform has its own preferred choice of basis it wants to be understood in. This distinguished basis is given by the basis of eigenvectors. In this basis, the matrix of linear transformation is diagonal.

Example 6.3.1:
Consider reflection across a line. Note that I did not specify the coordinate frame and hence did not give this line an equation. So, if we are to write the matrix of transformation, we have numerous choices. But there are some choices that are easier than others.

Pick the line of the reflection to be the x-axis and the line perpendicular to it to be the y-axis. In this coordinate frame, the linear transform is given by

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) . \tag{6.48}$$

Example 6.3.2:
Similarly, if we consider projection onto a plane, we may take a coordinate system where the x and y-axes are in the plane and the z-axis is orthogonal to the plane. In this coordinate frame, the matrix of transformation is given by

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) . \tag{6.49}$$

Exercise:
Write down the matrix of projection onto the plane $x+y+z=0$ by computing $X\Lambda X^{-1}$ for the appropriate X and Λ.

6.4 Applications

6.4.1 Markov matrices and processes

Note:
We now turn our attention to Markov matrices. Our goal is to model a random process in which a system transitions from one state to another in discrete time steps.

Assume that at each time step, there are n-states a system could be in. At time k, we model the system as a vector $\vec{x}_k \in \mathbb{R}^n$, whose components represent the probability of being in each of the n states. We denote the initial state by $\vec{x}_0$.

Definition 6.4.1:
We term a vector $\vec{x}_i$ whose components are non-negative and sum up to 1 a *probability vector*.

Example 6.4.1:
Let us model the evolution of population in a city and its suburbs, where migration to and from the city occurs. We assume that

$$\vec{x}_0 = \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}, \tag{6.50}$$

i.e. 60% live in the city and 40% in the suburbs. Say each year 5% of the city dwellers move to the suburbs and 3% of the suburbanites move to the city. The rest stay. We represent the population after k years/k steps as

$$\vec{x}_k = \begin{bmatrix} c_k \\ s_k \end{bmatrix}. \tag{6.51}$$

The migration information translates to the following matrix equation:

$$\begin{bmatrix} c_{k+1} \\ s_{k+1} \end{bmatrix} = \vec{x}_{k+1} = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \cdot \begin{bmatrix} c_k \\ s_k \end{bmatrix} \equiv M \cdot \vec{x}_k \, . \tag{6.52}$$

Then we see

$$\vec{x}_0 = \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix} \to \vec{x}_1 = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix} \to \vec{x}_2 = \begin{bmatrix} 0.56 \\ 0.44 \end{bmatrix} \to \vec{x}_3 = \begin{bmatrix} 0.47 \\ 0.53 \end{bmatrix} \to \dots \tag{6.53}$$

In particular, $\vec{x}_k = M^k \cdot \vec{x}_0$. It turns out that

$$\lim_{k\to\infty} \vec{x}_k = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}. \tag{6.54}$$

Thus, in the long run, 37.5% of the population will be living in the city, whereas 62.5% will be in the suburbs.

Definition 6.4.2:
A *Markov matrix* is a square matrix M whose columns are probability vectors. A *Markov chain* is a sequence of probability vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \dots$ such that

$$\vec{x}_{k+1} = M\vec{x}_k \, , \tag{6.55}$$

for a Markov matrix M. We refer to the limit $\lim\limits_{k\to\infty} \vec{x}_k$ – if it exists – as the *steady state vector*.

Note:
A steady state vector *necessarily* has the property $M\vec{x} = \vec{x}$, i.e. satisfies $(M - I)\vec{x} = \vec{0}$. Thus any steady state vector is an eigenvector to the eigenvalue 1.

Example 6.4.2:
In the previous example, we compute

$$(M - I)\vec{x} = \begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}\, . \tag{6.56}$$

This shows $0.03x_2 = 0.05x_1$. We also know $x_1+x_2 = 1$. Thus $x_1 = 0.375$ and $x_2 = 0.625$. This is exactly what was stated above.

Example 6.4.3:
Suppose we are interested in changes in voter preferences during each election cycle – say, among Democrats, Republicans and Liberals (DRL). We list the shifts in this order from left to right and top to bottom:

	D	R	L
D	0.70	0.10	0.30
R	0.20	0.80	0.30
L	0.10	0.10	0.40

(6.57)

Hence, the 0.20 says that 20% of the supporters of Democrats transition to the Republicans. Alternatively, we look at the following Markov matrix:

$$M = \begin{bmatrix} 0.70 & 0.10 & 0.30 \\ 0.20 & 0.80 & 0.30 \\ 0.10 & 0.10 & 0.40 \end{bmatrix} . \tag{6.58}$$

And of course, we can ask for a steady state vector.

Theorem 6.4.1:
If M is a Markov matrix, then there exists a vector $\vec{x} \neq \vec{0}$ such that $M\vec{x} = \vec{x}$.

Proof
We need to show that 1 is always an eigenvalue of M. In other words, we need to show that $M - I$ is singular. Set $A := M - I$ and note that

$$\begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \cdot A = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix} . \tag{6.59}$$

Therefore, $N(A^T)$ is non-trivial and by the rank-nullity theorem:

$$\mathrm{rk}(A) = \mathrm{rk}(A^T) = n - \dim_{\mathbb{R}}\left(N\left(A^T\right)\right) < n\,. \tag{6.60}$$

Therefore, A is singular. ■

Consequence:
We are thus always guaranteed that a *candidate* for a steady state vector does exist, i.e. a vector $\vec{x} \neq 0$ with $M\vec{x} = \vec{x}$. But this is not sufficient to conclude that the limit $\lim\limits_{k\to\infty} \vec{x}_k$ does exist!

As an example, consider the matrix

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{6.61}$$

This Markov matrix goes from state 1 to state 2 and vice versa with probability 1. Thus, there cannot be a steady state under these circumstances.

Question:
We may ask the following questions:

- Under what conditions is the steady state vector unique?
- Does the Markov chain attached to M always settle to a steady state vector?

Lemma 6.4.1:
If M is a Markov matrix, then M^k is a Markov matrix.

Exercise:
Prove this statement.

Claim 22:
Be $M \in \mathbb{M}(n \times n, \mathbb{R})$ a Markov matrix. Then M cannot have an eigenvalue λ with $|\lambda| > 1$.

Proof
Let us assume the contrary, i.e. suppose M is a Markov matrix which has an eigenvalue λ with $|\lambda| > 1$. This means that there exists a vector $\vec{v}$ with

$$M\vec{v} = \lambda\vec{v}\,. \tag{6.62}$$

Hence, $M^n\vec{v} = \lambda^n\vec{v}$, which implies

$$|M^n\vec{v}| = |\lambda|^n \cdot |\vec{v}|\ . \tag{6.63}$$

Since $|\lambda| > 1$, the length of the vector on the right grows to ∞ for $n \to \infty$. Thus, to mirror this on the LHS, also the entries of M^n have grow very large for $n \to \infty$. This contradicts with M being a Markov matrix. Namely, the entries of every column of M are non-negative and add to 1. ■

Theorem 6.4.2 (Perron-Frobenius)**:**
If $M \in \mathbb{M}(n \times n, \mathbb{R})$ is a *positive* (i.e. all entries strictly positive) Markov matrix, then $\lambda = 1$ is the unique largest eigenvalue.

Remark:
A Markov chain associated to a positive Markov matrix is termed *irreducible*.

Remark:
You may think that being a positive Markov matrix is a rather rigid requirement. In fact, one can establish the following stronger result.

Claim 23:
If $M \in \mathbb{M}(n \times n, \mathbb{R})$ is a Markov matrix such that some power M^k is positive, then the Perron-Frobenius theorem applies to M.

Example 6.4.4:
Consider the Markov matrix

$$M = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix}. \tag{6.64}$$

Then

$$M^2 = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}. \tag{6.65}$$

So the Perron-Frobenius does already apply to the Markov matrix M.

6.4.2 Page rank vector

Note:
Our next topic concerns searching on the web. Namely, given a search string, how should the search engine determine the order in which to rank the output? A naive approache is as follows:

1. Keep an index of all web pages.

2. Respond to a query by browsing through the index and list the webpages according to the number of times the search query appears on that webpage.

We can agree that this approach is not very smart. It is fairly easy to abuse the system to have a completely unimportant webpage appear as the first search result. Naive as it may sound, this was exactly the approach used by search engines in the 90s such as Altavista & Lycos. In a very simplistic viewpoint, Sergey Brin and Larry Page realized that the world wide web was a democracy, where someone linking to your webpage was a vote for your webpage. Thus, their idea was to rank the webpages according to the number of votes:

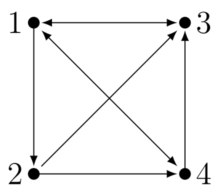
- If I create a webpage A and link to webpage B, that means I consider B relevant.

- Also, if B is considered important and it links to C, then it asserts, that C is important as well. Thus, B transfers its authority to C.

In the following, we want to quantify importance.

Question:
Given n interlinked webpages, rank them in order of importance. To this end, assign the pages importance scores $x_1, x_2, \ldots, x_n \geq 0$. Our insight is to use the existing link structure of the web to determine these scores.

Example 6.4.5:
Let us consider a simplified version of the world-wide-web:



(6.66)

Thus, there are 4 webpages. Each is represented by a node and a directed edge from node i to j represents a hyperlink on webpage i to j.

According to our model, each page transfers its importance *evenly* to the pages it links to. For instance, node 1 passes $\frac{1}{3}$ of its importance score to each of the three nodes it links to. Let us encode this information as a system of equations:

$$\begin{aligned} x_1 &= 1 \cdot x_3 + \frac{1}{2} x_4 \,, \\ x_2 &= \frac{1}{3} x_1 \,, \\ x_3 &= \frac{1}{3} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_4 \,, \\ x_4 &= \frac{1}{3} x_1 + \frac{1}{2} x_2 \,. \end{aligned} \tag{6.67}$$

As a matrix equation, this says

$$\vec{x} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \cdot \vec{x} \,. \tag{6.68}$$

Thus, the importance vector is an eigenvector of a certain Markov matrix! In this case, it is the unique steady state vector

$$\vec{x} = \frac{1}{31} \cdot \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix} \,. \tag{6.69}$$

It might appear a tiny bit magical, that such a vector indeed exists, especially as our rules for deriving importance appear contrived and self-referrential.

Note:
Here are two alternative approaches:

- Instead of finding the 1-eigenvector, one could start with a random assignment of importance scores and then update them according to our rules. Hence, one could start for example with

  $$\vec{x}_0 = \frac{1}{4} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \,. \tag{6.70}$$

  Then, we multiply with A repeatedly and find

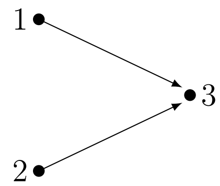
  $$A^8 \vec{x}_0 \sim \begin{bmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{bmatrix} \,. \tag{6.71}$$

- Brin and Page considered the ”random surfer model“. This involves a guy starting on a webpage and clicking one of the hyperlinks on the webpage *uniformly at random*. This creates a Markov chain whose Markov matrix is the same as the one given earlier. The components of the steady state vector of this matrix can now be interpreted as the amount of time one spends on a certain webpage. Or you could think of its components as giving you the probability of ending on a certain webpage in the long run.

Irrespective of the perspective we pick, we definitely get a sense of importance.

Remark:
Here are two potential issues:

1. Webpages that do not have any hyperlink:
   Consider the following network:

   (6.72)

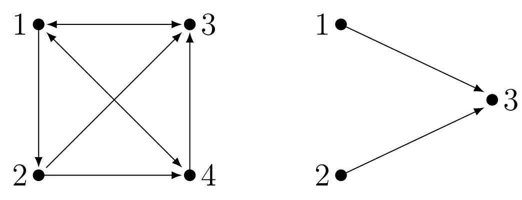
   This leads to the Markov matrix

   $$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}. \tag{6.73}$$

   In this case $M^2 = 0$ and the importance vector is identically zero, which does not really reflect the above network appropriately.

   An easy fix to this problem is to turn the third column to $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}^T$. This means, that anybody who is on webpage 3 ”restarts“ their browsing experience by picking a website uniformly at random.

2. Disconnected components:
   Let us consider the following network with two disconnectec components:

   (6.74)

   While our approach can compare the importance of webpage in one connected component, it seems not helpful in comparing webpages belonging to different connected components.

We would like our 1-eigenspace of the associated Markov matrix to be 1-dimensional. But convince yourself, that if the web has r connected components, then the 1-eigenspace must be at least r-dimensional. This does not play well with the fact that we would really like a unique steady state vector.

Here is the simple and ingenious solution by Brin and Page: We replace the Markov matrix M by the new matrix

$$G = (1-p) \cdot M + p \cdot \frac{1}{n} \cdot \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}. \tag{6.75}$$

The matrix on the right is called the *teleportation matrix.* n is the total number of webpages and $0 \leq p \leq 1$ a probability.

The probabilistic interpretation is as follows: With probability $(1-p)$ we follow the random surfer model from before. However, with probability p we open a random webpage amongst all possible webpages. Google originally chose $p = 0.15$.

The new matrix G above is referred to as the *Google matrix.* It is still Markov. Even more importantly, it is *positive.* Thus, it is guaranteed to have a unique steady state vector, the so-called *Page-Rank vector.*

Example 6.4.6:
For $p = 0.15$, we find for eq. (6.74) that

$$G = \begin{bmatrix} 0.0214286 & 0.0214286 & 0.871429 & 0.446429 & 0.0214286 & 0.0214286 & 0.0214286 \\ 0.304762 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 \\ 0.304762 & 0.446429 & 0.0214286 & 0.446429 & 0.0214286 & 0.0214286 & 0.0214286 \\ 0.304762 & 0.446429 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 \\ 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.304762 \\ 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.304762 \\ 0.0214286 & 0.0214286 & 0.0214286 & 0.0214286 & 0.871429 & 0.871429 & 0.304762 \end{bmatrix}. \tag{6.76}$$

By computing the eigenvector to the eigenvalue 1, which is normalized such that its entries add up to 1, we find the unique steady-state vector/the importance vector:

$$\begin{bmatrix} 0.20979 \\ 0.0810366 \\ 0.164541 \\ 0.11559 \\ 0.0913204 \\ 0.0913204 \\ 0.246401 \end{bmatrix}. \tag{6.77}$$

This is the *Page-Rank vector.*

6.4.3 Systems of ordinary differential equations

Differential equations are a power tool and appear in many areas. In physics, the equation of motion is central. In biology, the Lotka–Volterra equations (which describe dynamics among predators and prey) are being discussed. Economy and finances would employ statistical differential equations to model system dynamics. For example, Wiener- or Brownian-motions are used to simulate stock market dynamics. The central difference between the equation ($A \in \mathbb{M}(m \times n, \mathbb{R})$, $\vec{b} \in \mathbb{R}^m$)

$$A\vec{x} = \vec{b}\,, \tag{6.78}$$

to which a solution is a vector $\vec{x} \in \mathbb{R}^n$, the solution to a differential equation is a function. For example, we could focus on functions $f\colon \mathbb{R} \to \mathbb{R}$ and try to find those which satisfy certain properties. Typically, we would assume that f can be differentiated and then try to relate derivates of f with f. In this sense, the possibly simplest equation that can arise is to demand that for all $x \in \mathbb{R}$ it holds

$$f'(x) = f(x)\,. \tag{6.79}$$

The only solution to this equation is

$$f\colon \mathbb{R} \to \mathbb{R}\,,\ x \mapsto e^x\,. \tag{6.80}$$

In general, it is far from clear if differential equations admit solutions. And even if, it can be very tough to find all solutions. Additional constraints can arise once the domain of the allowed solution functions is constrained. For example, it becomes a lot less trivial to solve

$$f'(x) = f(x)^2\,. \tag{6.81}$$

This is an instance of a so-called Riccati differential equation. Most of what we will discuss in the following focuses on so-called *ordinary differential equations*. That is a differential equation of the form

$$\sum_{i=0}^{n} \alpha_i \cdot f^{(i)}(x) = 0\,, \qquad \alpha_i \in \mathbb{R}\,, \tag{6.82}$$

where $f^{(i)}$ denotes the i-th derivate of the function $f\colon \mathbb{R} \to \mathbb{R}$ (which we will assume to be smooth and can therefore be differentiated as many times as we like) and we assume $\alpha_n \neq 0$. For those differential equations, it is known that the space of solutions is a vector space of dimension n. In particular, if we fix $k \in \mathbb{R}$, then all solutions to

$$f''(x) = -k^2 \cdot f(x)\,, \tag{6.83}$$

are given by

$$f(x) = C_1 \cos(kx) + C_2 \sin(kx)\,, \tag{6.84}$$

where $(C_1, C_2) \in \mathbb{R}^2$ parametrize the 2-dimensional vector space of solutions.

