Counting massless matter in F-theory with CAP

Martin Bies

ULB Brussels

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Overview

Presentation based on work with . . .

- T. Weigand, C. Mayrhofer, C. Pehle

  1402.5144, 1706.04616, 1706.08528, 1802.08860

- M. Barakat, S. Gutsche, S. Posur, K. M. Saleh

  5 CAP-packages on *https://github.com/HereAround*

- K. Veschgini in progress

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Outline

- Motivation
- Introduction to F-theory
- G_4-flux and counting massless matter in F-theory
- Applications in F-theory GUT-models

Gravity + Standard Model = ?

String theory – a promising candidate

- Every consistency string theory contains a graviton
- D-branes carry gauge theories
- UV finite theory (at least up to 2-loop order)

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String theory – a promising candidate

- Every consistency string theory contains a graviton
- D-branes carry gauge theories
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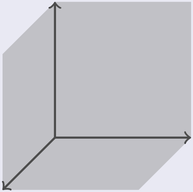
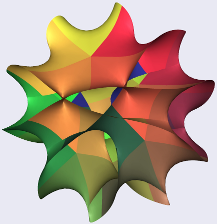
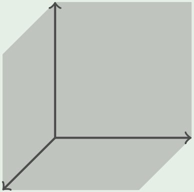
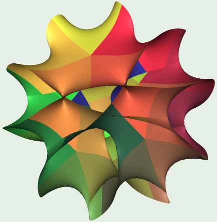
Drawback: Consistency requires 10d spacetime

$\mathcal{S} =$  $\times$ 

our 4-dim. world $\mathcal{W}$ ‘small’ 6-dim. manifold $\mathcal{B}_3$

Towards the string landscape

Ambiguity: Which manifold $\mathcal{B}_3$ (and substructure) to choose?

$\mathcal{S} \quad = $  $\times$ 

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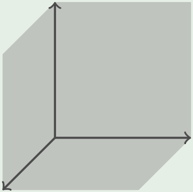

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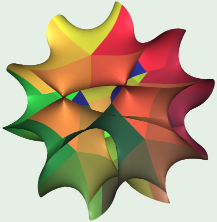
$\times$

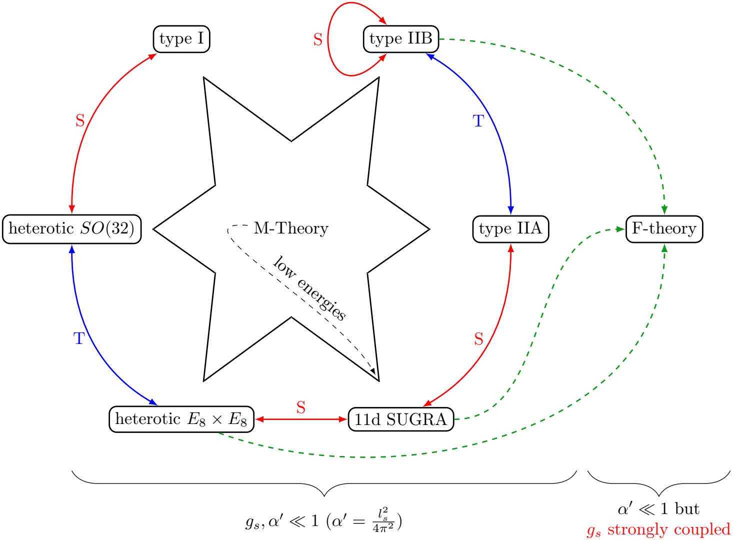


'small' 6-dim. manifold $\mathcal{B}_3$

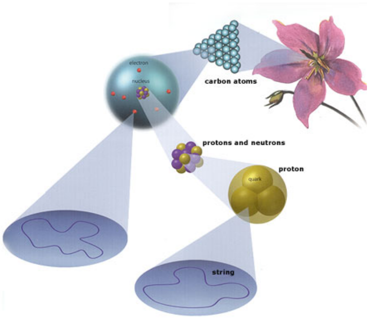
Choices – curse or blessing?

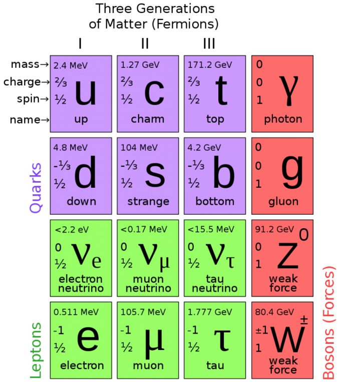
- Many possible choices for $\mathcal{B}_3$ (and substructure)
- Holy grail: Find $\mathcal{B}_3$ (and substructure) such that string theory on $\mathcal{S} = \mathcal{W} \times \mathcal{B}_3$ reproduces physics experienced in $\mathcal{W}$

Means of simplification: The M-theory star



Towards a quality check on F-theory vacua





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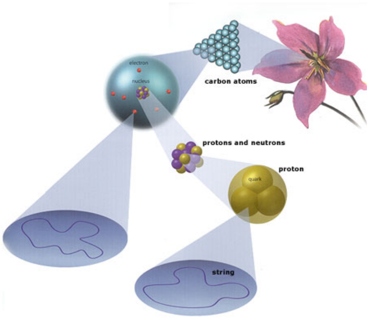
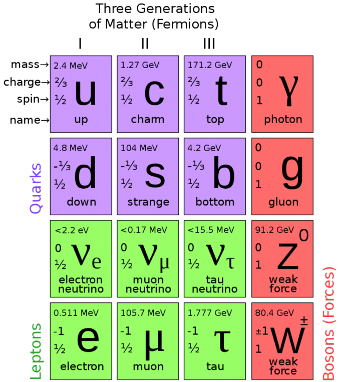
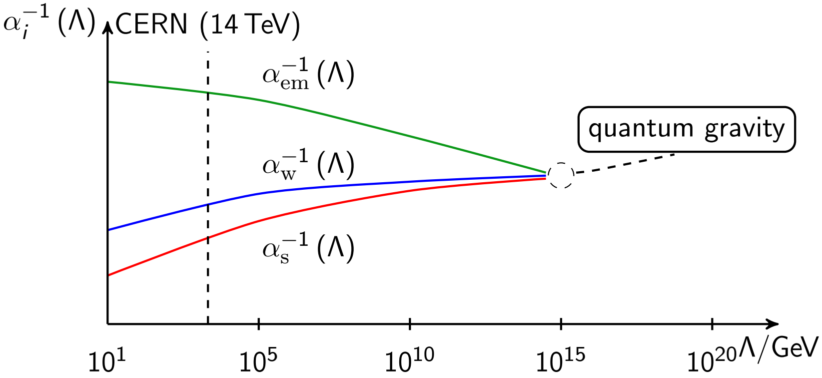




Proposal for quality criterion

Number of standard model particles

Towards a quality check on F-theory vacua



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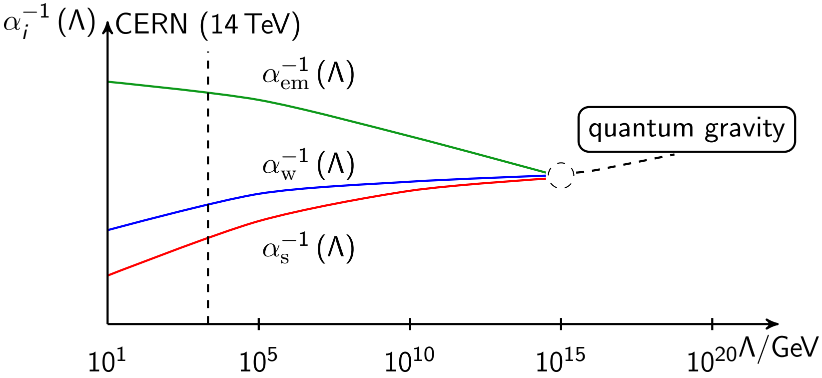
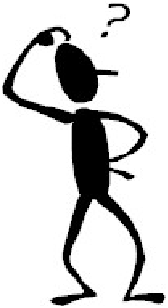


