Math 313/513, Spring 2022 **Martin Bies**

Final exam

Monday, May 2, 2022: 12:00 – 14:00 EST

Unless explicitly stated differently, justify all your answers!

Instructions

- Allowed materials: Pen and paper.
- Required materials: Penn card/ID.
- Forbidden materials: Anything not listed above.
- Fill in your information below.
- On each piece of paper, state your name and student ID.

Student information

First name ______________________

Last name ______________________

Penn ID ______________________

Result

Exercise	1	2	3	4	5	6	Σ
Points							

Problem 1: True or false? No justification necessary. [10 Points]
We consider $A \in \mathbb{M}(n \times n, \mathbb{R})$.

1. If $\det(A) = 0$, then there exists $\vec{b} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$ has no solution.
2. If $\text{ch}_A(\lambda) = \lambda^2 (\lambda + 4)^2 (\lambda - 3)$, then $\text{rk}(A) = 4$.
3. If A is a Markov matrix, then $N(A - I) \neq \{\vec{0}\}$.
4. The singular values of A coincide with the eigenvalues of A.

Problem 2: Singular value decomposition (SVD) [10 points]

1. Sketch an algorithm which computes the SVD.
2. How is the SVD related to the four fundamental vector spaces of a matrix?
3. Use the SVD to compute an approximation of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R})$.

Problem 3: Help the matrix police [10 Points]
The matrix police are seeking $A \in \mathbb{M}(m \times n, \mathbb{R})$ for speeding. A witness described A:

"$\dim(N(A^T)) = 0$, the singular values of A are $\sigma_1 = \sqrt{2}$, $\sigma_2 = 1$ and the singular vectors are $\vec{v}_1 = \frac{1}{5} \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Also, I clearly observed $A\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$."

Use this information to help the matrix police: Find all A with these properties.

Problem 4: The special orthogonal group [10 Points]
We consider the *special orthogonal group*

$$\text{SO}(n) = \left\{ A \in \mathbb{M}(n \times n, \mathbb{R}) \,|\, A^{-1} = A^T \text{ and } \det(A) = 1 \right\} . \tag{1}$$

1. If $R_1, R_2 \in \text{SO}(n)$, argue that $R_1 \cdot R_2 \in \text{SO}(n)$.
2. For all $\alpha \in [0, 2\pi)$, show that $R(\alpha) := \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \in \text{SO}(2)$.
3. Verify that $f \colon [0, 2\pi) \to \text{SO}(2)\,,\ \alpha \mapsto R(\alpha)$ is injective.
   Hint: You have to show that $R(\alpha_1) = R(\alpha_2)$ implies $\alpha_1 = \alpha_2$.
4. **Math 513:** Find **all** $\alpha \in [0, 2\pi)$ such that $R(\alpha) \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} R(\alpha)^{-1}$ is diagonal.

Problem 5: Damped spring-mass system [10 Points]

The dynamics of a damped spring-mass system – damping $\delta \in \mathbb{R}_{>0}$ – is described by

$$x''(t) = -2\delta x'(t) - \omega^2 x(t)\,. \tag{2}$$

1. Find $A \in \mathbb{M}(2 \times 2, \mathbb{R})$ such that $\vec{z}\,'(t) = A \cdot \vec{z}(t)$, where $\vec{z}(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$.

2. Let $\vec{z}_0 \in \mathbb{R}^2$. Show that $\vec{z}(t)\colon \mathbb{R} \to \mathbb{R}^2\,,\, t \mapsto e^{At} \cdot \vec{z}_0$ solves $\vec{z}\,'(t) = A \cdot \vec{z}(t)$.

For $\omega > \delta$ (i.e. weak damping) and $z_0 = \begin{bmatrix} C \\ D \end{bmatrix} \in \mathbb{R}^2$, an explicit computation shows

$$x(t) = e^{-\delta t}\left[C\cos(\widetilde{\omega}t) + \frac{\delta C + D}{\widetilde{\omega}} \cdot \sin(\widetilde{\omega}t)\right], \qquad \widetilde{\omega} = \sqrt{\omega^2 - \delta^2}\,. \tag{3}$$

We fix the initial configuration at time $t = 0$ such that the mass is at position $x_0 \in \mathbb{R}$ and moves with velocity $v_0 \in \mathbb{R}$.

3. Evaluate $x(t)$ and $x'(t)$ at time $t = 0$.
   Hint: You should find expressions linear in C and D.

4. **Math 513:** Find $C, D \in \mathbb{R}$ such that $x(0) = x_0$ and $x'(0) = v_0$.

Problem 6: Complex diagonalization [10 Points]

Consider $\omega, \delta \in \mathbb{R}$ with $\omega > \delta > 0$ and set $\widetilde{\omega} = \sqrt{\omega^2 - \delta^2}$. We want to exponentiate

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\delta \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{4}$$

1. Compute the characteristic polynomial $\mathrm{ch}_A(\lambda)$ of A.

2. Consider $\lambda_1 = -\delta + i\widetilde{\omega} \in \mathbb{C}$, $\lambda_2 = -\delta - i\widetilde{\omega} \in \mathbb{C}$. Show $\mathrm{ch}_A(\lambda_1) = \mathrm{ch}_A(\lambda_2) = 0$.
   Hint: The complex imaginary unit i satisfies $i^2 = -1$.

3. Verify that

$$A \cdot \begin{bmatrix} \lambda_2 \\ \omega^2 \end{bmatrix} = \lambda_1 \cdot \begin{bmatrix} \lambda_2 \\ \omega^2 \end{bmatrix}, \qquad A \cdot \begin{bmatrix} \lambda_1 \\ \omega^2 \end{bmatrix} = \lambda_2 \cdot \begin{bmatrix} \lambda_1 \\ \omega^2 \end{bmatrix}. \tag{5}$$

   Show also that

$$A = \begin{bmatrix} \lambda_2 & \lambda_1 \\ \omega^2 & \omega^2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot \frac{i}{2\widetilde{\omega}} \cdot \begin{bmatrix} 1 & -\frac{\lambda_1}{\omega^2} \\ -1 & \frac{\lambda_2}{\omega^2} \end{bmatrix}. \tag{6}$$

4. For $C, D \in \mathbb{R}$, show that the first component of $e^{At} \cdot \begin{bmatrix} C \\ D \end{bmatrix}$ matches eq. (3).
   Hint: Without proof, you may use that $e^{ia} = \cos(a) + i\sin(a)$ for $a \in \mathbb{R}$.