In many situations, systems of coupled differential equations arise. In the physics, this happens for example if the motion of one object influences that of another (e.g. in a collision). In biology, one typically has many predators and prays. Their behaviors influence each other (for example if one predator eats all prey, then the others starve). In the stock market, the action of a single trader typically influence that of other traders. In the latter case, it is virtually impossible to capture all influences, which is why there one switches to statistical differential equations. Still, in well-behaved circumstances, one can study systems of coupled differential equations. An example would be

$$\begin{aligned} u_1'(t) &= -1 \cdot u_1(t) + 2 \cdot u_2(t)\,, \\ u_2'(t) &= 1 \cdot u_1(t) - 2 \cdot u_2(t)\,, \end{aligned} \tag{6.85}$$

where we are trying to find the functions $u_1, u_2 \colon \mathbb{R} \to \mathbb{R}$. One goal of this section is to show that this can be achieved by computing so-called matrix exponentials. Another goal will be to use linear algebra to uncover the dynamics of two masses coupled to springs. The latter is governed by

$$\begin{aligned} m_a \cdot x_a''(t) &= -k_1 x_a(t) + k_2 \left(x_b(t) - x_a(t)\right)\,, \\ m_b \cdot x_b''(t) &= -k_1 x_b(t) + k_2 \left(x_a(t) - x_b(t)\right)\,. \end{aligned} \tag{6.86}$$

The upshot is that we can use diagonalization to express this complicated system by two equations of the form eq. (6.83), which is arguably much easier to solve.

A first encounter

Example 6.4.7:
Let us consider two functions

$$u_1 \colon \mathbb{R} \to \mathbb{C}\,,\ t \mapsto u_1(t)\,, \qquad u_2 \colon \mathbb{R} \to \mathbb{C}\,,\ t \mapsto u_1(t)\,. \tag{6.87}$$

We encode their behaviour with time by the following equations:

$$\begin{aligned} u_1'(t) &= -1 \cdot u_1(t) + 2 \cdot u_2(t)\,, \\ u_2'(t) &= 1 \cdot u_1(t) - 2 \cdot u_2(t)\,. \end{aligned} \tag{6.88}$$

You can consider this to be a continuous analogue of Markov chains discussed in the previous section. In particular, we need to provide an initial condition, say $\vec{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

To understand $u(t)$ as a function of time, we will need to understand the eigenvalues and eigenvectors of the matrix formed by the coefficients. In the case at hand, this matrix is

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.89}$$

Thus, we are interested in solving

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} . \tag{6.90}$$

We know how to proceed if A is a diagonal matrix by simply utilizing the fact that the solution to the differential equation

$$f'(t) = \lambda \cdot f(t) , \tag{6.91}$$

is given by $f(t) = C \cdot e^{\lambda t} + D$. Thus, if we want to solve the system for a non-diagonal matrix A, then we should perhaps try and alter the system so that it becomes diagonal. Hence, let us compute the eigenvalues and eigenvectors of A. Since A is singular, we know that $\lambda_1 = 0$ is an eigenvalue. Since the trace is -3, we know that $\lambda_2 = -3$ is the other eigenvalue. For the eigenspaces, we find

$$\mathrm{Eig}_{\mathbb{R}}\,(A, 0) = \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} , \qquad \mathrm{Eig}_{\mathbb{R}}\,(A, -3) = \mathrm{Span}_{\mathbb{R}} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} . \tag{6.92}$$

Let us set

$$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} , \qquad \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} . \tag{6.93}$$

Then, a quick check shows that $\vec{u} = c_1 \cdot e^{\lambda_1 t} \cdot \vec{x}_1$ satisfies $\vec{u}'(t) = A\vec{u}(t)$. Similarly, $\vec{u} = c_2 \cdot e^{\lambda_2 t} \cdot \vec{x}_2$ satisfies the same differential equation. It follows, that any linear combination of these two special solutions does satisfy this differential equation. By plugging in the values $\lambda_1 = 0$ and $\lambda_2 = -3$, it follows

$$\vec{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = c_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} . \tag{6.94}$$

Finally, we use the initial condition to compute c_1 and c_2. This gives $c_1 = c_2 = \frac{1}{3}$.

Remark:
As time goes to infinity, the term involving e^{-3t} shrinks to 0. Thus, $\lim_{t\to\infty} \vec{u}(t) = \frac{1}{3}\vec{x}_1$. One says the system approaches a steady state. This is similar to the steady states of a Markov chain.

Note:
It is instructive to compare the solution of the continuous version $\vec{u}'(t) = A \cdot \vec{u}(t)$ with its discrete analogue $\vec{u}_{k+1} = A \cdot \vec{u}_k$. In the former, a general solution is given by

$$\vec{u}(t) = c_1 e^{\lambda_1 t} \vec{x}_1 + c_2 e^{\lambda_2 t} \vec{x}_2 , \tag{6.95}$$

and in the latter, a general solution is given by

$$\vec{u}_k = c_1 \lambda_1^k \vec{x}_1 + c_2 \lambda_2^k \vec{x}_2 . \tag{6.96}$$

Remark:
Let us discuss various aspects of the solutions to $\vec{u}'(t) = A\vec{u}(t)$:

- Stability:
  A is stable if $\vec{u}(t)$ approaches 0 as $t \to \infty$. Under what condition on the eigenvalues of A are we guaranteed stability? Clearly, we want the exponentials to decay. That happens iff *the real part of all eigenvalues is negative.* It is useful to recall that $\left|e^{a+ib}\right| = |e^a|$.

- Steady state:
  Under what condition on the eigenvalues does $\vec{u}(t)$ approach a fixed vector as $t \to \infty$? For this, we can allow some eigenvalues to vanish, i.e. $\lambda_i = 0$. However, all other eigenvalues must have negative real part, so that they decay as $t \to \infty$.

- Decoupling:
  Assume that A is diagonalizable, i.e. $A = X\Lambda X^{-1}$ with Λ the (diagonal) eigenvalue matrix. Thus, we have

  $$\vec{u}'(t) = A\vec{u}(t) \qquad \Leftrightarrow \qquad \vec{u}'(t) = X\Lambda X^{-1}\vec{u}(t)\,. \tag{6.97}$$

  We set $\vec{v}(t) = X^{-1}\vec{u}(t)$. Then, we see

  $$\vec{u}'(t) = A\vec{u}(t) \qquad \Leftrightarrow \qquad X\vec{v}'(t) = X\Lambda\vec{v}(t) \qquad \Leftrightarrow \qquad \vec{v}'(t) = \Lambda\vec{v}(t)\,. \tag{6.98}$$

  Since Λ is diagonal, the equation $\vec{v}'(t) = \Lambda\vec{v}(t)$ describes a system of n independent ordinary differential equation. One says, the coupled system $\vec{u}'(t) = A\vec{u}(t)$ becomes uncoupled or decouples.

  The advantage of uncoupling the system is, that each of its equations is of the form $v_i'(t) = \lambda_i v_i(t)$ which can be solved readily. This is exactly how one obtains the solution in general form.

Matrix exponentials

Above, we studied an example of a couple system of two differential equations, which was represented by a matrix A which is diagonalizable. This begs to look into the situation of a non-diagonalizable A. In this case, we have to work a little harder. Our intention is to write the solution of $\vec{u}'(t) = A\vec{u}(t)$ as $\vec{u}(t) = e^{A\cdot t}\cdot\vec{u}(0)$. This intuition/desire derives from mimicking the case of a single differential equation. Hence, we must investigate the meaning of the expression $e^{A\cdot t}$.

Definition 6.4.3 (Matrix exponential)**:**
We define

$$e^{A\cdot t} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + A\cdot t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\,. \tag{6.99}$$

Remark (Differentiation)**:**
We may wonder what happens when we differentiate e^{At} with respect to t:

$$\left(\frac{d}{dt}\left(e^{At}\right)\right)(t) = A + A^2 t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \dots$$
$$= A \cdot \left(I + A \cdot t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right) \tag{6.100}$$
$$= A \cdot e^{At}.$$

Note:
We interchanged differentiation and the infinite sum. This is not always allowed. You want to revise your calculus class and absolute convergence.

Note:
The eigenvalues of e^{At} are closely related to those of A. Namely, suppose that $\vec{x}$ is an eigenvector of A with eigenvalue λ. Then it holds

$$e^{At}\vec{x} = \left(I + A \cdot t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right) \cdot \vec{x}$$
$$= \left(1 + \lambda \cdot t + \frac{\lambda^2t^2}{2!} + \frac{\lambda^3t^3}{3!} + \dots\right) \cdot \vec{x} \tag{6.101}$$
$$= e^{\lambda t} \cdot \vec{x}.$$

Hence, the eigenvalues of e^{At} are given by $e^{\lambda t}$ as λ ranges over all eigenvalues of A.

Example 6.4.8:
Let us compute e^{At} when A is diagonalizable, i.e. $A = X\Lambda X^{-1}$ with diagonal eigenvalue matrix Λ. Then:

$$e^{At} = I + A \cdot t + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$
$$= I + X\Lambda X^{-1} \cdot t + \frac{\left(X\Lambda X^{-1}\right)^2 t^2}{2!} + \frac{\left(X\Lambda X^{-1}\right)^3 t^3}{3!} + \dots \tag{6.102}$$
$$= X \cdot \left(I + \Lambda \cdot t + \frac{\Lambda^2t^2}{2!} + \frac{\Lambda^3t^3}{3!} + \dots\right) X^{-1}$$
$$= X \cdot e^{\Lambda t} X^{-1}.$$

Example 6.4.9:
Let us return to our opening example and see if $e^{At} \cdot \vec{u}(0)$ indeed does give the same solution as obtained earlier. Recall that we considered

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}. \tag{6.103}$$

It holds $A = X\Lambda X^{-1}$ with

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}, \qquad X = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \qquad X^{-1} = \frac{-1}{3} \cdot \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}. \tag{6.104}$$

Now an easy computation shows

$$e^{At}\vec{u}(0) = X \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix} \cdot X^{-1} \cdot \vec{u}(0) = \frac{1}{3} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}e^{-3t} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \qquad (6.105)$$

This is exactly the same answer as before.

Exercise:
In general, for two matrices A, B it holds

$$e^A \cdot e^B \neq e^B \cdot e^A \neq e^{A+B}\,! \qquad (6.106)$$

Convince yourself:

- Find two matrices A and B with $e^A \cdot e^B \neq e^B \cdot e^A \neq e^{A+B}$.
- Find two matrices A and B with $e^A \cdot e^B = e^B \cdot e^A = e^{A+B}$.

The general case has been investigated in large detail and resulted in the famous Baker-Campbell-Hausdorff formula. A special instances of this formula is that if $AB = BA$, then $e^A \cdot e^B = e^B \cdot e^A = e^{A+B}$. Prove this result.

Example 6.4.10:
Let us compute e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. To this end, we note that

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (6.107)$$

From this it follows $e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$. This is a rotation matrix, i.e. an element of the group $SU(2)$. We observe the following:

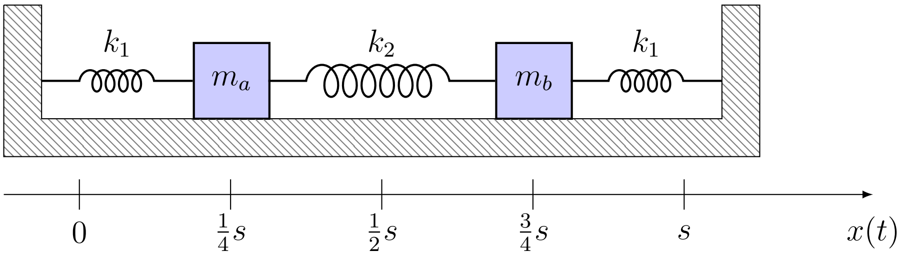
- Our A is skey-symmetric and e^{At} is orthogonal. This holds in general.
- The eigenvalues of A are i and $-i$. The eigenvalues of e^{At} are e^{it} and e^{-it}.

Remark:
If you study Lie groups, you will find that the matrix A is an element of the Lie algebra $\mathfrak{su}(2)$ and that there is a map, called the *exponential map*, $\mathfrak{su}(2) \to SU(2)$. It is given by exponentiation of matrices and allows to describe elements of $SU(2)$ by elements of the Lie algebra $\mathfrak{su}(2)$. This insight from group theory is particular important when it comes to representations of groups. The latter are, for example, omnipresent in qantum mechanics and quantum field theory.

Note:
For $x \in \mathbb{R}$, e^x is never zero and therefore has a (multiplicative) inverse. A similar fact holds true for matrices. If $A \in \mathbb{M}(n \times n, \mathbb{R})$, then e^{At} always has the inverse e^{-At}. This is true even if A is not invertible!

System of coupled masses

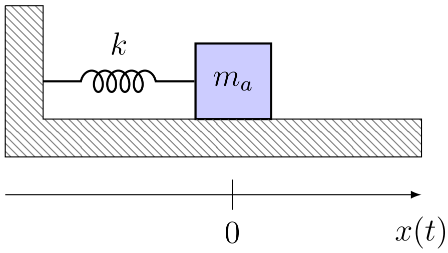
We complete this section by applying this technology to a real-life example, namely the following system of two masses coupled by springs:



(6.108)

The dynamics of this system is described by a coupled system of ordinary differential equations (ODEs). It involved the masses $m_a, m_b > 0$ as well as the Hook constants $k_1, k_2 > 0$ of the springs. The latter measure how hard it is to compress the springs.