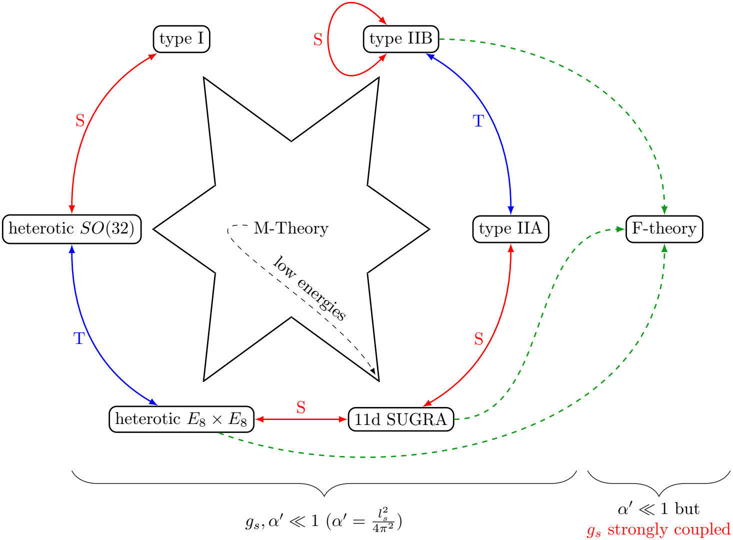
Consequence: Modified quality criterion

Number of massless particles in F-theory vacuum

Questions so far?



Approaching F-theory from IIB string theory



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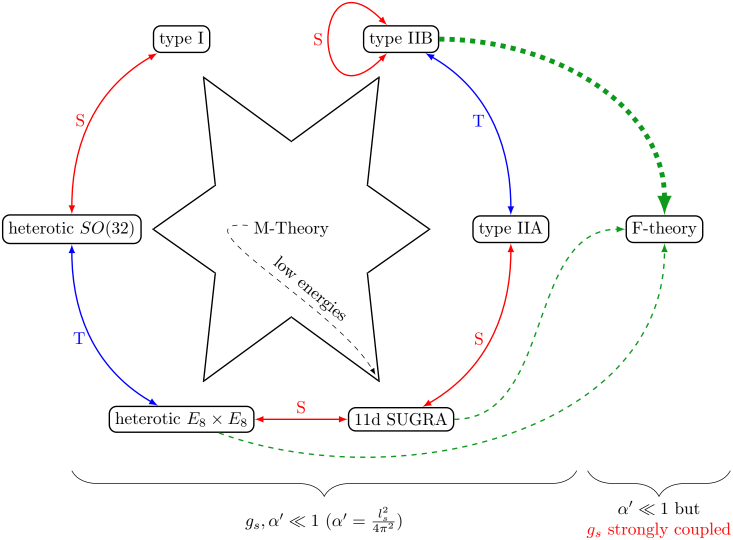

Revision: IIB Supergravity (10D)

Bosonic field content

field	symbol	type
dilaton	ϕ	scalar
metric	$G_{\mu\nu}$	symmetric 2-tensor
B-field	B_2	2-form
RR 0-form	C_0	0-form
RR 2-form	C_2	2-form
RR 4-form	C_4	4-form

New fields

- Axio dilaton $\tau := C_0 + ie^{-\phi}$
- $H_3 := dB_2$
- $G_3 := F_3 - \tau H_3$
- $\ldots$

Action

$$S_{IIB} = \frac{2\pi}{l_s^8} \int_{M_{10}} d^{10}x \left[\sqrt{-G} R - \frac{d\tau \wedge \star d\overline{\tau}}{2\left(\Im\tau\right)^2} + \frac{dG_3 \wedge \star d\overline{G_3}}{\Im\tau} + \ldots \right]$$

SL(2, $\mathbb{Z}$) invariance of IIB-SUGRA

Classical symmetry: SL(2, $\mathbb{R}$)

Given $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$, S_{IIB} is invariant under

$$\begin{pmatrix} C_4 \\ G \end{pmatrix} \mapsto \begin{pmatrix} C_4 \\ G \end{pmatrix}, \quad \tau = C_0 + ie^{-\phi} \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}.$$

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Reduced symmetry in quantised IIB SUGRA

- Partition function contains factor $\exp(2\pi i \tau)$

$\Rightarrow$ Invariant only if τ is transformed by SL$(2, \mathbb{Z})$

$\Rightarrow$ Quantised IIB SUGRA has SL$(2, \mathbb{Z})$-symmetry

Backreaction of D7-branes

Magnetic charge of D7-brane under C_0

- D7-brane has 8-dimensional world-volume $\mathcal{D}_8$
- IIB SUGRA contains 0-form field C_0

$\Rightarrow$ $C_0 \to F_1 = dC_0 \overset{*}{\to} F_9 = d\widetilde{C_8} \to \widetilde{C_8}$

$\Rightarrow$ Magnetic coupling $S_{\text{magnetic}} = \int\limits_{\mathcal{D}_8} \widetilde{C}_8$

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Consequence: Backreaction

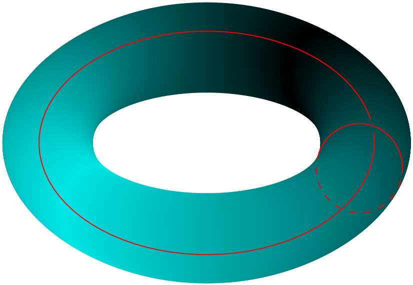
- Two dimensional space orthogonal to D7-brane

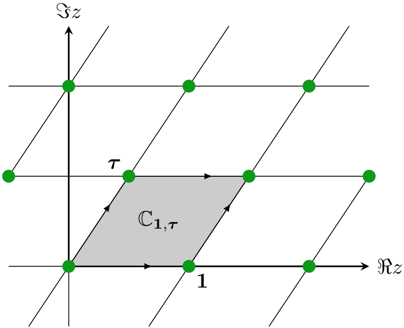
$\Rightarrow$ W.l.o.g. complex plane $\mathbb{C}$ with D7-brane at position z_0

$\Rightarrow$ Since D7-brane is source, C_0 satisfies $\Delta C_0 = \delta(z - z_0)$

$\Rightarrow$ $\tau(z) = C_0(z) + ie^{-\phi(z)} = \frac{1}{2\pi i}\log(z - z_0) + \ldots$

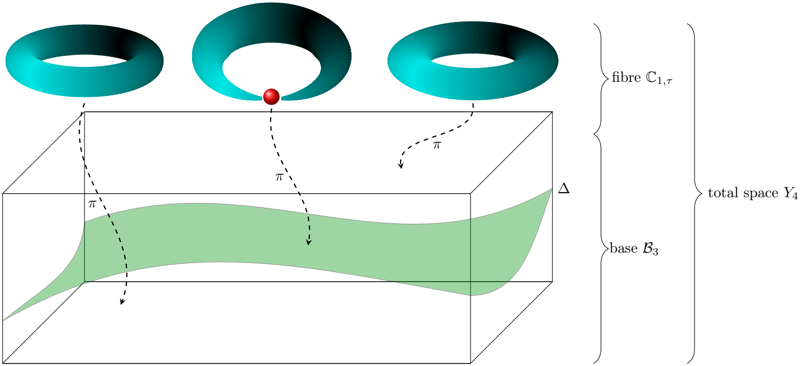
Geometrising the $\mathrm{SL}(2,\mathbb{Z})$-invariance