Before we study the coupled system, we look at one mass m attached to a spring of Hook constant k:



(6.109)

By the Hook law, if we compress the spring $x(t)$, then it will generate a repulsive force to expand again. This repulsive force is proportional to the size of compression and the factor of proportionality is the Hook constant. Hence, if the mass m is at position $x(t)$, then this repulsive force is given by

$$F_{\text{spring}} = -kx(t)\,. \tag{6.110}$$

Note that if $x(t)$ is positive, then this will be a force of contraction. So the spring always wants to go back to its initial configuration. If we compress it, it wants to extend. And if we extend it, it wants to shrink. This force accelerates the mass m which by Newton's laws leads to the differential equation:

$$m \cdot x''(t) = -k \cdot x(t)\,. \tag{6.111}$$

In this expression, $x''(t)$ is exactly the acceleration of the mass m. And this acceleration is smaller for larger masses: ”heavier objects are harder to move“. The general solution to this equation is

$$x(t) = A \cdot \cos(\omega t) + B \cdot \sin(\omega t)\,, \tag{6.112}$$

where $\omega = \sqrt{\frac{k}{m}}$. By plugging this into eq. (6.111), it is not too hard to verify that it indeed solves the equation. That this is the most general solution is not at all obvious, but nonetheless true. Based on physical intuition, we could give a heuristic argument by saying that the mass on the spring must oscillate back and forth (as long as we ignore frictions, as we do here), and the two functions that simulate oscillations are sinus and cosine. So, in this spirit, we should anticipate from physical experience, that a linear combination of sinus and cosine generates the most general solution. (But of course, there is a more solid mathematical argument beyond this heuristic motivation, which we will not go into in this course.)

If we employ eq. (6.112), then we can link the constants A and B to the initial configuration of the system. Namely,

$$x(0) = A\,. \tag{6.113}$$

Hence, the position of the mass m at time $t = 0$ was $x_0 = x(0) = A$. Similarly, we have

$$x'(0) = B\omega\,. \tag{6.114}$$

Hence, we conclude that the velocity, with which the mass m was moving at time $t = 0$ is given by $v_0 = x'(0) = B\omega$. We can thus rewrite this solution as:

$$x(t) = x_0 \cdot \cos(\omega t) + \frac{v_0}{\omega} \cdot \sin(\omega t)\,. \tag{6.115}$$

With these insights, let us return to eq. (6.108). Note that we choose the coordinate system such that at time t, the mass m_a is at position $\frac{s}{4} + x_a(t)$ and m_b at position $\frac{3s}{4} + x_b(t)$. The displacements $x_a(t)$, $x_b(t)$ are governed by the system of ODEs similar to eq. (6.111), namely

$$\begin{aligned} m_a \cdot x_a''(t) &= -k_1 x_a(t) + k_2\left(x_b(t) - x_a(t)\right)\,, \\ m_b \cdot x_b''(t) &= -k_1 x_b(t) + k_2\left(x_a(t) - x_b(t)\right)\,. \end{aligned} \tag{6.116}$$

To solve this system, we define

$$\vec{x}(t) = \begin{bmatrix} x_a(t) \\ x_b(t) \end{bmatrix}\,. \tag{6.117}$$

Also, to simplify the computation, let us assume $m = m_a = m_b$. A more general setup can for example be investigate with `Python`. With that said, we can rewrite eq. (6.116) as

$$\vec{x}''(t) = A \cdot \vec{x}(t)\,, \qquad A = \frac{1}{m} \cdot \begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_1 + k_2) \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{6.118}$$

The matrix A can be diagonalized:

$$A = S\Lambda S^{-1}\,, \qquad S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\,, \qquad \Lambda = -\frac{1}{m}\begin{bmatrix} k_1 & 0 \\ 0 & k_1 + 2k_2 \end{bmatrix}\,. \tag{6.119}$$

So instead of solving this system directly, let us plug this diagonalization into it:

$$\vec{x}''(t) = S\Lambda S^{-1} \cdot \vec{x}(t) \tag{6.120}$$

$$\Leftrightarrow \qquad \left(S^{-1}\vec{x}(t)\right)'' = \Lambda \cdot \left(S^{-1}\vec{x}(t)\right) . \tag{6.121}$$

Hence, $\vec{y}(t) = S^{-1}\vec{x}(t)$ obeys a very simple system of differential equations:

$$\vec{y}''(t) = \begin{bmatrix} -\frac{k_1}{m} & 0 \\ 0 & -\frac{k_1+2k_2}{m} \end{bmatrix} \cdot \vec{y}(t) . \tag{6.122}$$

In other words, the system decoupled and the components of $\vec{y}(t)$ each satisfy an equation of the type in eq. (6.111), whose solution we know are eq. (6.112). Hence, by setting $\omega_1 = \sqrt{\frac{k_1}{m}}$, $\omega_2 = \sqrt{\frac{k_1+2k_2}{m}}$, we find:

$$\vec{y}(t) = \begin{bmatrix} c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t) \\ d_1 \cos(\omega_2 t) + d_2 \sin(\omega_2 t) \end{bmatrix} . \tag{6.123}$$

The constants $c_1, c_2, d_1, d_2 \in \mathbb{R}$ can be fixed with the initial conditions. To this end, we first compute $\vec{x}(t)$:

$$\vec{x}(t) = S \cdot \vec{y}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t) \\ d_1 \cos(\omega_2 t) + d_2 \sin(\omega_2 t) \end{bmatrix} , \tag{6.124}$$

$$= \begin{bmatrix} c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t) - d_1 \cos(\omega_2 t) - d_2 \sin(\omega_2 t) \\ c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t) + d_1 \cos(\omega_2 t) + d_2 \sin(\omega_2 t) \end{bmatrix} . \tag{6.125}$$

Subsequently, we fix the constants by fixing initial conditions for the masses m_a and m_b:

$$x_a = x_a(0) , \qquad v_a = x_a'(0) , \qquad x_b = x_b(0) , \qquad v_b = x_b'(0) . \tag{6.126}$$

Explicitly,

$$\vec{x}(0) = \begin{bmatrix} c_1 - d_1 \\ c_1 + d_1 \end{bmatrix} , \qquad \vec{x}'(0) = \begin{bmatrix} c_2\omega_1 - d_2\omega_2 \\ c_2\omega_1 + d_2\omega_2 \end{bmatrix} . \tag{6.127}$$

This gives

$$c_1 = \frac{x_b + x_a}{2} , \qquad d_1 = \frac{x_b - x_a}{2} , \tag{6.128}$$

$$c_2 = \frac{v_a + v_b}{2\omega_1} , \qquad d_2 = \frac{v_b - v_a}{2\omega_2} , \tag{6.129}$$

which one can plug back into eq. (6.125) to obtain the final solution. The dynamics of this system are rich and are best seen by playing with its parameters and plotting the results. Therefore, this is left as an exercise on a homework assignment.

Remark:
In a similar spirit, we can solve the equation $y''(t) - 2y'(t) + y(t) = 0$ for given initial values $y(0)$ and $y'(0)$. The first step is to realize that, even though we have been given one equation, there are secretly two equations. To see this, set $\vec{u}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Then, we have

$$\begin{aligned} \left(\frac{d}{dt}y\right)(t) &= y'(t)\,, \\ \left(\frac{d}{dt}y'\right)(t) &= y''(t) = 2y'(t) - y(t)\,. \end{aligned} \tag{6.130}$$

This we can encode in the equation

$$\left(\frac{d}{dt}\vec{u}\right)(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \cdot \vec{u}(t) \equiv A \cdot \vec{u}(t)\,. \tag{6.131}$$

Note that the matrix A is *not* diagonalizable. Namely, the characteristic polynomial has repeated root 1 and the 1-eigenspace is spanned only by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. That is, the algebraic and geometric multiplicity do not coincide, showing that A is not diagonalizable.

Hence, we compute e^{At} from its definition as infinite series. The key fact which makes this computation easy is $(A - I)^2 = 0$. We use this by writing

$$e^{At} = e^{It} \cdot e^{(A-I)t} = e^{It} \cdot (I + (A - I) \cdot t) = e^{t} \cdot \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}\,. \tag{6.132}$$

Therefore, the general solution is given by

$$\vec{u}(t) = e^{t} \cdot \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \cdot \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}\,. \tag{6.133}$$

And thus $y(t)$ is given by the first component of $\vec{u}(t)$.

6.5 Eigenvalues and eigenvectors of real, symmetric matrices

6.5.1 The spectral theorem

Note:
We now focus on eigenvalues and eigenvectors of real, symmetric matrices. We already discussed special instances of such matrices, namely reflections and projections. In both cases, we had a basis of eigenvectors. Even more – we had a basis of *orthogonal* eigenvectors! We can thus wonder if this is true in general and what can be said about the eigenvalues. It turns out, a whole lot!

Remark:
Let us see what diagonalizability implies in the context of symmetric matrices. So take $S \in \mathbb{M}(n \times n, \mathbb{R})$ with $S = S^T$ and assume $S = X\Lambda X^{-1}$. Then $S^T = \left(X^{-1}\right)^T \Lambda^T X^T$. Since $S = S^T$, we may hope that $X^{-1} = X^T$, or equivalently $X^T X = I$. This in turn means that X better be orthogonal! Indeed, diagonalization acquires a really nice form in the setting of symmetric matrices.

Theorem 6.5.1 (Spectral theorem)**:**
Every symmetric matrix $S \in \mathbb{M}(n \times n, \mathbb{R})$ has real eigenvalues. It admits a factorization

$$S = Q\Lambda Q^T \,, \tag{6.134}$$

with the eigenvalues $\lambda_1, \ldots, \lambda_n$ of S along the diagonal of Λ (all other entries of Λ are zero). Furthermore, there is a basis of $\mathbb{R}^n$ formed from orthonormal eigenvectors of S. Such eigenvectors, with eigenvalues $\lambda_1, \ldots, \lambda_n$, form the columns of Q. In particular, Q is orthogonal, i.e. $Q^T Q = I = QQ^T$.

Remark:
Let us emphasize again, that for a real, symmetric matrix $S \in \mathbb{M}(n \times n, \mathbb{R})$, it holds:

- All eigenvalues of S are real.
- There exists a basis of $\mathbb{R}^n$ formed from *orthonormal* eigenvectors of S.

Example 6.5.1:
Let us study an example. We consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{6.135}$$

Then we find $\mathrm{ch}_A(\lambda) = -\left(\lambda - 5\right) \cdot \left(\lambda + 1\right)^2$. Hence, the eigenvalues of A are 5, -1. By use of the Gram-Schmidt procedure, we find

$$\begin{aligned} \mathrm{Eig}_{\mathbb{R}}\left(A, -1\right) &= \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\} \\ \mathrm{Eig}_{\mathbb{R}}\left(A, 5\right) &= \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \end{aligned} \tag{6.136}$$

This leads to the following orthogonal basis of $\mathbb{R}^3$ from eigenvectors of A:

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{6.137}$$

We can normalize this basis, to find

$$\mathcal{B}_0 = \left\{ \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} . \tag{6.138}$$

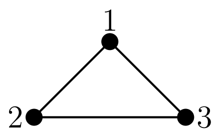
This is an orthonormal basis of $\mathbb{R}^3$ from eigenvectors of A.

Exercise:
Let Q be the matrix whose columns are the above 3 eigenvectors of A. Check that

$$Q^T A Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} . \tag{6.139}$$

Note:
No inverses are needed for the diagonalization of symmetric matrices, but we make use of the Gram-Schmidt procedure.

Remark:
Symmetric matrices appear in many important applications. One is the Hessian matrix that allows to investigate the type of local extrema of a function $f\colon \mathbb{R}^n \to \mathbb{R}$. Another instance are *adjacency matrices* in graph theory. If node i connects to node j, then we record a 1 in column i row j. Otherwise, we record a 0. For example, the graph



(6.140)

gives the *adjacency matrix*

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) . \tag{6.141}$$

Consequently, adjancency matrices are always symmetric. Thus, the spectral theorem applies and we conclude that all eigenvalues are real. This bit is very important when studying connectivity properties of *sparse graphs*. The keyword you want to look up is the *spectral gap*. These questions are to be studied when designing robust networks, which should be so-called *expander graphs*.

Claim 24:
Be $S \in \mathbb{M}(n \times n, \mathbb{R})$ a symmetric matrix. Then its eigenvalues are real.

Proof
Suppose S is symmetric and $S\vec{x} = \lambda\vec{x}$. A priori, $\vec{x}$ and λ might be complex valued. Complex conjugation yields $S\overline{\vec{x}} = \overline{\lambda} \cdot \overline{\vec{x}}$ where we used $\overline{S} = S$ since $S \in \mathbb{M}(n \times n, \mathbb{R})$.

Now, transposition gives $\overline{\vec{x}}^T \cdot S = \overline{\vec{x}}^T \cdot \overline{\lambda}$ where we used that S is symmetric, i.e. $S^T = S$. Right multipication with $\vec{x}$ gives

$$\overline{\vec{x}}^T \cdot S \cdot \vec{x} = \overline{\vec{x}}^T \cdot \overline{\lambda} \cdot \vec{x}\,. \tag{6.142}$$

But note, that left multiplication of $S\vec{x} = \lambda\vec{x}$ with $\overline{\vec{x}}$ gives

$$\overline{\vec{x}}^T \cdot S \cdot \vec{x} = \overline{\vec{x}}^T \cdot \lambda \cdot \vec{x}\,. \tag{6.143}$$

Hence, by comparing eq. (6.142) and eq. (6.143), we find

$$\overline{\lambda} \cdot \vec{x}^T\vec{x} = \lambda \cdot \vec{x}^T\vec{x}\,. \tag{6.144}$$

Recall that we assumed that $\vec{x}$ is an eigenvector. Hence, $\vec{x} \neq 0$ and thus $\overline{\vec{x}}^T \cdot \vec{x} \in \mathbb{R}_{>0}$. Therefore $\lambda = \overline{\lambda}$, which completes this proof. ■

Claim 25:
Be $S \in \mathbb{M}(n \times n, \mathbb{R})$ a symmetric matrix, $\vec{x}$, $\vec{y}$ two eigenvectors of S with different eigenvalues. Then $\vec{x}^T\vec{y} = 0$, i.e. $\vec{x}\perp\vec{y}$.

Proof
Suppose that $S\vec{x} = \lambda_1\vec{x}$ and $S\vec{y} = \lambda_2\vec{y}$ with $\lambda_1 \neq \lambda_2$. Then

$$\lambda_1 \cdot \vec{x}^T \cdot \vec{y} = (\lambda_1\vec{x})^T \cdot \vec{y} = (S\vec{x})^T \cdot \vec{y} = \vec{x}^T S^T \vec{y} = \vec{x}^T S\vec{y} = \lambda_2 \cdot \vec{x}^T \cdot \vec{y}\,. \tag{6.145}$$

Thus, since $\lambda_1 \neq \lambda_2$, we must have $\vec{x}^T\vec{y} = 0$ and $\vec{x}\perp\vec{y}$. ■

Note:
We will omit the argument which implies that any symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$ does admit enough eigenvectors to form a basis of $\mathbb{R}^n$. Rather, we take this as faith. Then, in summary, our observations imply that we can write

$$S = Q \cdot \Lambda Q^T\,, \tag{6.146}$$

with $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_i \in \mathbb{R}$ are the eigenvalues of S and the columns of Q are eigenvectors of S which furnish an orthonormal basis of $\mathbb{R}^n$.

Consequence (Interpretation of the spectral theorem)**:**
Let us write $S = Q \cdot \Lambda Q^T$ explicitly:

$$S = \begin{bmatrix} \vec{q}_1 & \ldots & \vec{q}_n \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \cdot \begin{bmatrix} \vec{q}_1^T \\ \vdots \\ \vec{q}_n^T \end{bmatrix} = \sum_{i=1}^{n} \lambda_i \cdot \left(\vec{q}_i\vec{q}_i^T\right). \tag{6.147}$$

Note that, since $\vec{q}_i$ is a vector of length 1, the matrices $\vec{q}_i \cdot \vec{q}_i^T$ are projection matrices. Hence, the spectral theorem says, that any symmetric, real matrix is a linear combination of projection matrices.

6.5.2 Definitness of matrices

Note:
We encountered the definiteness of matrices when studying the type of local extrema of functions $f: \mathbb{R}^n \to \mathbb{R}$. Namely, we found that a local extremum is a local minimum/maximum if the Hessian matrix of f is positive/negative definite. Other applications include the study of quadrics, e.g. ellipses/parabola. We develop the notion of definiteness of matrices before we exemplify its applications to the topics of quadrics.

Definition 6.5.1:
A symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$ with eigenvalues $\lambda_i \in \mathbb{R}$ is termed

- positive semi-definite iff all $\lambda_i \geq 0$,
- positive definite iff all $\lambda_i > 0$,
- negative semi-definite iff all $\lambda_i \leq 0$,
- negative definite iff all $\lambda_i < 0$,
- indefinite, if there is (at least) one positive and one negative eigenvalue.

Comment:
We will focus on positive definite matrices. But many of the following statements extend negative (semi-)definite matrices.

Note:
Our first task is to be able to tell when a matrix is positive-definite. One potential approach would be to compute the roots of the characteristic polynomial and then check the signs. However, this approach is not very smart. Namely, computing the roots of polynomials is not an easy task, expecially as numerical approaches are prone to error. So we would ideally like to avoid computing the roots. After all, we are solely interested in their signs. Luckily for us, there are various ways.

Example 6.5.2:
Let us consider $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})$. The eigenvalues are $\lambda_1 = \det(S)$ and $\lambda_2 = \operatorname{tr}(S)$. Hence, this matrix is positive definite iff

$$a + c > 0 \qquad \text{and} \qquad ac - b^2 > 0\,. \tag{6.148}$$

Equivalently, we can write ($ac > b^2$ requires that a, c have the same sign and it is positive since $a + c > 0$)

$$a > 0 \qquad \text{and} \qquad ac - b^2 > 0\,. \tag{6.149}$$

Note that A is the determinant of the top-left submatrix $\tilde{S} = [a]$ of S and that $ac - b^2$ is the determinant of S. Alternatively, we can take the viewpoint of pivots:

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \to \begin{bmatrix} a & b \\ 0 & \frac{ac-b^2}{a} \end{bmatrix}\,. \tag{6.150}$$

We thus see that positivity of the pivots a and $\frac{ac-b^2}{a}$ guarantees that all eigenvalues are positive as well. Both of these criteria do apply to real, symmetric matrices.

Corollary 6.5.1:
For a symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$, the following are equivalent:

- S is positive definite,
- all upper left determinants of S are positive,
- all pivots of S are positive.

Example 6.5.3:
Consider the matrix

$$S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{6.151}$$

The upper left determinants of S are 2, 3 and 4. Thus, we see that this matrix is positive definite. Alternatively, we find the row echelon form of S:

$$S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} \end{bmatrix}\,. \tag{6.152}$$

We thus see that all pivots are positive. Consequently, S is positive definite.

Note:
Let us look at the eigenvector equation $S\vec{x} = \lambda\vec{x}$. Then

$$\vec{x}^T S \vec{x} = \lambda \cdot \vec{x}^T \vec{x}\,. \tag{6.153}$$

So, for a positive definite matrix S, the RHS is positive. In fact, this is true for all non-zero vectors $\vec{x}$ if S is positive definite.

Corollary 6.5.2:
For a symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$, the following are equivalent:

- S is positive definite,
- $\vec{x}^T S \vec{x} > 0$ for all $\vec{x} \in \mathbb{R}^n \setminus \vec{0}$.

Example 6.5.4:
Let us consider $\vec{x}^T = \begin{bmatrix} x & y \end{bmatrix}$ and $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})$. Then we conclude

$$\vec{x}^T S \vec{x} = ax^2 + 2bxy + cy^2\,. \tag{6.154}$$

Let us discuss what the positivity of this expression means for all $(x, y) \in \mathbb{R}^2 - \{(0,0)\}$. W.l.o.g., let us assume $y \neq 0$. Then we have

$$a\left(\frac{x}{y}\right)^2 + 2b \cdot \left(\frac{x}{y}\right) + c > 0\,. \tag{6.155}$$

let us set $z = \frac{x}{y}$. Then, what do we know about a, b, c if the parabola $az^2 + bz + c$ is positive for all $z \in \mathbb{R}$? Since this parabola must be concave up, we get that $a > 0$. Since we do not want any roots to $az^2 + bz + c = 0$, we must have $4\left(b^2 - ac\right) < 0$. Note that these are precisely the conditions that guarantee positive definiteness! Of course, this analysis becomes a little more challenging for S of larger dimension.

Remark:
Observe the surprising fact, that if S and T are positive definite, then so is $S{+}T$. Realize, that proving this fact by way of positivity of pivots or upper left determinants is nearly impossible. But with our third criterion, it is completely straightforward. Indeed, for any non-zero $\vec{x}$ we find

$$\vec{x}^T\left(S + T\right)\vec{x} = \vec{x}^T S\vec{x} + \vec{x}^T T\vec{x} > 0\,. \tag{6.156}$$

Note:
There is yet another criterion for definiteness. Let $A \in \mathbb{M}(m \times n, \mathbb{R})$ a rectangular, real matrix. Then $S = A^T A \in \mathbb{M}(n \times n, \mathbb{R})$ is symmetric. It turns out that S is positive definite, provided A has linearly independent columns. Namely, under this assumption we know $A\vec{x} \neq 0$ for any $\vec{x} \neq 0$. Consequently,

$$\vec{x}^T S\vec{x} = \vec{x}^T A^T A\vec{x} = \left(A\vec{x}\right)^T\left(A\vec{x}\right) = \left|A\vec{x}\right|^2 > 0\,. \tag{6.157}$$

Consequence:
We have established that for symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$, the following are equivalent:

- S is positive definite,
- all upper left determinants of S are positive,
- all pivots of S are positive,
- $\vec{x}^T S\vec{x} > 0$ for all $\vec{x} \neq 0$,
- $S = A^T A$ for some real matrix A with linearly independent columns.

Note:
We may wonder how we can express a symmetric $S \in \mathbb{M}(n \times n, \mathbb{R})$ as $S = A^T A$. To this end we note that $S = LDL^T$ is the symmetric version of the LU-factorization (cf. section 2.5). Furthermore, since S is positive definite, we know that D has positive entries λ_i along the diagonal. Therefore, we can consider $\sqrt{D} := \operatorname{diag}\left(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}\right)$. Now, for $A^T = L\sqrt{D}$ we find

$$A^T A = L \cdot \sqrt{D} \cdot \sqrt{D} \cdot L^T = LDL^T = S\,. \tag{6.158}$$

Definition 6.5.2:
For a symmetric, positive definite $S \in \mathbb{M}(n \times n, \mathbb{R})$, we term $A = \left(L\sqrt{D}\right)^T$ the *Cholesky factor* and $S = A^T A$ the *Cholesky decomposition* of S.

Remark:
The *Cholesky factor* is triangular but involves square roots. The latter can at times be undesirable.

Note:
We can also use the eigenvalue matrix of S instead of D. Say $S = Q\Lambda Q^T$, where Q is orthogonal. Since S is positive definite, Λ has positive entries and hence we can take square roots. Consequently

$$S = Q\Lambda Q^T = \left(Q\sqrt{\Lambda}Q^T\right)^T \cdot \left(Q\sqrt{\Lambda}Q^T\right) . \tag{6.159}$$

Thus, with $A = Q\sqrt{\Lambda}Q^T$, we again find $S = A^T A$.

Example 6.5.5:
Let us consider

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) . \tag{6.160}$$

Convince yourself, with any of the above tests, that S is indeed positive definite. In particular, we find

$$\vec{x}^T S \vec{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 . \tag{6.161}$$

Since S is positive definite, this quantity is positive whenever $(x_1, x_2, x_3) \neq (0, 0, 0)$. To see this, we want to write this quantity as a sum of squares. If we achieve $S = A^T A$, then indeed we have $\vec{x}^T S \vec{x} = |A\vec{x}|^2$. Explicitly:

- If you use $S = LDL^T$, then you find

$$\vec{x}^T S \vec{x} = 2\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}x_3^2 . \tag{6.162}$$

- If you use $S = Q\Lambda Q^T$, then you find

$$\vec{x}^T S \vec{x} = \lambda_1 \cdot \left(\vec{q}_1^T \vec{x}\right)^2 + \lambda_2 \cdot \left(\vec{q}_2^T \vec{x}\right)^2 + \lambda_3 \cdot \left(\vec{q}_3^T \vec{x}\right)^2 . \tag{6.163}$$

Remark:
As already mentioned, closely related to positive definite matrices are *positive semi-definite matrices.*, for which the eigenvalues are constrained to be non-negative. By replacing all ">0" above by "≥ 0", you can find criteria for semi-definiteness. In particular, positive semi-definite matrices possess a factorization $S = A^T A$ where A is allowed to have linearly dependent columns.

6.5.3 Application: Quadratic forms

Note:
The decomposition $S = Q\Lambda Q^T$ has a meaning for quadratic forms. This is what we turn to next.

Example 6.5.6:
Let us consider the ellipse

$$E = \left\{(x, y) \in \mathbb{R}^2 \,,\, ax^2 + 2bxy + cy^2 = 1\right\} \,. \tag{6.164}$$

Note that we can write

$$ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \equiv \vec{x}^T S \vec{x} \,. \tag{6.165}$$

When S is positive definite, then E is indeed an ellipse, otherwise not. It thus follows, that there is a relation between

- 2×2 positive definite matrices,
- ellipses in $\mathbb{R}^2$.

Exercise:
Verify that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives a circle.

Example 6.5.7:
Let us now find the axes of the tilted ellipse given by

$$E = \left\{(x, y) \in \mathbb{R}^2 \,,\, 5x^2 + 8xy + 5y^2 = 1\right\} \,. \tag{6.166}$$

To this end we first notice that

$$5x^2 + 8xy + 5y^2 = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \equiv \vec{x}^T S \vec{x} \,. \tag{6.167}$$

If is readily verified that the eigenvalues are $1, 9$. Moreover, we have

$$\begin{aligned} \text{Eig}\,(S, 1) &= \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \,, \\ \text{Eig}\,(S, 9) &= \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \,. \end{aligned} \tag{6.168}$$

We know that these two eigenvectors are orthogonal as S is symmetric. Let us make them orthonormal. Then we have

$$S = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \equiv Q\Lambda Q^T \,. \tag{6.169}$$

Thereby, we find

$$5x^2 + 8xy + 5y^2 = 1 \cdot \left(\frac{-x+y}{\sqrt{2}} \right)^2 + 9 \cdot \left(\frac{x+y}{\sqrt{2}} \right)^2 . \tag{6.170}$$

If we introduce new coordinates by

$$X := \frac{x+y}{\sqrt{2}} , \qquad Y := \frac{-x+y}{\sqrt{2}} , \tag{6.171}$$

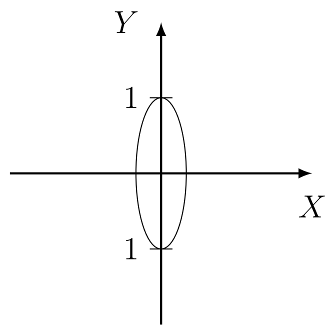
then we can view the ellipse as

$$\tilde{E} = \left\{ (X, Y) \in \mathbb{R}^2 , \, 9X^2 + Y^2 = 1 \right\} . \tag{6.172}$$

Hence, the original ellipse E is really $\tilde{E}$ after a certain coordinate transformation, namely

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{\sqrt{2}} \\ \frac{-x+y}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \equiv R \cdot \begin{bmatrix} x \\ y \end{bmatrix} . \tag{6.173}$$

Note that R is a rotation of the x-y plane by $\frac{\pi}{4}$ clockwise. Hence, E becomes $\tilde{E}$ upon rotation by 45 degrees clockwise:



(6.174)

Consequence:
The eigenvectors give us the directions of the major and minor axis of the ellipise. For this very reason, the factorization for $S = Q\Lambda Q^T$ is sometimes referred to as the *principal axis theorem.*

Remark:
In general, $\vec{x}^T S \vec{x} = 1$ describes an ellipsoid in $\mathbb{R}^n$, provided that S is positive definite.

7 Further topics

7.1 Singular value decomposition (SVD)

We will now discuss a very interesting factorization, that applies to any matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$. Hence, this matrix need not be square or even diagonalizable. Still, one can always write A as

$$A = U \Sigma V^T \,, \tag{7.1}$$

where $U \in \mathbb{M}(m \times m, \mathbb{R}$ and $V \in \mathbb{M}(n \times n, \mathbb{R})$ are orthogonal and $\Sigma \in \mathbb{M}(m \times n, \mathbb{R})$ similar to the eigenvalue matrix of a diagonalizable matrix. In particular, most entries of Σ will be zero and its non-zero entries along the diagonal are called the singular values of A. Such a factorization is called a *singular value decomposition.*

The singular values measure, in a certain way, important contributions to the matrix A. Therefore, by taking only the most important singular values, we can approximate the matrix A. This is for example employed in image compression. Another rather important application is the *principal component analysis* (PCA).

7.1.1 Existence and details of the singular value decomposition

Remark:
We begin by recalling the four fundamental vector spaces of a matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$. This we summarized most efficiently in the image-coimage factorization of the linear map $\varphi_A \colon \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$. Namely, recall that any such map can be factored by the four linear subspaces $N(A)$, $R(A)$, $C(A)$, $N(A^T)$:

$$\begin{array}{ccccccc}
\ker(\varphi_A) \cong N(A) & \xhookrightarrow{\varphi_K} & \mathbb{R}^n & \xrightarrow{\varphi_A} & \mathbb{R}^m & \xrightarrow{\varphi_P} & N(A^T) \cong \operatorname{coker}(\varphi_A) \\
 & & \Big\downarrow{\scriptstyle \varphi_{M_1}} & & {\scriptstyle \varphi_{M_2}}\Big\uparrow & & \\
 & \operatorname{coim}(\varphi_A) \cong & R(A) & \xrightarrow{\varphi_X} & C(A) & \cong \operatorname{im}(\varphi_A) &
\end{array} \tag{7.2}$$

To some extend, one could view the singular value decomposition as a matrix version of this diagram. We will come back to this momentarily.