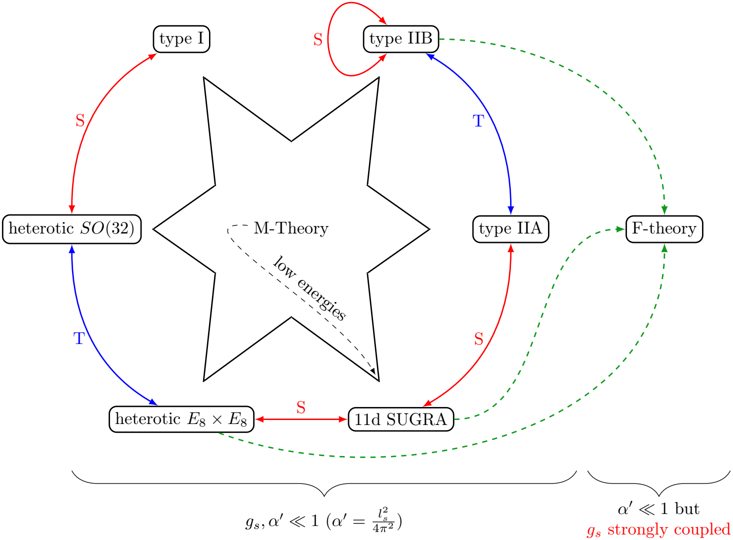
IIB-SUGRA	Torus $\mathbb{C}_{1,\tau}$
Axio-dilaton $\tau(z)$	complex structure
$SL(2,\mathbb{Z})$-invariance	Symmetry group
D7-brane: $\tau(z) = \frac{\log(z-z_0)}{2\pi i} + \ldots$	Singular torus

Geometric 'book-keeping device'



IIB-SUGRA	Geometry
union of loci of D7-branes in IIB-compactification	Singular locus Δ of elliptic fibration $\mathbb{C}_{1,\tau} \hookrightarrow Y_4 \stackrel{\pi}{\twoheadrightarrow} \mathcal{B}_6$

Approaching F-theory from M-theory



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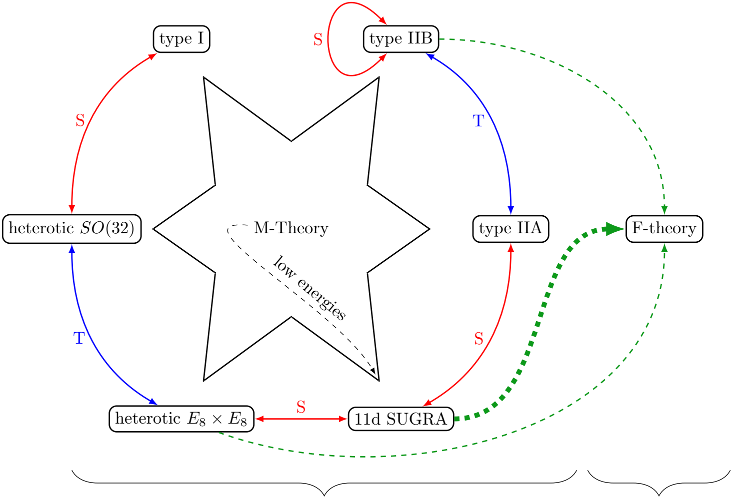


$g_s, \alpha' \ll 1$ $(\alpha' = \frac{l_s^2}{4\pi^2})$

$\alpha' \ll 1$ but
g_s strongly coupled

Defining F-theory from M-theory

11d SUGRA on $\mathcal{M}_{1,2} \times Y_4$:
$\pi \colon Y_4 \twoheadrightarrow \mathcal{B}_3$ and $\pi^{-1}\{p\} \cong \mathbb{C}_{\mathbf{1},\tau}$ with 1-cycles $S^1_A(p)$, $S^1_B(p)$

Shrink $S^1_A(p)$ to zero

IIA string theory on $\mathcal{M}_{1,2} \times X$:
$\pi_X \colon X \twoheadrightarrow \mathcal{B}_3$ and $\pi_X^{-1}\{p\} \cong S^1_B$

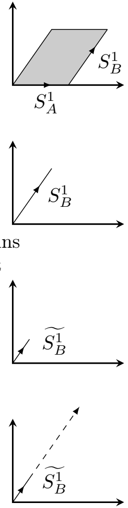
T-duality along $S^1_B(p)$

IIB string theory on $\mathcal{M}_{1,2} \times \widetilde{X}$:
$\widetilde{\pi_X} \colon \widetilde{X} \twoheadrightarrow \mathcal{B}_3$ and $\widetilde{\pi_X}^{-1}\{p\} \cong \widetilde{S}^1_B(p)$ with $\widetilde{R}_B(p) = \frac{\alpha'}{R_B(p)}$

Decompactify $\widetilde{S}^1_B(p)$, i.e. $\widetilde{R}_B(p) \to \infty$

IIB string theory on $\mathcal{M}_{1,2} \times \mathbb{R} \times \mathcal{B}_3 \equiv \mathcal{M}_{1,3} \times \mathcal{B}_3$

$\tau(p)$ remains constant



Towards a dictionary between physics and geometry

Strategy and problems

- Use definition of F-theory as M-theory limit

$\Rightarrow$ Compare physics of 11D SUGRA compactification and geometry of elliptic fibration $\mathbb{C}_{1,\tau} \hookrightarrow Y_4 \twoheadrightarrow \mathcal{B}_3$

- Y_4 is singular (over D7-branes), so singularities are important

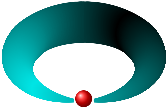
$\Rightarrow$ Two approaches:

1. Work with singular Y_4 (e. g. 1310.1931, 1410.4867, 1603.00062)
2. Resolve singularities and work with smooth space $\hat{Y}_4$ (e. g. 1109.3454, 1202.3138)

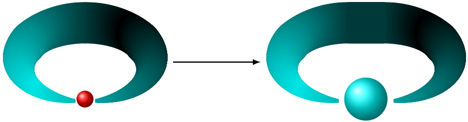
Choice in this talk

We work with **smooth space** $\hat{Y}_4$

Blow-up resolution in a cartoon



Cartoon on blow-up resolution



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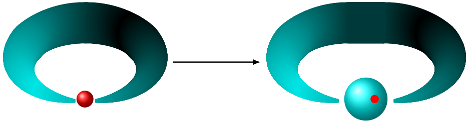

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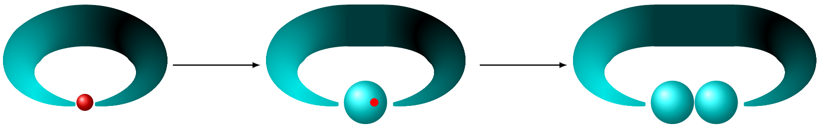

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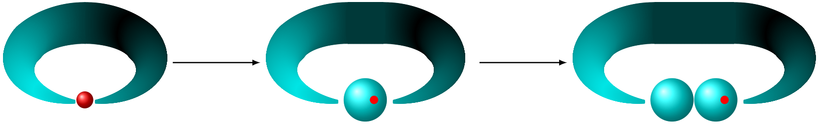

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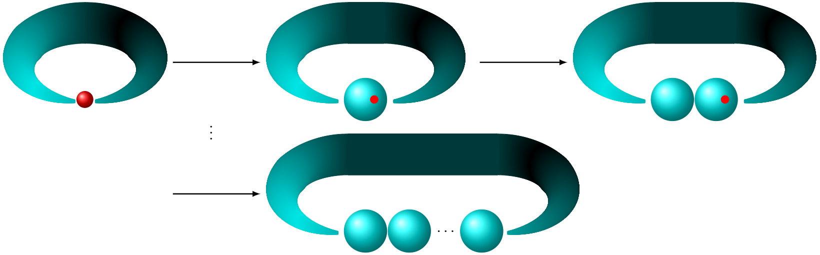

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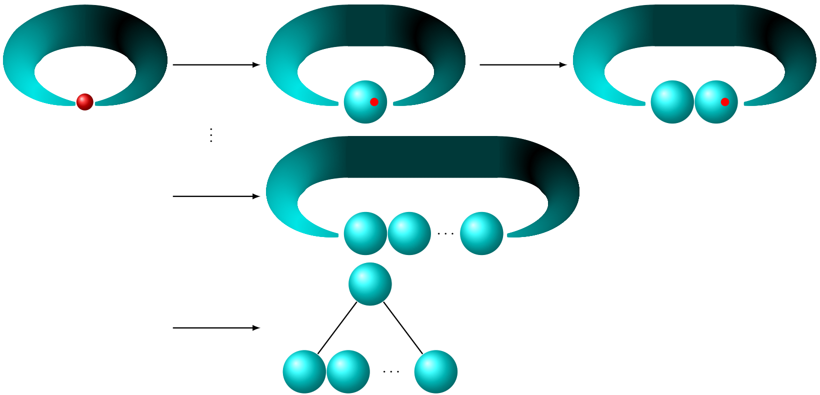

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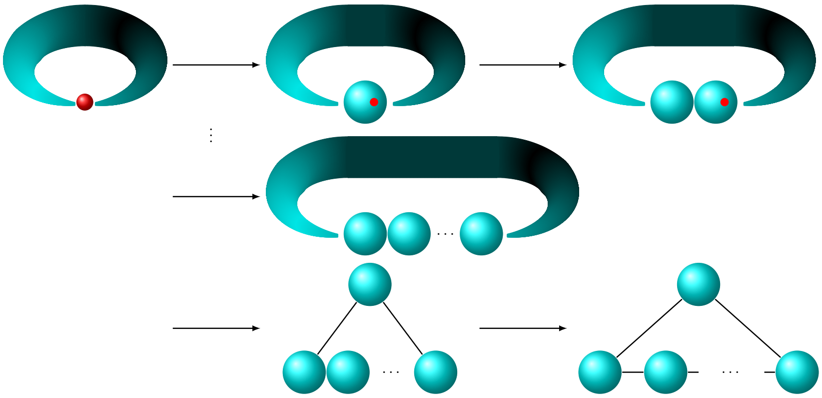

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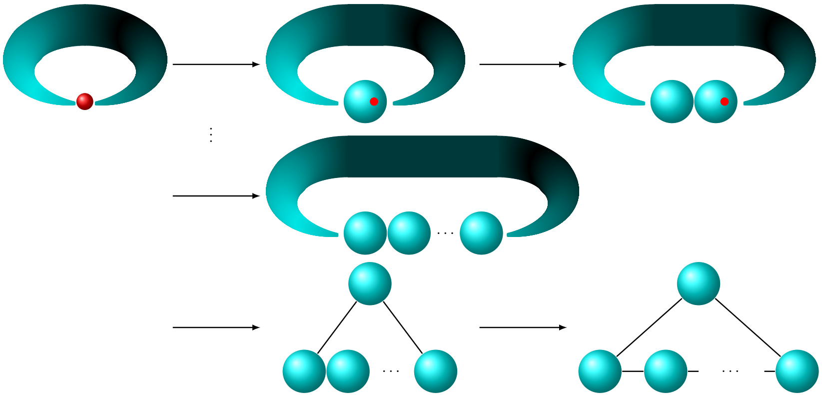
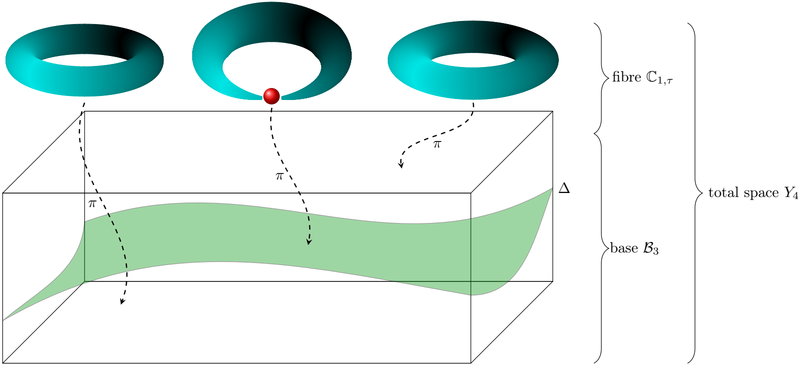


In general obtain ...

... **affine Dynkin diagrams** of A-, B-, C-, D-, E-, F_4 and G_2-type

non-Abelian gauge theories and massless matter



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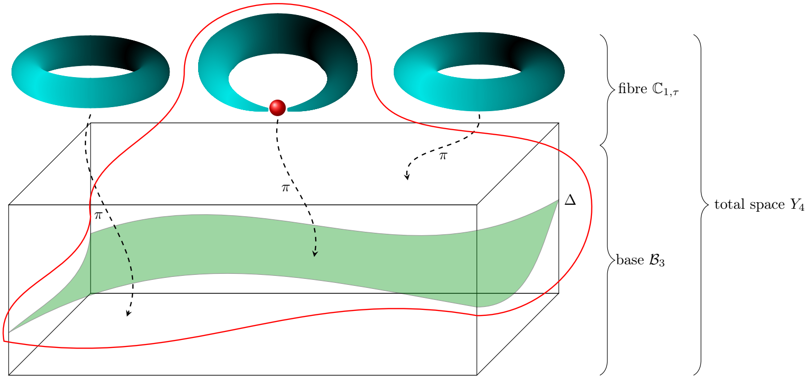

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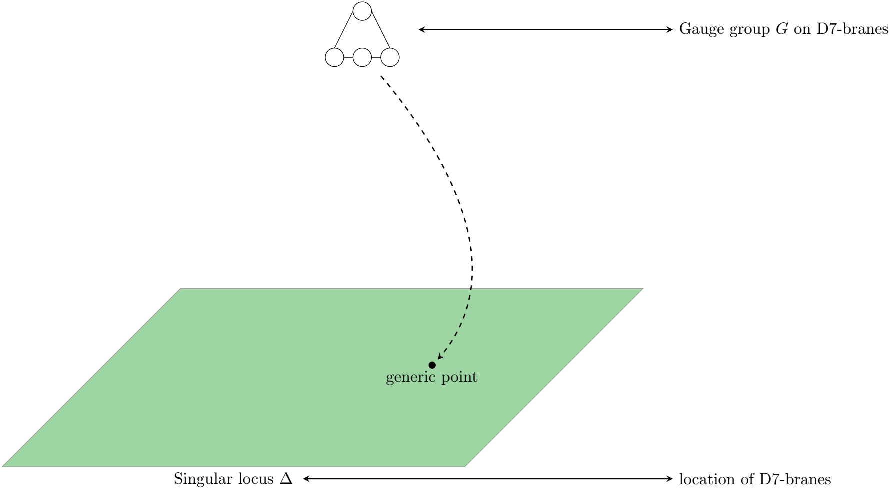

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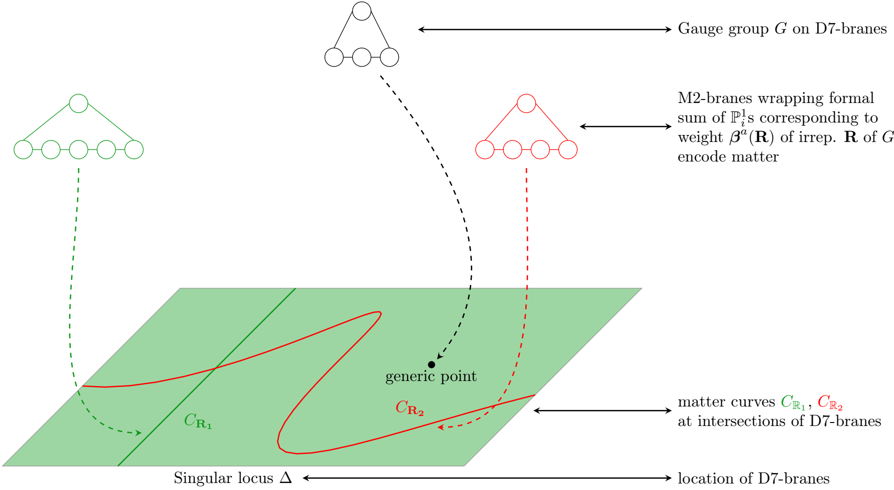