Note:
If $A \in \mathbb{M}(m \times n, \mathbb{R})$, then $A^T A \in \mathbb{M}(n \times n, \mathbb{R})$ is a symmetric and positive semi-definite matrix. Hence, it admits a basis

$$\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\} \,, \tag{7.3}$$

of $\mathbb{R}^n$ of orthonormal eigenvectors with non-negative eigenvalues $\lambda_1, \dots, \lambda_n$. We will assume that the eigenvectors are ordered such that their eigenvalues are non-ascending. In order words, we assume

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \,. \tag{7.4}$$

Example 7.1.1:
Let us consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R}) \,. \tag{7.5}$$

Then we have

$$A^T A = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) \,. \tag{7.6}$$

We find $\mathrm{ch}_{A^TA}(\lambda) = -\lambda \cdot (10 - \lambda) \cdot (12 - \lambda)$. Hence, the eigenvalues are 0, 10 and 12. So we order them as follows:

$$\lambda_1 = 12 \,, \qquad \lambda_2 = 10 \,, \qquad \lambda_3 = 0 \,. \tag{7.7}$$

The eigenspaces are as follows:

$$\begin{aligned} \mathrm{Eig}_{\mathbb{R}}(A^T A, 12) &= \left\{ c \cdot \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \,, c \in \mathbb{R} \right\} \,, \\ \mathrm{Eig}_{\mathbb{R}}(A^T A, 10) &= \left\{ c \cdot \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \,, c \in \mathbb{R} \right\} \,, \\ \mathrm{Eig}_{\mathbb{R}}(A^T A, 0) &= \left\{ c \cdot \frac{1}{\sqrt{30}} \cdot \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \,, c \in \mathbb{R} \right\} \,, \end{aligned} \tag{7.8}$$

Then the orthonormal eigenvectors $\vec{v}_i$ are as follows:

$$\vec{v}_1 = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \,, \qquad \vec{v}_2 = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \,, \qquad \vec{v}_3 = \frac{1}{\sqrt{30}} \cdot \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \,. \tag{7.9}$$

Remark:
We distinguish two types of eigenvectors of $A^T A$:

- Eigenvalue $\lambda = 0$:
  Suppose that $\vec{x}$ is an eigenvector of $A^T A$ with eigenvalue 0. Then $A^T A \vec{x} = 0$. Since $\vec{x} \neq \vec{0}$, this is equivalent to

$$0 = \vec{x}^T A^T A \vec{x} = (A\vec{x})^2 \,. \tag{7.10}$$

Hence $A\vec{x} = 0$ and therefore $\vec{x} \in N(A)$. Conversely, let $\vec{x} \in N(A)$. Then $A^T A\vec{x} = 0$ and thus $\vec{x}$ is an eigenvector of $A^T A$ with eigenvalue 0. Consequently, the eigenvectors to $A^T A$ with eigenvalue 0 span $N(A)$.

- Eigenvalue $\lambda > 0$:
  Since $\mathbb{R}^n = N(A) \oplus R(A)$, we then conclude that the eigenvectors of $A^T A$ with eigenvalue $\lambda_i > 0$ span $R(A)$.

Example 7.1.2 (Continuation)**:**
Recall that for

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R}), \tag{7.11}$$

we found

$$\vec{v}_1 = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \vec{v}_3 = \frac{1}{\sqrt{30}} \cdot \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}. \tag{7.12}$$

with $\lambda_1 = 12$, $\lambda_2 = 10$ and $\lambda_3 = 0$. Indeed, it holds

$$R(A) = \mathrm{Span}_{\mathbb{R}}(\vec{v}_1, \vec{v}_2), \qquad N(A) = \mathrm{Span}_{\mathbb{R}}(\vec{v}_3). \tag{7.13}$$

Note:
We form a matrix from the vector $\vec{v}_i$, namely

$$V = \begin{bmatrix} | & | & \dots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & \dots & | \end{bmatrix} \in \mathbb{M}(n \times n, \mathbb{R}). \tag{7.14}$$

Let us take r to be rank of the matrix A. Then $\{\vec{v}_1, \dots, \vec{v}_r\}$ is a basis of $R(A)$ and the remaining vectors span $N(A)$. Now suppose that $\vec{x} \in N(A)$. Then the first r-components of $V^T\vec{x}$ vanish. Conversely, if $\vec{x} \notin N(A)$, then at least one of the first r components of $V^T\vec{x}$ is non-zero. We take V^T to represent the left-half of the diagram in eq. (7.2).

Example 7.1.3 (Continuation)**:**
For the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R})$ we have

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}). \tag{7.15}$$

Construction 7.1.1:
We now turn to the isomorphism and the right-half of eq. (7.2). We recall that $R(A) \cong C(A)$. In particular, given that we have an orthonormal basis $\vec{v}_1, \dots, \vec{v}_r$ of $R(A)$, a

generating set of $C(A)$ is given by $A\vec{v}_1, \ldots, A\vec{v}_r$. We make a slight modification. Namely, we identified the eigenvalues $\lambda_1, \ldots, \lambda_r > 0$ of $A^T A$ above. Motivated by the fact that $A^T A$ looks like a square of the matrix A, it is tempting to consider $\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}$ as "eigenvalues" of A. Indeed, those are exactly the singular values of A. We thereby define a basis of $C(A)$ by

$$\{\vec{u}_1\,, \ldots, \vec{u}_r\} = \left\{\frac{A\vec{v}_1}{\sigma_1}\,, \ldots, \frac{A\vec{v}_r}{\sigma_r}\right\} \subset \mathbb{R}^m\,. \tag{7.16}$$

Note that these vectors are orthonormal since $\vec{v}_i$ are! Namely,

$$\begin{aligned} \vec{u}_i^T \vec{u}_j &= \left(\frac{A\vec{v}_i}{\sigma_i}\right)^T \cdot \left(\frac{A\vec{v}_j}{\sigma_j}\right) = \frac{\vec{v}_i^T A^T A\vec{v}_j}{\sigma_i\sigma_j} = \frac{\vec{v}_i^T \sigma_j^2 \cdot \vec{v}_j}{\sigma_i\sigma_j} = 0\,, \\ \vec{u}_i^T \vec{u}_i &= \left(\frac{A\vec{v}_i}{\sigma_i}\right)^T \cdot \left(\frac{A\vec{v}_i}{\sigma_i}\right) = \frac{\vec{v}_i^T A^T A\vec{v}_i}{\sigma_i^2} = \frac{\vec{v}_i^T \sigma_i^2 \cdot \vec{v}_i}{\sigma_i^2} = \vec{v}_i^T \vec{v}_i = 1\,. \end{aligned} \tag{7.17}$$

Hence, by extending those vectors $\vec{u}_i$ with an orthonormal basis $\{\vec{u}_{r+1}\,, \ldots, \vec{u}_m\}$ of $N(A^T)$ we obtain an orthonormal basis of $\mathbb{R}^m$. This allows to define the orthogonal matrix

$$U = \begin{bmatrix} | & | & \cdots & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & \cdots & | \end{bmatrix} \in \mathbb{M}(m \times m, \mathbb{R})\,. \tag{7.18}$$

This matrix represents the right-half of eq. (7.2). The isomorphism at the bottom-middle of this diagram can be represented by the block-diagonal matrix

$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ \hline & & & \end{array}\right] \in \mathbb{M}(m \times n, \mathbb{R})\,, \tag{7.19}$$

where we only display its non-trivial entries. Hence, at the top-left, there is a diagonal matrix with $\sigma_1, \ldots, \sigma_r$ on its diagonal. Then we extend this to an $m \times n$-matrix by adding sufficiently many zeros.

Example 7.1.4:
Let us again return to the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R})\,. \tag{7.20}$$

Recall that

$$
\begin{aligned}
\mathrm{Eig}_{\mathbb{R}}(A^T A, 12) &= \left\{ c \cdot \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} , c \in \mathbb{R} \right\} , \\
\mathrm{Eig}_{\mathbb{R}}(A^T A, 10) &= \left\{ c \cdot \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} , c \in \mathbb{R} \right\} , \\
\mathrm{Eig}_{\mathbb{R}}(A^T A, 0) &= \left\{ c \cdot \frac{1}{\sqrt{30}} \cdot \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} , c \in \mathbb{R} \right\} ,
\end{aligned}
\tag{7.21}
$$

from which we formed the matrix

$$
V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) . \tag{7.22}
$$

To find the matrix $U \in \mathbb{M}(2 \times 2, \mathbb{R})$, we recall that σ_i is the (positive) square roots of the non-zero eigenvalues, namely $\sigma_1 = \sqrt{12}$ and $\sigma_2 = \sqrt{10}$. Then, by the above definition

$$
\begin{aligned}
\vec{u}_1 &= \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{\sqrt{12}} \cdot \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \\
\vec{u}_2 &= \frac{A\vec{v}_2}{\sigma_2} = \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} .
\end{aligned}
\tag{7.23}
$$

Here, $m = 2$ and we do not have to add any vectors $\vec{u}_i$. Or put differently, $N(A^T) = \{\vec{0}\}$. Hence, we find

$$
\begin{aligned}
U &= \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) , \\
\Sigma &= \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R}) .
\end{aligned}
\tag{7.24}
$$

Consequence:
You may by now wonder: Why do we construct these matrices? Anything special about them? The answer is yes. To see this, let us work out the meaning of $U\Sigma V^T$. By the above definitions, it holds

$$
U\Sigma V^T = \sum_{i=1}^{r} \sigma_i \cdot \vec{u}_i \vec{v}_i^T . \tag{7.25}
$$

By recall that $\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$. Hence, we find

$$
U\Sigma V^T = \sum_{i=1}^{r} A \cdot \vec{v}_i \vec{v}_i^T = A \cdot \sum_{i=1}^{r} \vec{v}_i \vec{v}_i^T . \tag{7.26}
$$

Now take $\vec{x} in \mathbb{R}^n$. Therefore, we can write

$$\vec{x} = \sum_{j=1}^{n} \lambda_j \vec{v}_j \,. \tag{7.27}$$

Consequently, we have

$$U\Sigma V^T \vec{x} = \sum_{i=1}^{n} \lambda_i A\vec{v}_i = A\vec{x} \,. \tag{7.28}$$

Hence $A = U\Sigma V^T$! This is the singular value decomposition of A.

Corollary 7.1.1:
Every $A \in \mathbb{M}(m \times n, \mathbb{R})$ admits a singular value decomposition $A = U\Sigma V^T$. We can write is also as

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T \,. \tag{7.29}$$

Hence, the singular value decomposition expresses A as a sum of r matrices of rank 1. This identifies r as the rank of A. Also, recall that $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0$. Suppose that $\sigma_1 \gg \sigma_2$. Then

$$A \approx \sigma_1 \vec{u}_1 \vec{v}_1^T \,. \tag{7.30}$$

Hence, we can then approximate the matrix A. This is the key insight for many applications of the singular value decomposition.

Example 7.1.5:
Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R}) \,. \tag{7.31}$$

Then we found

$$\begin{aligned} V &= \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) \,, \\ U &= \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) \,, \\ \Sigma &= \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R}) \,. \end{aligned} \tag{7.32}$$

An explicit computation shows that $A = U\Sigma V^T$. One approximation of A is given by only keeping only the largest singular value. Hence,

$$A \approx \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}^T \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} . \quad (7.33)$$

This may not seem too impressive. The problem is that $\sigma_1 \sim \sigma_2$. We will come back to this momentarily.

Example 7.1.6:
As another example take

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) . \quad (7.34)$$

Then the singular value decomposition of A is (numeric approximation) given by

$$\begin{aligned} A = &\begin{bmatrix} 0.38509 & 0.827671 & 0.408248 \\ 0.55951 & 0.142414 & -0.816497 \\ 0.733931 & -0.542844 & 0.408248 \end{bmatrix} \\ &\cdot \begin{bmatrix} 9.62348 & 0. & 0. \\ 0. & 0.623475 & 0. \\ 0. & 0. & 0. \end{bmatrix} \cdot \begin{bmatrix} 0.38509 & -0.827671 & 0.408248 \\ 0.55951 & -0.142414 & -0.816497 \\ 0.733931 & 0.542844 & 0.408248 \end{bmatrix}^T . \end{aligned} \quad (7.35)$$

In this case, the two singular values are $\sigma_1 \sim 9.6$ and $\sigma_2 \sim 0.62$. So in this case, we should expect to obtain a much better approximation by merely keeping σ_1. Indeed,

$$U \cdot \begin{bmatrix} 9.62348 & 0. & 0. \\ 0. & 0 & 0. \\ 0. & 0. & 0. \end{bmatrix} \cdot V^T = \begin{pmatrix} 1.42711 & 2.07349 & 2.71988 \\ 2.07349 & 3.01265 & 3.9518 \\ 2.71988 & 3.9518 & 5.18373 \end{pmatrix} . \quad (7.36)$$

Indeed, this is not too bad an approximation of the matrix A above!

Remark:

- For $S \in \mathbb{M}(n \times n, \mathbb{R})$ symmetric and positive-definite, the singular value decomposition of A coincides with what you obtain from the spectral theorem.
- An important application of the SVD is the so-called principal component analysis. Strang's book has a detailed description of this.
- There is an analogue of the singular value decomposition (as well as the spectral theorem) for complex valued matrices.

7.1.2 Application: Image compression

Let us focus on an image in gray-scale. This we can think of as $A \in \mathbb{M}(m \times n, \mathbb{R})$ where m is the number of horizontal pixels across and n the number of pixels from top to bottom. Gray-scales are measures as integers from 0 to 255 (2^8, corresponding to 8 bytes). So in a first approximation, we should expect to require $m \times n \times 8$ bytes of memory space of save this image on a hard drive. However, the colors of neighboring pixels are typically similar. Therefore, we should expect that we can do better.

Our approach will be to compute a singular value decomposition of the matrix A in question. We then plot the singular values σ_1 of A and only keep the largest ones. Suppose that we keep k singular values. Then we have

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T \approx \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^T \,. \tag{7.37}$$

This means that we only need to save $k \times (m+n)$ integers. Typically, an image matrix will have full rank. So $r = \min(m, n)$. In fortunate cases, we can take $k \ll r$ and we can save a lot of memory.