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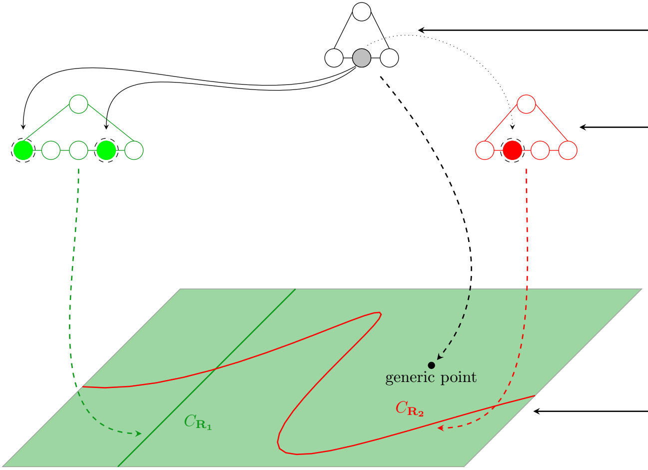


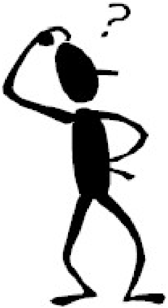
Gauge group G on D7-branes

M2-branes wrapping formal sum of $\mathbb{P}^1_i$s corresponding to weight $\beta^a(\mathbf{R})$ of irrep. $\mathbf{R}$ of G encode matter

matter curves $C_{\mathbb{R}_1}$, $C_{\mathbb{R}_2}$ at intersections of D7-branes

location of D7-branes

Questions?



Strategy and disclaimer

Strategy

Step	Mathematics
Parametrise G_4-fluxes	Chow group $\mathrm{CH}^2(\hat{Y}_4)$
Describe massless matter	Sheaf cohomology
Count zero modes with CAP	Exts of f. p. graded S-modules

Strategy and disclaimer

Strategy

Step	Mathematics
Parametrise G_4-fluxes	Chow group $\mathrm{CH}^2(\hat{Y}_4)$
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Count zero modes with `CAP`	Exts of f. p. graded S-modules

Disclaimer

1. We choose to resolve singular Y_4 to obtain smooth $\hat{Y}_4$
   - $\Rightarrow$ Can only detect **Abelian** gauge backgrounds
   - $\Rightarrow$ Formulation of non-Abelian gauge fluxes might depart from 1310.1931, 1410.4867, 1603.00062
2. Description of G_4-flux on smooth $\hat{Y}_4$ exists in language of Cheeger-Simons cohomology 0312069, 0409135, 0409158

Origin of G_4-flux in F-theory

11d SUGRA action ($G_4 = dC_3$)

$$S_{11D} = \frac{M_{11D}^9}{2} \int\limits_{M_{11}} d^{11}x \left(\sqrt{-\det G} R - \frac{G_4 \wedge * G_4}{2} - \frac{C_3 \wedge G_4 \wedge G_4}{6} \right)$$

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Consequence

- M2-branes couple electrically to 3-form gauge potential C_3
- $G_4 = dC_3 \in H^{2,2}(\hat{Y}_4)$ is field strength

An different way to think of G_4

Theorem of de Rham (1931): Duality of differential forms and cycles

- M **compact**, $C_r(M)$ its r-chains and $\Omega^r(M)$ its r-forms
- Inner product

$$\langle \cdot, \cdot \rangle \colon C_r(M) \times \Omega^r(M) \to \mathbb{R} \, , \; (c, \omega) \mapsto \langle c, \omega \rangle = \int_c \omega$$

$\Rightarrow$ Extends to inner product $\langle \cdot, \cdot \rangle \colon H_r(M) \times H^r(M)$

- de Rham proved that $\langle \cdot, \cdot \rangle$ is bilinear and non-degenerate

$\Rightarrow$ $H^r(M) \cong H_r^\vee(M)$ (dual vector spaces)

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Consequence

$G_4 \in H^{2,2}(\hat{Y}_4, \mathbb{Z})$ can be represented by complex 2-cycle A

Gauge backgrounds and Deligne cohomology

Questions

- What specifies gauge date C_3 beyond field strength G_4?

$\Rightarrow$ Look for structure which combines information on

- field strength $G_4 \in H^{2,2}_{\mathbb{Z}}(\hat{Y}_4)$
- Wilson line d.o.f. $\oint C_3$

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Natural candidate in mathematics 9801057, 9802093, 0312069, 0409135, 0409158, 1104.2610, 1203.6662, 1212.4505, 1310.1931, 1402.5144

$$0 \rightarrow J^2(\hat{Y}_4) \hookrightarrow H^4_D(\hat{Y}_4, \mathbb{Z}(2)) \twoheadrightarrow H^{2,2}_{\mathbb{Z}}(\hat{Y}_4) \rightarrow 0$$

H. Esnault, E. Viehweg – 'Beilinson's conjectures on special values of L-functions' 1988

Intermediate Jacobian $J^2(\hat{Y}_4) \simeq \frac{H^3(\hat{Y}_4, \mathbb{C})}{H^{2,1}(\hat{Y}_4) + H^3(\hat{Y}_4, \mathbb{Z}))}$	$\leftrightarrow$	Wilson lines $\oint C_3$
Deligne cohomology $H^4_D(\hat{Y}_4, \mathbb{Z}(2))$	$\leftrightarrow$	full gauge data
$H^{2,2}_{\mathbb{Z}}(\hat{Y}_4)$	$\leftrightarrow$	field strength G_4

Practical representation: Chow group

Motivation

- $H^4_D(\hat{Y}_4, \mathbb{Z}(2))$ is hard to handle (practically)

$\Rightarrow$ Easy-to-work-with parametrisation (of subset) from $\mathrm{CH}^2(\hat{Y}_4)$

M. Green, J. Murre, C. Voisin – 'Algebraic Cycles and Hode Theory', 1994

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Basics on the Chow group $\mathrm{CH}^k(X)$

- Rational equivalence:
  $C_1 \sim C_2 \in Z_p(X)$ iff $C_1 - C_2$ is zero/pole of a **rational** function defined on $p+1$-dim. irreducible subspace of X

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Consequence

Full G_4-gauge data $\leftrightarrow A \in \mathrm{CH}^2(\hat{Y}_4)$ – equ. class of 2-cycle

Local picture: Twisted Theory on D7-branes 0802.2969, 0802.3391

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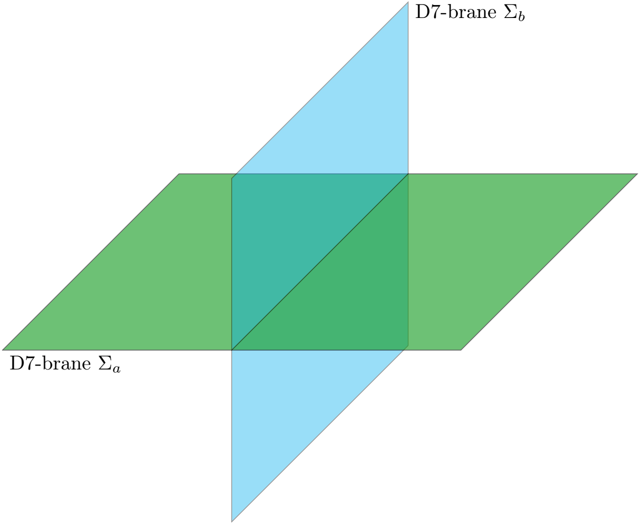

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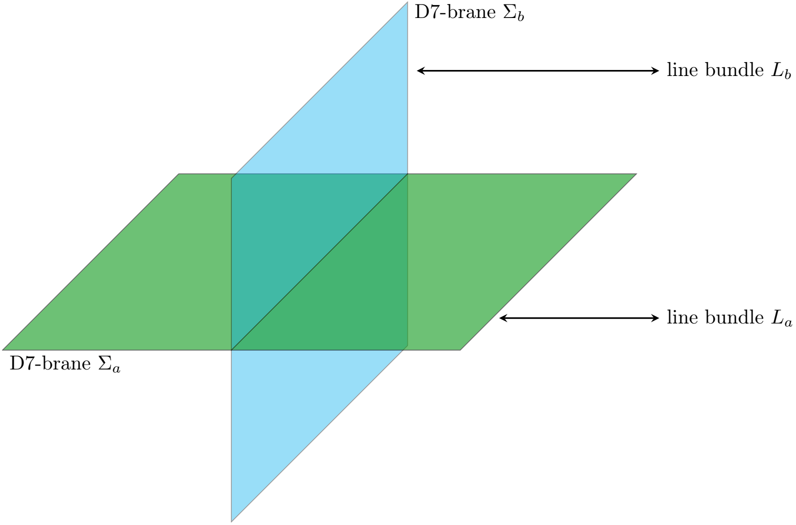

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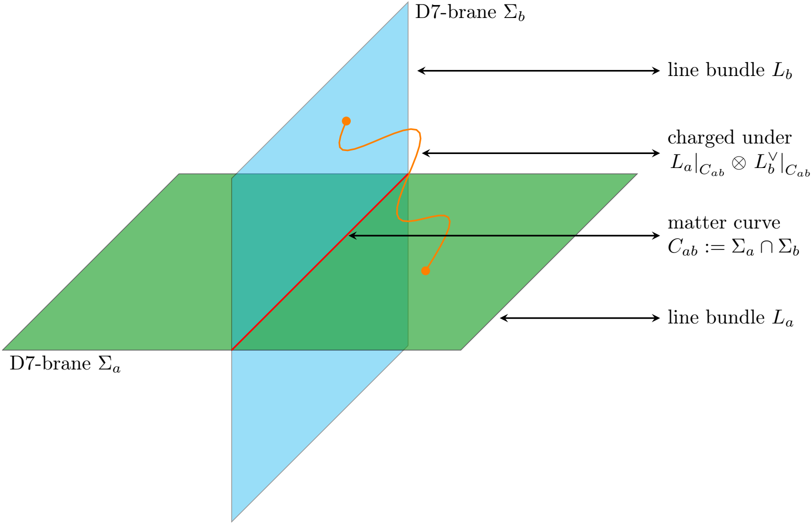

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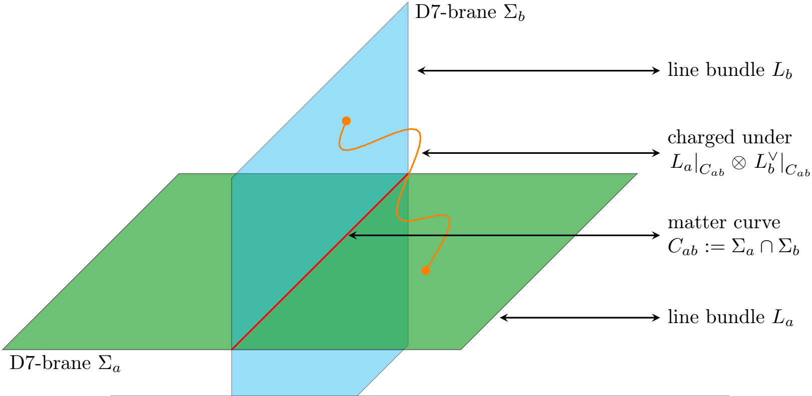
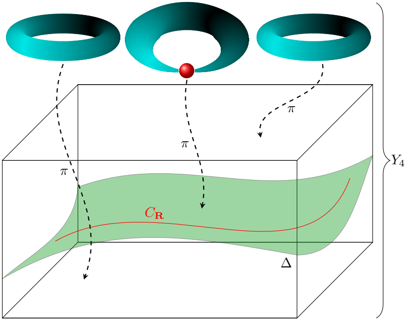


Chiral $\mathcal{N}=1$ multiplet on C_{ab}	$\leftrightarrow$	$H^0(C_{ab}, L_{ab} \otimes \sqrt{K_{C_{ab}}})$
Anti-chiral $\mathcal{N}=1$ multiplet on C_{ab}	$\leftrightarrow$	$H^1(C_{ab}, L_{ab} \otimes \sqrt{K_{C_{ab}}})$
chiral index	$\leftrightarrow$	$\chi(\mathbf{R}_{ab}) = \int_{C_{ab}} c_1(L_{ab})$

($\sqrt{K_{C_{ab}}}$ spin bundle induced by holomorphic embedding of on C_{ab})

Matching local picture with global data 1706.04616



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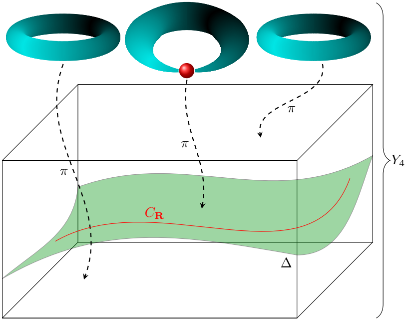

1. State in irrep. $\mathbf{R}$ (weight $\beta^a(\mathbf{R})$)

$$\leftrightarrow S^a_{\mathbf{R}} = \sum_{i=1}^{n} n_i^a \mathbb{P}^1_i \left(C_{\mathbf{R}} \right) \in \mathrm{CH}^2(\hat{Y}_4)$$

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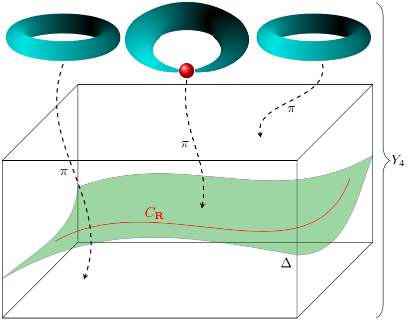

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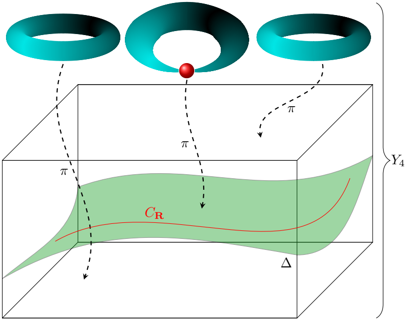

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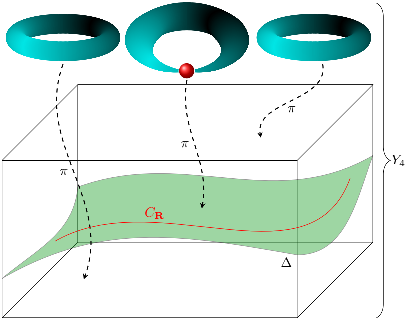

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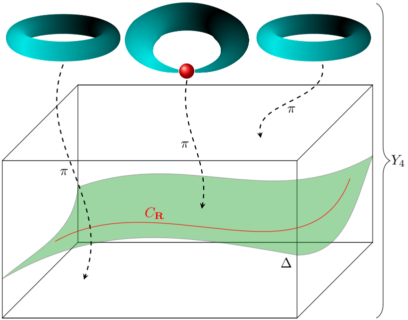

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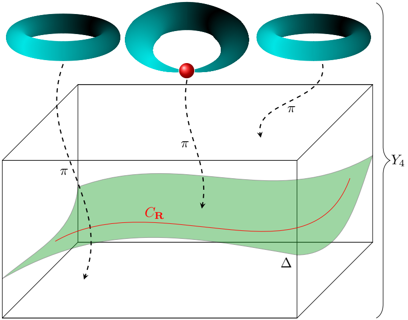

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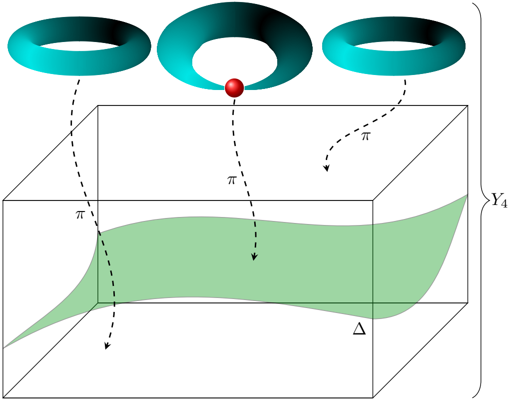
Consequence

$$\begin{array}{rcc} \mathcal{N}=1 \text{ chiral multiplets} & \leftrightarrow & H^0\left(C_{\mathbf{R}}, L\left(S^a_{\mathbf{R}}, A\right)\right) \\ \mathcal{N}=1 \text{ anti-chiral multiplets} & \leftrightarrow & H^1\left(C_{\mathbf{R}}, L\left(S^a_{\mathbf{R}}, A\right)\right) \\ \text{chiral index} & \leftrightarrow & \int_{S^a_{\mathbf{R}}} G_4 \end{array}$$

How to count massless matter? Our strategy is . . .