Suppose for example that $m = 1024$ and $n = 768$. Then we should expect $r \sim 768$. Our naive memory estimate it thus

$$N_{\text{naive}} = 8 \times 1024 \times 768 = 6291456 \sim 6.3 \times 10^6 \,. \tag{7.38}$$

So we would expect that it takes $6MB$ to save this image. Now suppose that we can take $k = 100$. Then, the singular value decomposition allows us to consume only

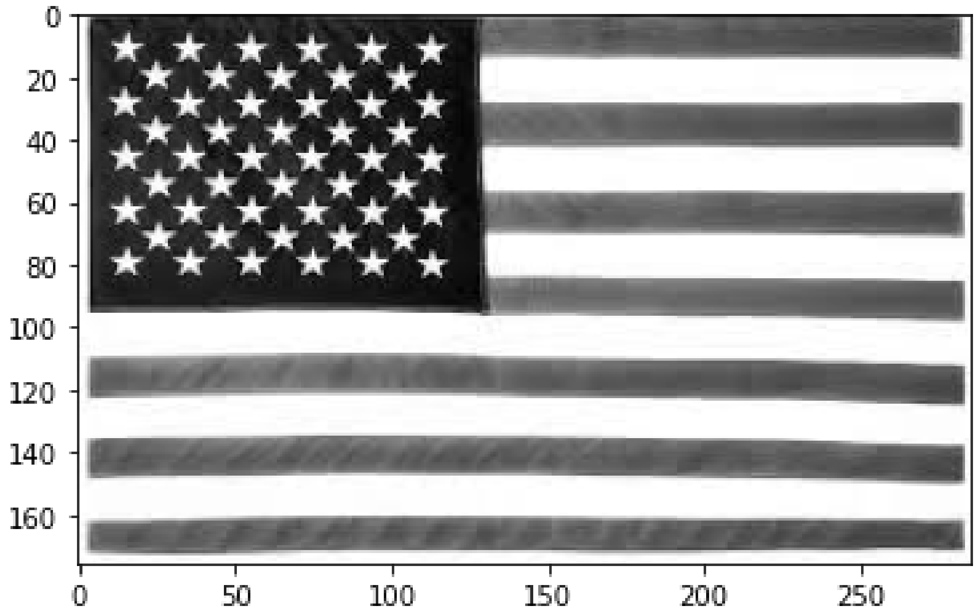
$$N_{\text{SVD}} = 8 \times 100 \times (1024 + 768) = 1433600 \sim 1.4 \times 10^6 \,. \tag{7.39}$$

If we could even take $k = 20$, then

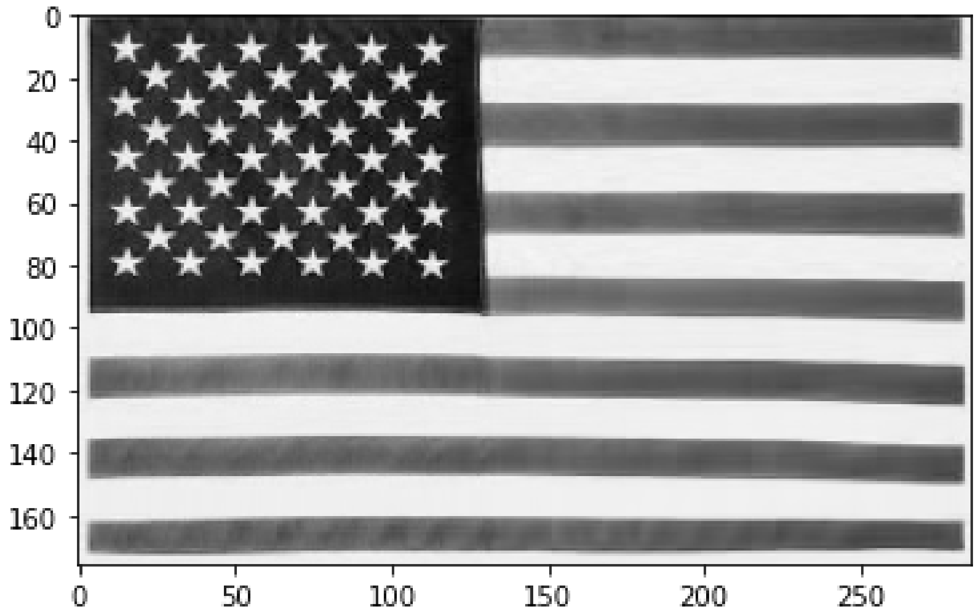
$$N_{\text{SVD}}^{(2)} = 8 \times 20 \times (1024 + 768) = 286720 \sim 0.3 \times 10^6 \,. \tag{7.40}$$

So we can do with a fraction!

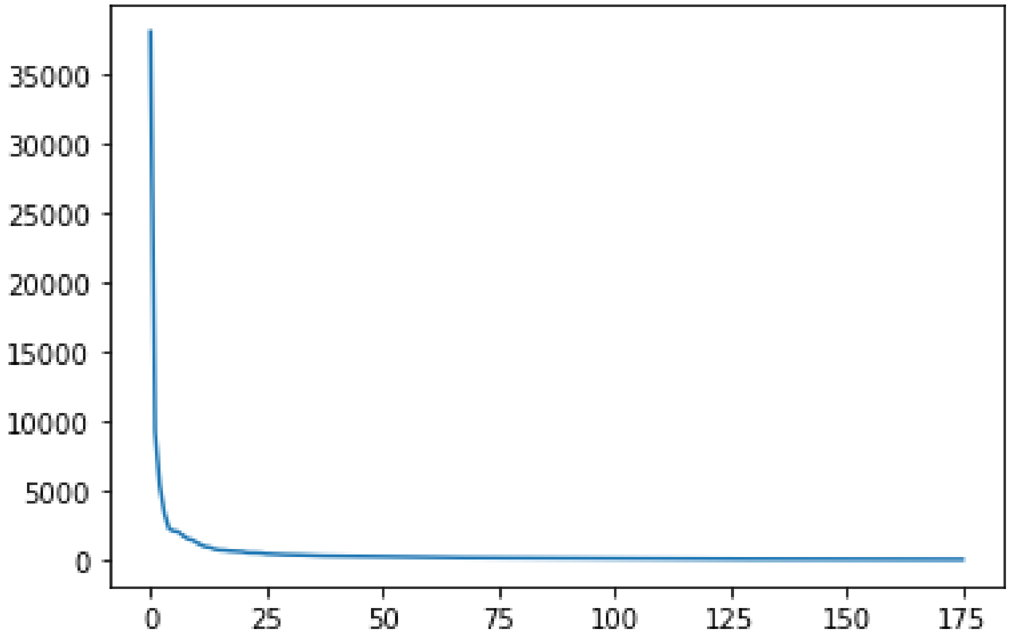
Example 7.1.7:
Let us look at a picture of the US flag. In gray-scales, it looks as follows:



This corresponds to a matrix $A \in \mathbb{M}(176 \times 286, \mathbb{R})$. Its rank indeed turns out to be $r = 176$ and, with a simple computation in `Python` we find its singular value decomposition. In keeping only the first 50 singular values, we achieve an image compression of about 50%. The resulting picture looks as follows:



Can you see the difference to the picture above? To understand why this works so well here, let us plot the singular values of this matrix/picture:



They drop very fast. So in keeping say the first 50 singular values, we keep the most significant contributions. This is why this worked so well!

7.1.3 Application: Principal component analysis

Principal component analysis is yet another important application of the singular value decomposition. In lose terms, it can be used to tell how data is clustered. Specifically, the data that we have in mind is acquired by measuring certain quantities for n samples. Here are examples of that sort.

Example 7.1.8:
10 students take a Math, a History and a Biology exam. Here are the results:

Student	1	2	3	...	9	10
Math	70	75	77	...	97	95
History	72	70	75	...	92	93
Biology	63	77	71	...	95	96

(7.41)

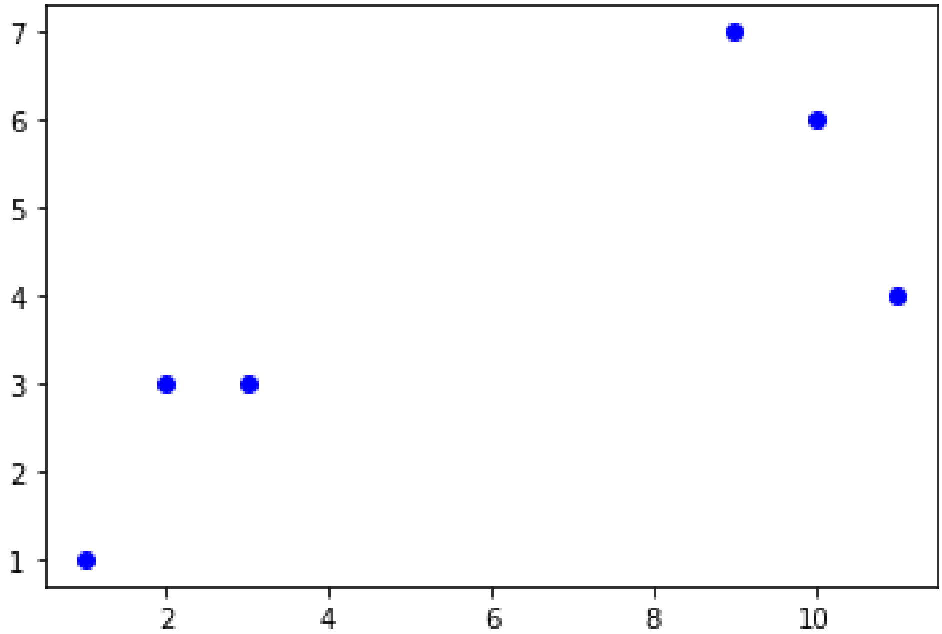
With a human eye, we would say that student 9, 10 are similar and the students 1,2,3 are similar in performance. This pattern we want principal component analysis to detect.

Example 7.1.9:
Here is a simpler set of data. Namely, for 6 different people, we list the number of T-shirts and shoes they own.

Person	1	2	3	4	5	6
# T-Shirts	10	11	9	3	2	1
# Shoes	6	4	7	3	3	1

(7.42)

Again, with a human eye, it would seem that the first three people are more simliar than the last three. Indeed, this we immediately see once we plot the data:



They question is, can we "see" this with linear algebra?

Definition 7.1.1:
We collect our data in the data matrix $A^{(0)} \in \mathbb{M}(m \times n, \mathbb{R})$, such that the j-th column represents the data collected from the j-th sample. In other words, for n-samples we measure m-features.

This data matrix $A^{(0)}$ we now prepared towards our application of the singular value decomposition. Namely, principal component analysis "only" tells us about relative positions of the data points, not their absolute location. We thus prepare our data by removing the average of every row. Specifically, for every $1 \leq i \leq m$ we set

$$\mu_i = \frac{\sum_{j=1}^{n} A_{ij}^{(0)}}{n} . \tag{7.43}$$

This we use to form a new matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$ with

$$A = \begin{bmatrix} A_{11}^{(0)} - \mu_1 & A_{12}^{(0)} - \mu_1 & \dots & A_{1n}^{(0)} - \mu_1 \\ A_{21}^{(0)} - \mu_2 & A_{22}^{(0)} - \mu_2 & \dots & A_{2n}^{(0)} - \mu_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}^{(0)} - \mu_m & A_{m2}^{(0)} - \mu_m & \dots & A_{mn}^{(0)} - \mu_m \end{bmatrix} . \tag{7.44}$$

The data listed in A is centered at the origin $\vec{0}$, which is (in general) not the case for the data in $A^{(0)}$.

Example 7.1.10:
Let us do this for the data

Person	1	2	3	4	5	6
# T-Shirts	10	11	9	3	2	1
# Shoes	6	4	7	3	3	1

(7.45)

Then we have

$$A^{(0)} = \begin{bmatrix} 10 & 11 & 9 & 3 & 2 & 1 \\ 6 & 4 & 7 & 3 & 3 & 1 \end{bmatrix}. \tag{7.46}$$

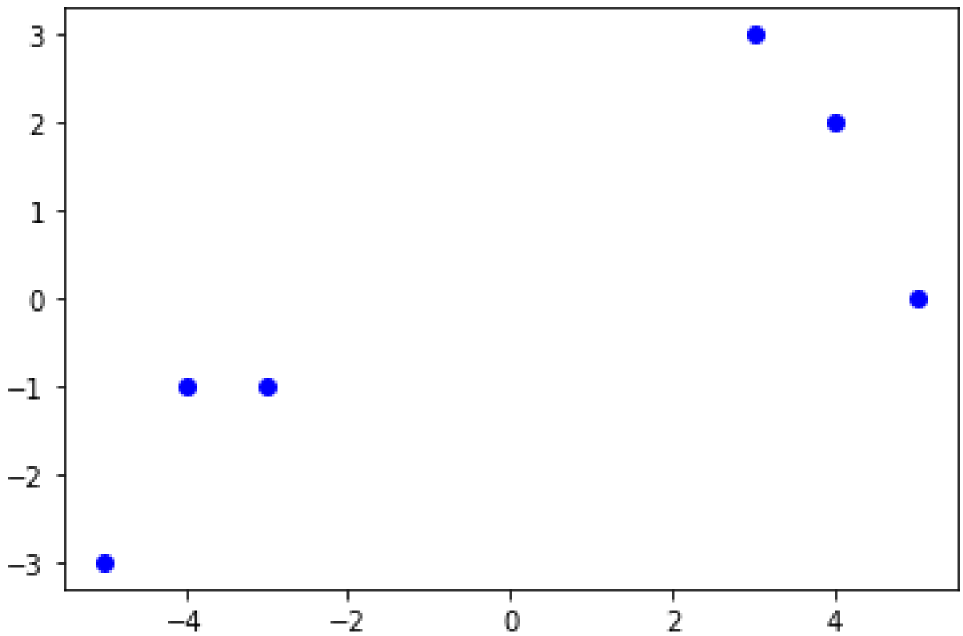
It is not too hard to see that

$$\mu_1 = \frac{36}{6} = 6\,, \qquad \mu_2 = \frac{24}{6} = 4\,. \tag{7.47}$$

Therefore, the normalized matrix A is given by

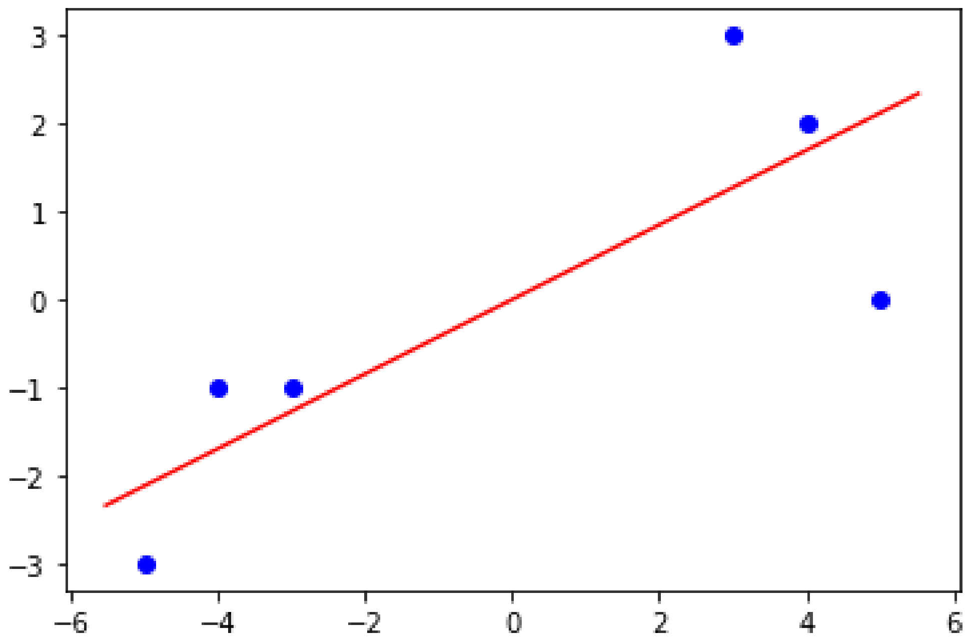
$$A = \begin{bmatrix} 4 & 5 & 3 & -3 & -4 & -5 \\ 2 & 0 & 3 & -1 & -1 & -3 \end{bmatrix}. \tag{7.48}$$

We can also plot this shifted data:



Note that the relative positions have not changed. All that we achieved in passing from $A^{(0)}$ to A is to center our data at the origin $\vec{0}$.

Remark:
As mentioned, our goal is to find patterns that explain how the data is clustered. In staying with the above example, the data is maximally spread-out along the line which best fits the data:



One way to find this line is to use linear regression, as discussed in section 4.4. Let us compare this with a singular value decomposition.

Example 7.1.11:
Let us compute the singular value decomposition of

$$A = \begin{bmatrix} 4 & 5 & 3 & -3 & -4 & -5 \\ 2 & 0 & 3 & -1 & -1 & -3 \end{bmatrix} \in \mathbb{M}(2 \times 6, \mathbb{R}) . \tag{7.49}$$

This can be achieved with the following `Python3`-code:

```
import numpy as np
A = np.array([[4,5,3,-3,-4,-5],[2,0,3,-1,-1,-3]])
U, D, V = np.linalg.svd(A)
V = np.transpose(V)
Sigma = np.zeros((A.shape[0], A.shape[1]))
for i in range(len(D)):
    Sigma[i,i] = D[i]
```

In particular, we find (rounded to two decimal places)

$$U = \begin{bmatrix} 0.92 & -0.39 \\ 0.39 & 0.92 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) . \tag{7.50}$$

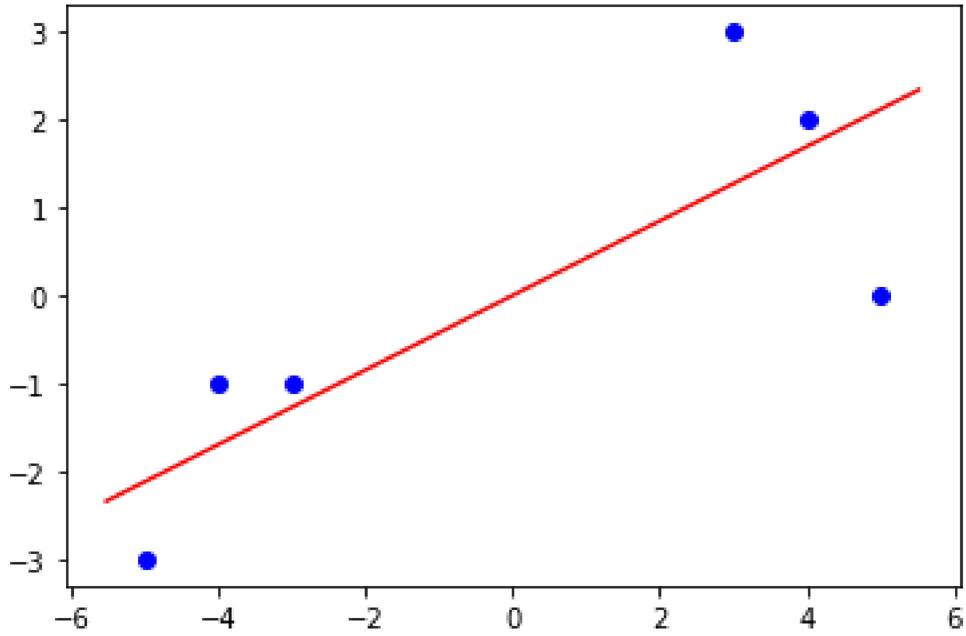
In other words, we have obtained the ONB $\{\vec{u}_1, \vec{u}_2\}$ of $\mathbb{R}^2$ with

$$\vec{u}_1 = \begin{bmatrix} 0.92 \\ 0.39 \end{bmatrix} , \qquad \vec{u}_2 = \begin{bmatrix} -0.39 \\ 0.92 \end{bmatrix} . \tag{7.51}$$

The corresponding singular values are (rounded to one decimal place)

$$\sigma_1 = 10.8\,, \qquad \sigma_2 = 2.7\,. \tag{7.52}$$

So, we expect that σ_1 and therefore also $\vec{u}_1$ is a lot more important to the matrix A than $\vec{u}_2$. So let us plot the normalized data and the line in direction $\vec{u}_1$:



This is exactly the line of best approximation! This line is called *principal component 1* or for short PC_1. We should wonder why this works.

Lemma 7.1.1:
Let $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then A^TA and AA^T has the same non-zero eigenvalues.

Proof

- $\Rightarrow$:
  Let $\vec{x}$ and eigenvector to A^TA with eigenvalue $\lambda \neq 0$. This means

  $$A^TA\vec{x} = \lambda\vec{x}\,. \tag{7.53}$$

  In particular, $A\vec{x} \neq \vec{0}$, since we assume $\lambda \neq 0$. Multiply with A from the left:

  $$AA^T(A\vec{x}) = \lambda(A\vec{x})\,. \tag{7.54}$$

  Consequently, AA^T has eigenvalue λ.

- $\Leftarrow$:
  Let $\vec{x}$ and eigenvector to AA with eigenvalue $\lambda \neq 0$. This means

  $$AA^T\vec{x} = \lambda\vec{x}\,. \tag{7.55}$$

In particular, $A^T\vec{x} \neq \vec{0}$, since we assume $\lambda \neq 0$. Multiply with A^T from the left:

$$A^T A(A^T\vec{x}) = \lambda(A^T\vec{x})\,. \tag{7.56}$$

Consequently, $A^T A$ has eigenvalue λ.

■

Consequence:
The singular values of $A \in \mathbb{M}(m \times n, \mathbb{R})$ are the non-zero eigenvalues of $A^T A$ or, equivalently, the non-zero eigenvalues of AA^T.

Definition 7.1.2:
Let $A \in \mathbb{M}(m \times n, \mathbb{R})$ be a matrix with data centered at $\vec{0}$. Then $S = \frac{AA^T}{n-1}$ is the so-called *sample covariance matrix*. In particular,

$$S_{kk} = \frac{\sum_{i=1}^{n}\left(A_{1k}^{(0)} - \mu_k\right)^2}{n-1}\,. \tag{7.57}$$

Remark:
The reasons why one divides by $n-1$ rather than n are best known to statisticians. We will not discuss this further here.

Consequence:
The reason why the first vector in U picks up the direction in which the data is maximally spread-out is as follows:

- Instead of minimizing the length of the error vector, as we did in section 4.4, one can also maximize the distance of the projected points from the origin.

- The latter is picked up by the eigenvalues of the sample covariance matrix S. The fact that its diagonal entries are sums of squares serves as motivation. But this can be made more rigorous.

⇒ The vector $\vec{u}_i$ with largest singular value points in the direction which maximizes the sum of the squared distances of the projections of the sample points to the line of best fit.

Example 7.1.12:
In returning to the opening example of the number of t-shirts and shoes for 6 people:

Person	1	2	3	4	5	6
# T-Shirts	10	11	9	3	2	1
# Shoes	6	4	7	3	3	1

(7.58)

We have thus found that the first principal component PC_1 is given by the so-called *singular vector*

$$\vec{u}_1 = \begin{bmatrix} 0.92 \\ 0.39 \end{bmatrix} . \tag{7.59}$$

This means that the data is maximally spread-out in this direction. The coefficients of this vector give us an impression whether the number of t-shirts of the number of shoes is more important. In the case at hand, the number of t-shirts is more important than the number of shoes since $0.92 > 0.39$.

Definition 7.1.3:
The components of the singular vector of a principal component are the *loading scores*.

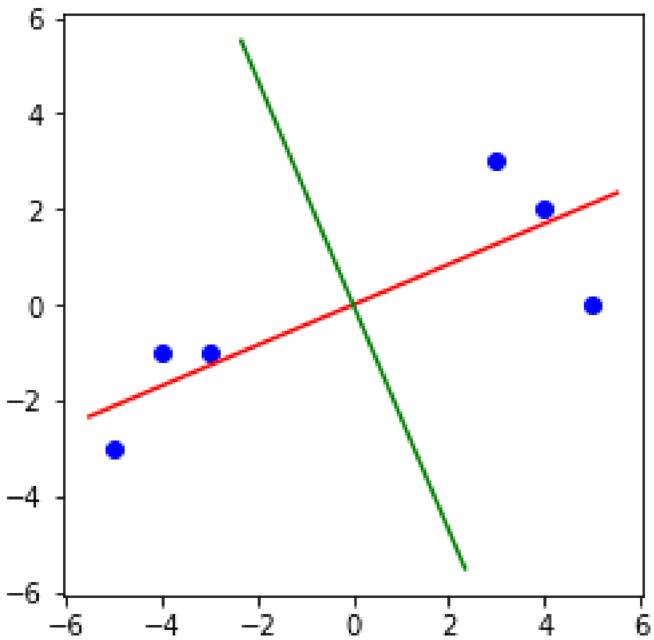
Example 7.1.13:
In our example, $\texttt{PC}_1$ is given by the singular vector

$$\vec{u}_1 = \begin{bmatrix} 0.92 \\ 0.39 \end{bmatrix} . \tag{7.60}$$

The loading score for the number of t-shirts is 0.92 and 0.39 for the number of shoes. Recall that we also found

$$\vec{u}_2 = \begin{bmatrix} -0.39 \\ 0.92 \end{bmatrix} . \tag{7.61}$$

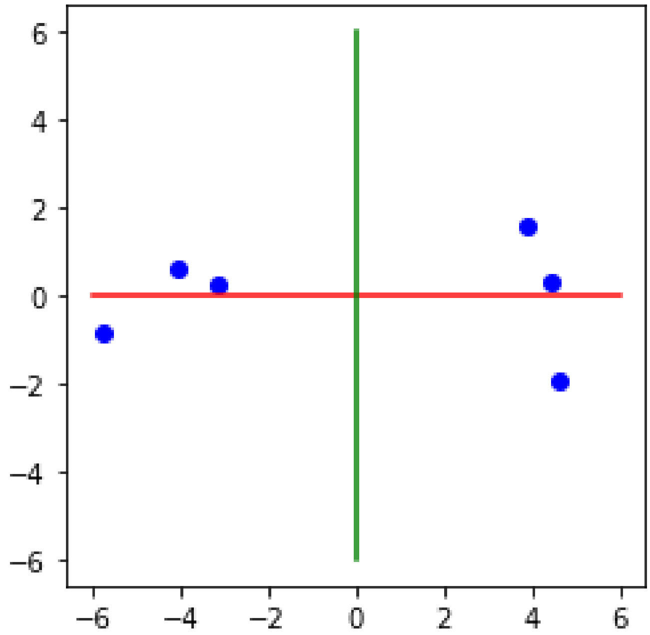
This gives us the second principal component PC_2. Of course, we can draw $\vec{u}_1$ and $\vec{u}_2$. In particular, they define an orthogonal basis of $\mathbb{R}^2$:



It makes sense to rotate this picture, such that the red line is horizontal and the green one vertical. To this end, we simply compute the matrix:

$$A_{\text{rotated}} = U^{-1}A = \begin{bmatrix} 4.46 & 4.61 & 3.93 & -3.15 & -4.0 & -5.77 \\ 0.29 & -1.94 & 1.60 & 0.24 & 0.63 & -0.82 \end{bmatrix}. \tag{7.62}$$

This yields the so-called *PCA*-plot:



The red horizontal line represents PC_1 and the vertical green line PC_2.

Remark:
The *variation* (of our data) about the origin is the sum of the squared distances divided by $n-1$ (= 5 in our case). Those distances are most easily accessible by looking at eq. (7.62). For one data point $p = (p_r, p_g)$, with p_r the component along the red (first) axis and p_g the component along the green (seond) axis, this distance is

$$|p|^2 = p_r^2 + p_g^2\,. \tag{7.63}$$

The total variation is thus obtained by summing all these squares. Here, we have

- Sum of squares of components along red axis:

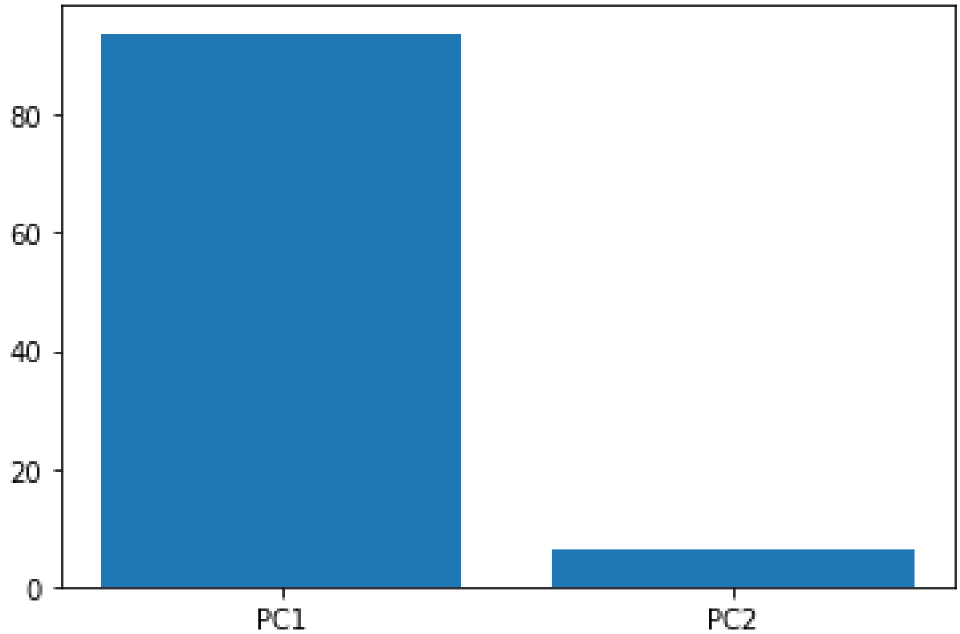
$$\mathrm{var}_r = \frac{4.46^2 + 4.61^2 + \cdots + (-5.77)^2}{5} \sim 23.3\,. \tag{7.64}$$

- Similarly, $\mathrm{var}_g = \frac{0.29^2 + (-1.94)^2 + \cdots + (-0.82)^2}{5} \sim 1.5$.