How to count massless matter? Our strategy is ...



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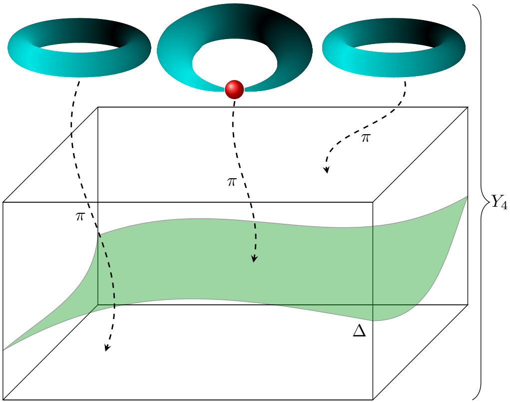

1. Pick ‘nice’ geometry

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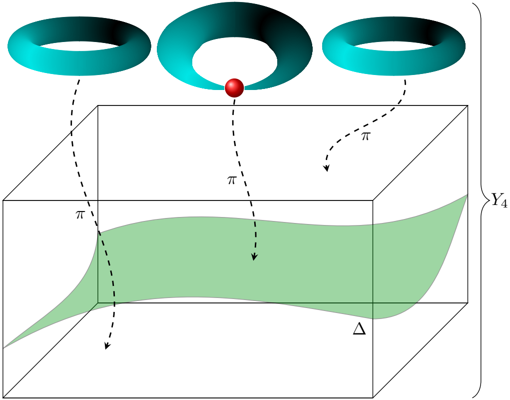

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$\Rightarrow$ Toric ambient space X_Σ

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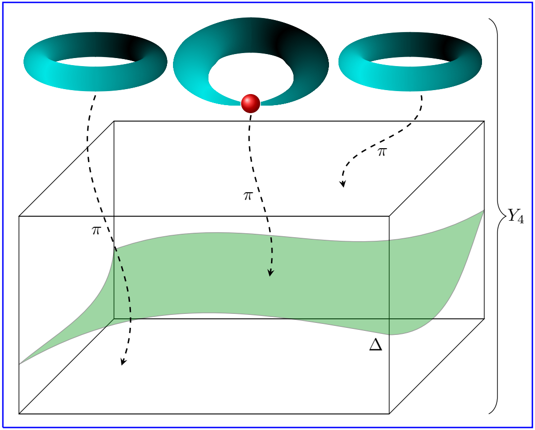

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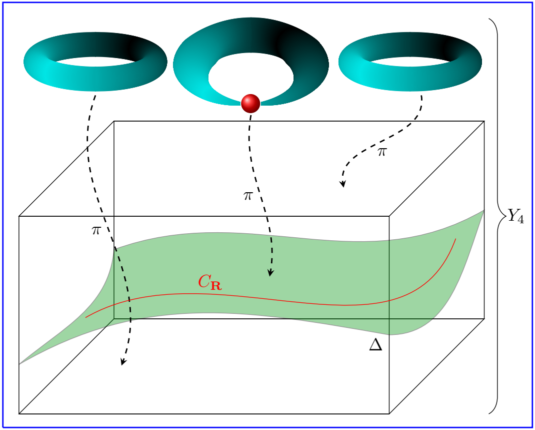

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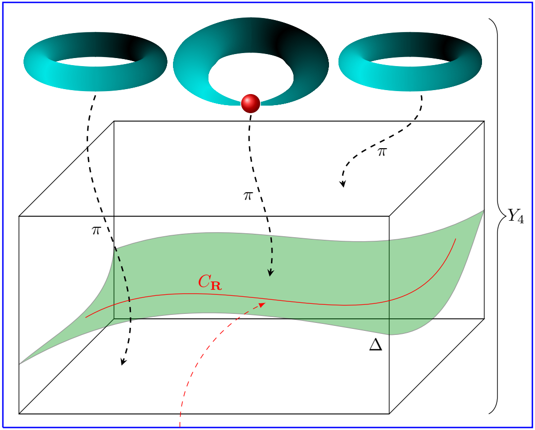

ambient space X_Σ

line bundle $L(S^a_{\mathbf{R}}, A)$ only defined on matter curve $C_{\mathbf{R}}$

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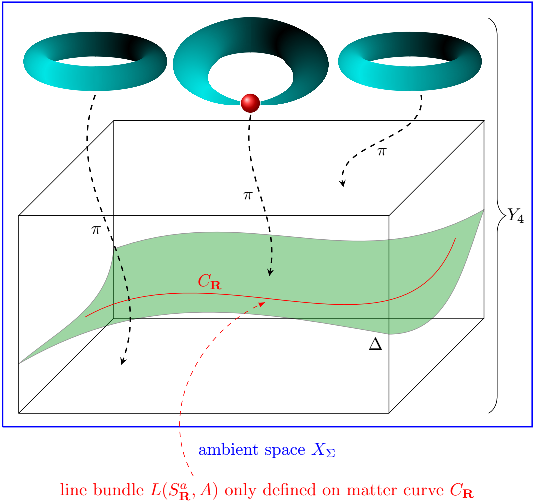

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2. Extend $L(S^a_{\mathbf{R}}, A)$ to X_Σ

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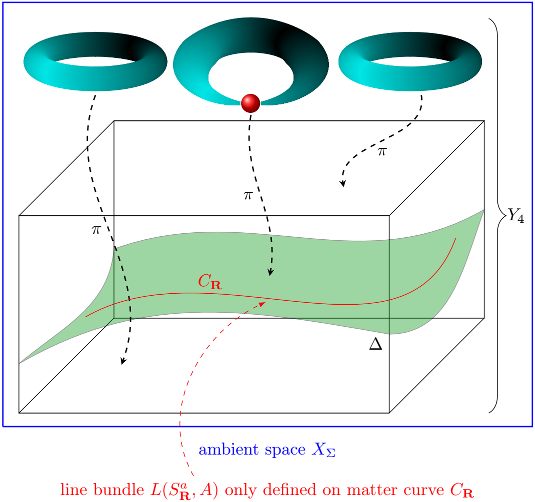

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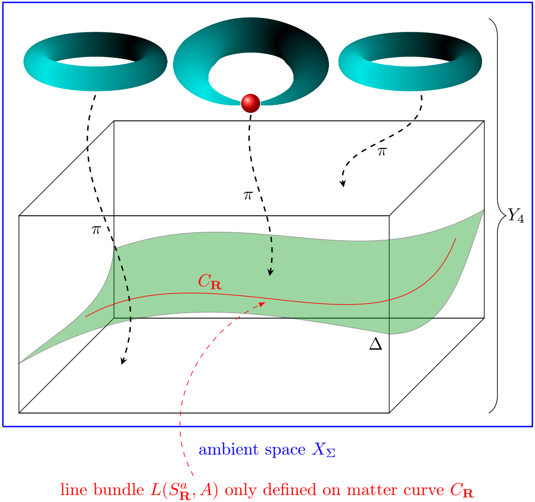

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3. Find computer models for $\mathfrak{Coh}(X_\Sigma)$