Crucially, the sum of the squares match the squares of the singular values. Hence, we could also have computed:

$$\text{var}_r = \frac{\sigma_1^2}{5} \sim 23.3\,, \qquad \text{var}_g = \frac{\sigma_2^2}{5} \sim 1.5\,. \tag{7.65}$$

In any case, the total variation of our data about the origin is roughly $23.3 + 1.5 = 24.8$. In particular, PC_1 accounts for $\frac{23.3}{24.8} \sim 93.9\%$ of the variation and PC_2 for the remaining 6.1%. This is often represented in a so-called *scree plot*:



Example 7.1.14:
Let us repeat these steps in a slightly more complicated example. We begin with the following data, which list exam scores for 10 students in 3 different exams:

Student	1	2	3	4	5	6	7	8	9	10
Math	70	75	77	50	55	35	98	63	97	95
History	72	70	75	95	97	50	33	93	92	93
Biology	63	77	71	35	42	85	97	95	89	96

(7.66)

This we do in `Python` and find the principal components:

$$\vec{u}_1 = \begin{bmatrix} 0.55 \\ -0.44 \\ 0.71 \end{bmatrix}, \qquad \vec{u}_2 = \begin{bmatrix} -0.53 \\ -0.84 \\ -0.12 \end{bmatrix}, \qquad \vec{u}_3 = \begin{bmatrix} -0.65 \\ 0.31 \\ 0.69 \end{bmatrix}. \tag{7.67}$$

The loading scores are simply the components of these vectors. The singular values found by `Python` are

$$\sigma_1 = 85.52\,, \qquad \sigma_2 = 64.11\,, \qquad \sigma_3 = 42.04\,. \tag{7.68}$$

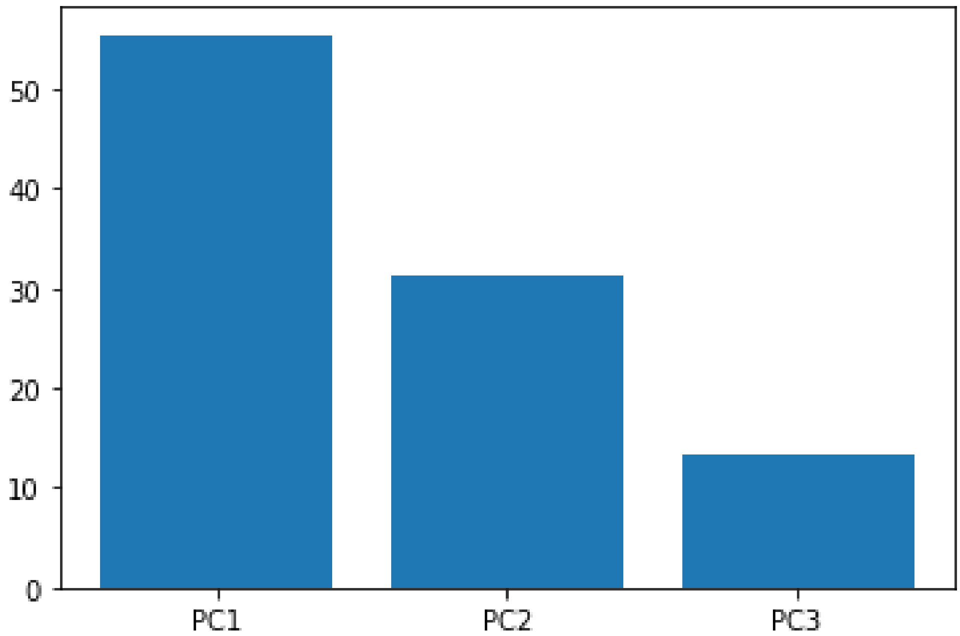
Recall that $\text{var}_i = \frac{\sigma_i^2}{n-1}$ and that the total variation is the sum of these variations. Here:

$$\text{var}_1 \sim 812\,, \qquad \text{var}_2 \sim 457\,, \qquad \text{var}_3 \sim 196\,. \tag{7.69}$$

Correspondingly, we find:

- PC_1 explains $\frac{812}{812+457+196} \sim 55.4\%$ of the total variation.
- PC_2 explains 31.2% of the total variation.
- PC_3 explains 13.4% of the total variation.

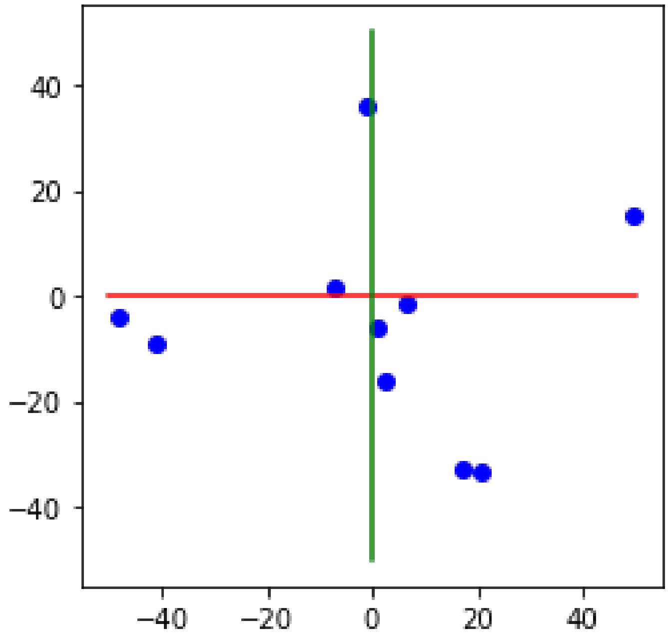
The corresponding scree plot looks as follows:



Suppose that we ignore PC_3, then we can obtain a 2-dimensional representation of the original 3-dimensional data by orthogonally projecting the data points onto PC_1 and PC_2. As above, this is achieved (since U is orthogonal) by looking at the row of the matrix

$$U^T A\,. \tag{7.70}$$

You want to recall that you obtain A by subtracting the averages of the rows in the initial data eq. (7.66). Then the first row of $U^T A$ represents the coordinates along PC_1, the second row of $U^T A$ represents the coordinates along PC_2 and the third row lists the coordinates along PC_3. If we ignore the third row, then we obtain the following 2-dimensional plot:



As above, the red line represents PC_1 and the green vertical line PC_2. Together, those lines account for $55.4\% + 31.2\% = 86.6\%$ of the variation in the data. This is not too bad as approximation. Also, we see from this plot that the data is significantly spread-out along PC_1 and PC_2. This reflects the fact that both matter here significantly. In particular, PC_2 accounts for 31.2% of the variation in the data.

7.2 Complex Vectors and Matrices

Note:
We will now briefly discuss Hermitian and unitary matrices. They arise for example in quantum mechanics (Hermitian matrices are then so-called observables) or in complex Fourier transform, most notably in the so-called *Fast-Fourier transform*. You can find more information on the latter in Strang's book.

Definition 7.2.1:
For a vector $\vec{z} \in \mathbb{C}^n$ with components z_i, we define $\overline{\vec{z}}^T := \begin{bmatrix} \overline{z_1} & \dots & \overline{z_n} \end{bmatrix}$.

Note:
This conjugate transpose is the appropriate analogue of transposition of real vectors.

Definition 7.2.2:
For $A = [z_{ij}] \in \mathbb{M}(n \times n, \mathbb{C})$ the *Hermitian conjugate* or *adjoint* matrix is $A^H = [\overline{z}_{ji}]$.

Example 7.2.1:
Consider the matrix

$$A = \begin{bmatrix} 1 & 1+i \\ 0 & 1-2i \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{C})\,. \tag{7.71}$$

Then the Hermitian conjugate matrix is given by

$$A^H = \begin{bmatrix} 1 & 0 \\ 1-i & 1+2i \end{bmatrix} \in \mathbb{M}(3 \times 2, \mathbb{C})\,. \tag{7.72}$$

Consequence:
For any $A \in \mathbb{M}(n \times n, \mathbb{C})$, it holds $\left(A^H\right)^H = A$.

Construction 7.2.1:
How should one define the dot product of two vectors with complex entries? You probably think we should do it in the same way as we do it for vectors in $\mathbb{R}^n$. Here is one reason why it needs adjusting. Namely, let us consider the vector

$$\vec{z} := \begin{bmatrix} 1 \\ i \end{bmatrix}\,. \tag{7.73}$$

Then we find that $\vec{z}^T\vec{z} = 1^2 + i^2 = 0$. Clearly, we want $\vec{z}^T\vec{z}$ to coincide with the square of the length. Therefore, let us consider $\overline{\vec{z}}^T\vec{z}$ instead:

$$\overline{\vec{z}}^T\vec{z} = \sum_{i=1}^{n} \overline{z}_i z_i = \sum_{i=1}^{n} |z_i|^2\,. \tag{7.74}$$

In particular, we obtain for $\vec{z} := \begin{bmatrix} 1 & i \end{bmatrix}^T$ that $\overline{\vec{z}}^T\vec{z} = 2$, which is far more reasonable.

Definition 7.2.3 (Inner product)**:**
The *inner product* of two complex vectors $\vec{u}, \vec{z} \in \mathbb{C}^n$ is defined as

$$(\vec{u}, \vec{v}) = \vec{u}^H \cdot \vec{v} = \overline{\vec{u}}^T \cdot \vec{v} = \sum_{i=1}^{n} \overline{u_i} \cdot v_i\,. \tag{7.75}$$

Note:
It holds $\vec{u}^H\vec{v} \neq \vec{v}^H\vec{u}$. Rather $\vec{u}^H\vec{v} = \left(\vec{v}^H\vec{u}\right)^H$.

Example 7.2.2:
We consider the vectors

$$\vec{u} := \begin{bmatrix} 1+2i \\ 2-i \end{bmatrix}, \qquad \vec{v} := \begin{bmatrix} 1 \\ i \end{bmatrix}\,. \tag{7.76}$$

Then it holds $\vec{u}^H \cdot \vec{v} = 0$ and $\vec{v}^H \cdot \vec{u} = 0$.

Definition 7.2.4:
Two vectors $\vec{u}, \vec{v}$ are orthogonal iff $\vec{u}^H\vec{v} = 0$.

Example 7.2.3:
The following two vectors are orthogonal:

$$\vec{u} := \begin{bmatrix} 1+2i \\ 2-i \end{bmatrix}, \qquad \vec{v} := \begin{bmatrix} 1 \\ i \end{bmatrix}\,. \tag{7.77}$$

Corollary 7.2.1:
Here are two simple consequence from the above:

- For any two $A, B \in \mathbb{M}(n \times n, \mathbb{C})$ it holds $(AB)^H = B^H A^H$.
- For any $A \in \mathbb{M}(n \times n, \mathbb{C})$ and $\vec{u}, \vec{v} \in \mathbb{C}^n$ it holds $(A\vec{u})^H \cdot \vec{v} = \vec{u}^H \cdot \left(A^H \vec{v}\right)$.

Exercise:
Prove these statements.

Definition 7.2.5:
A matrix $H \in \mathbb{M}(n \times n, \mathbb{C})$ with $A = A^H$ is termed a *Hermitian* matrix.

Note:
Hermitian matrices are the complex analogue of real symmetric matrices. Thus, Hermitian matrices have similar properties as their real symmetric counterparts.

Claim 26:
Every eigenvalue λ of a Hermitian matrix $S \in \mathbb{M}(n \times n, \mathbb{C})$ is real.

Proof
We first note that for any vector $\vec{z}$ it holds:

$$\left(\vec{z}^H S \vec{z}\right)^H = \vec{z}^H S^H \vec{z} = \vec{z}^H S \vec{z}. \tag{7.78}$$

Thus, $\vec{z}^H S \vec{z} \in \mathbb{R}$. Let us now apply this for an eigenvector of S with eigenavlue λ, i.e. $S\vec{z} = \lambda \vec{z}$. Thus

$$\vec{z}^H S \vec{z} = \lambda \cdot \vec{z}^H \vec{z}. \tag{7.79}$$

Since $\vec{z}^H S \vec{z}, \vec{z}^H \vec{z} \in \mathbb{R}$ it follows $\lambda \in \mathbb{R}$. ■

Note:
Hermitian matrices have a basis of orthogonal eigenvectors, which can in turn be normalized to unit length of vectors. This leads to the following

Theorem 7.2.1 (Spectral theorem)**:**
Any Hermitian matrix $S \in \mathbb{M}(n \times n, \mathbb{C})$ can be written as

$$S = U \Lambda U^H \tag{7.80}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with the real eigenvalues $\lambda_i \in \mathbb{R}$ of S and the columns of U are an orthonormal basis of $\mathbb{C}^n$ from eigenvectors of S.

Definition 7.2.6:
$U \in \mathbb{M}(n \times n, \mathbb{C})$ is termed a *unitary matrix* iff $U^H U = I$.

Consequence:
The columns of a unitary matrix U are an orthonormal basis of $\mathbb{C}^n$. They are the analogue of orthogonal matrices.

Example 7.2.4:
Let us consider the Hermitian matrix

$$S = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{C}) . \tag{7.81}$$

Its characteristic polynomial is $\mathrm{ch}_S(\lambda) = (\lambda - 8) \cdot (\lambda + 1)$. Hence, the eigenvalues are 8 and -1 and the corresponding eigenspaces are found to be

$$\begin{aligned} \mathrm{Eig}_{\mathbb{C}}(S, 8) &= \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \right\} , \\ \mathrm{Eig}_{\mathbb{C}}(S, -1) &= \mathrm{Span}\left\{ \begin{bmatrix} 1-i \\ -1 \end{bmatrix} \right\} . \end{aligned} \tag{7.82}$$

Note that the eigenvectors are orthogonal. We can normalize them to find the following orthonormal basis of $\mathbb{C}^2$:

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 \\ 1+i \end{bmatrix} , \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1-i \\ -1 \end{bmatrix} \right\} . \tag{7.83}$$

Therefore, we can write

$$S = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \cdot \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} . \tag{7.84}$$

Note that the matrix

$$U = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} , \tag{7.85}$$

is unitary. Even more, it is even Hermitian. This is usually not the case, i.e. the base changes for the spectra theorem of Hermitian matrices are in general only unitary and not Hermitian.

Claim 27:
For $S \in \mathbb{M}(n \times n, \mathbb{C})$, both unitary and Hermitian, the eigenvalues satisfy $\lambda_i \in \{-1, 1\}$.

Proof
Since S is Hermitian, it holds $\lambda_i \in \mathbb{R}$. Further, by the spectral theorem we can write

$$S = U \Lambda U^H . \tag{7.86}$$

Since the matrix U is unitary, it holds $U^{-1} = U^H$. This implies

$$S^{-1} = \left(U \Lambda U^H\right)^{-1} = \left(U^H\right)^{-1} \Lambda^{-1} U^{-1} = U \Lambda^{-1} U^H . \tag{7.87}$$

But recall that the matrix S is unitary itself. Hence,

$$S^{-1} = S^H = \left(U \Lambda U^H\right)^H = \left(U^H\right)^H \Lambda^H U^H = U \Lambda U^H . \tag{7.88}$$

In the last step we have used that $\Lambda = \mathrm{diag}\left(\lambda_1, \dots, \lambda_n\right)$ with $\lambda_i \in \mathbb{R}$. Consequently, by comparing eq. (7.87) and eq. (7.88) we find $\Lambda^{-1} = \Lambda$. This in turn implies $\lambda_i^{-1} = \lambda_i$. Hence, since $\lambda_i \in \mathbb{R}$, it follows $\lambda_i \in \{-1, 1\}$. This completes the proof. ■

Note:
One of the most important computational application of these theoretical insights is the *Fast Fourier transform*. Strang's book has a detailed exposition on this topic (cf. library course resource in Canvas).