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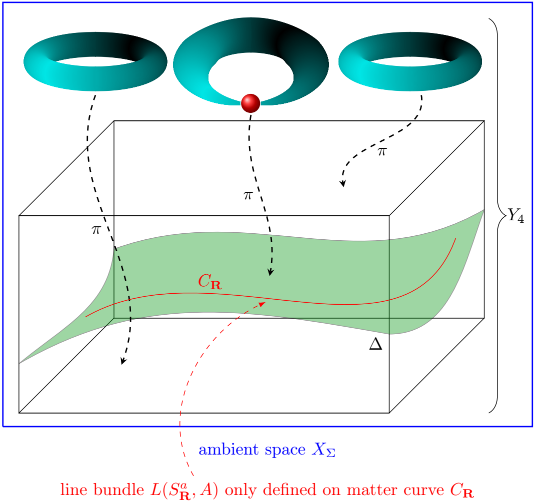

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4. Use these models to compute sheaf cohomology

Simple (ambient) spaces – toric varieties

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Remarks

- In this talk, all toric varieties are smooth and complete
- More background in [CoxLittleSchenk2011]

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Example: Projective space $\mathbb{P}^2_{\mathbb{Q}}$

- $S = \mathbb{Q}[x_1, x_2, x_3]$ and $\deg(x_i) = 1$
- $I_{SR} = \langle x_1 \cdot x_2 \cdot x_3 \rangle$

Coherent sheaves on a toric variety X_Σ (with Cox ring S)

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Sheafification functor

- $S\text{-fpgrmod}$: **category** of finitely presented graded S-modules
- $\mathfrak{Coh} X_\Sigma$: **category** of coherent sheaves on X_Σ

$\Rightarrow$ There exists the sheafification functor

$$\widetilde{\ } : S\text{-fpgrmod} \to \mathfrak{Coh} X_\Sigma \, , \; M \mapsto \widetilde{M}$$

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Computer models for coherent sheaves

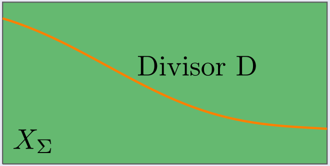
- The category $S\text{-fpgrmod}$ can be handled with CAP

$\Rightarrow$ $S\text{-fpgrmod}$ can serve as computer models for coherent sheaves

1003.1943, 1202.3337, 1210.1425, 1212.4068, 1409.2028, 1409.6100, 1712.03492

From Points to Coherent Sheaves

How to encode $\mathcal{O}_{X_\Sigma}(-D)$?



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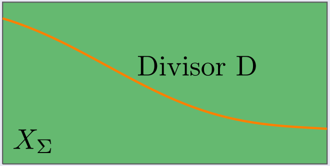

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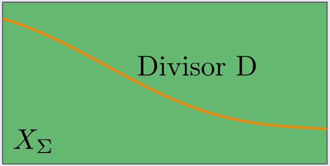

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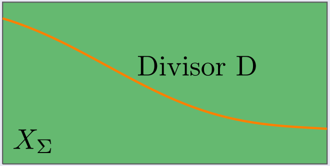

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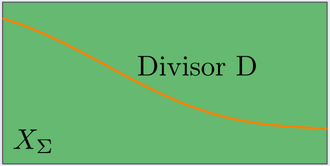

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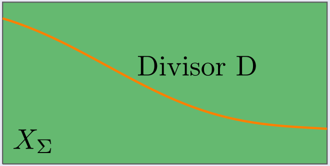

Answer

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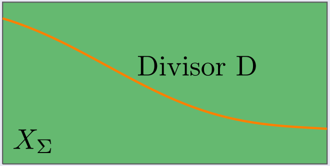

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$$\bigoplus_{j=1}^{R_2} S(e_j) \xrightarrow{A} \bigoplus_{i=1}^{R_1} S(d_i) \twoheadrightarrow M \to 0$$

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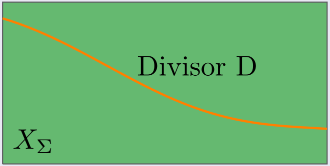

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$\Rightarrow$ $\widetilde{M} \cong \mathcal{O}_{X_\Sigma}(-D)$, so M is computer model for $\mathcal{O}_{X_\Sigma}(-D)$

Implemented Algorithm

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Input and Output

- smooth, complete toric variety X_Σ
- $F \in S\text{-fpgrmod}$

$$\Longrightarrow \quad h^i\left(X_\Sigma, \widetilde{F}\right)$$

Implemented Algorithm

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Step-by-step (References in two slides)

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$$V^k(X_\Sigma) := \left\{ L \in \mathrm{Pic}(X_\Sigma)\ ,\ h^k(X_\Sigma, L) = 0 \right\}$$

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3. Compute $\mathbb{Q}$-dimension of $\mathrm{Ext}^i_S(I, F)_0$

Example computation from 1706.04616

Input and Output

- $C_{\mathbf{5}_{-2}} \subseteq \mathbb{P}^2_{\mathbb{Q}}$
- $L_{\mathbf{5}_{-2}} \leftrightarrow F$ and F defined by $S(-36) \oplus S(-39) \oplus S(-41) \oplus S(-23) \oplus S(-38) \to S(-6) \oplus S(-21) \twoheadrightarrow F \to 0$

$\Longrightarrow$

$$h^1\left(\mathbb{P}^2_{\mathbb{Q}}, \widetilde{F}\right) = ?$$

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Apply Algorithm

1. Compute vanishing sets via *cohomCalg*:
   $V^0(\mathbb{P}^2_{\mathbb{Q}}) = (-\infty, -1]_{\mathbb{Z}}, \; V^1(\mathbb{P}^2_{\mathbb{Q}}) = \mathbb{Z}, \; V^2(\mathbb{P}^2_{\mathbb{Q}}) = [-2, \infty)_{\mathbb{Z}}$

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2. Use vanishing sets to find ideal I (along idea of G. Smith):
   $I = B_{\Sigma}^{(44)} \equiv \left\langle x_0^{44}, x_1^{44}, x_2^{44} \right\rangle$

Example computation from 1706.04616

Input and Output

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$h^1\left(\mathbb{P}^2_{\mathbb{Q}}, \widetilde{F}\right) = ?$

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2. $I = B_\Sigma^{(44)} \equiv \left\langle x_0^{44}, x_1^{44}, x_2^{44} \right\rangle$
3. Compute presentation of $\mathrm{Ext}^1_S\left(B_\Sigma^{(44)}, F\right)_0$:

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3. $\mathbb{Q}^{37425} \to \mathbb{Q}^{27201} \twoheadrightarrow \mathrm{Ext}^1_S\left(B_\Sigma^{(44)}, F\right)_0 \to 0$

$$\Rightarrow 28 = \dim_{\mathbb{Q}}\left[\mathrm{Ext}^1_S\left(B_\Sigma^{(44)}, F\right)_0\right] = h^1\left(\mathbb{P}^2_{\mathbb{Q}}, \widetilde{F}\right)$$

Summary on implementation in CAP

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- Have combined
  - *cohomCalg* by R. Blumenhagen et al.

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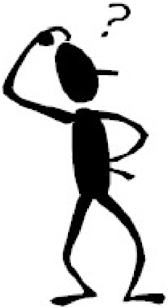

Hypercharge flux in F-theory GUT models

How to break $SU(5)$ to $SU(3) \times SU(2) \times U(1)$?

- Higgs effect
    - Requires knowledge of Higgs potential V
    - $\Rightarrow$ String theory: Derive V from geometry of $\mathcal{B}_3$
    - $\Rightarrow$ Fairly involved, so typically V is not known
- Alternative: Hypercharge flux 0802.2969, 0802.3391, ...
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Non-trivial check for CAP-performance

- Hypercharge flux A_Y **never pullback** of line bundle from Δ

$\Rightarrow$ A_Y requires full power of algorithms in CAP

$\Rightarrow$ Computation of massless spectrum possible 1706.04616, 1802.08860

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- $\Rightarrow$ A_Y requires full power of algorithms in `CAP`
- $\Rightarrow$ Computation of massless spectrum possible 1706.04616, 1802.08860
- **Can study moduli dependence of massless spectrum**

How the complex structure moduli enter

Moduli in $SU(5) \times U(1)_X$-Tate model from 1706.04616

- Matter curve $\supset C_{\mathbf{5}_{-2}} = V(a_{1,0} \cdot a_{4,3} - a_{3,2} \cdot a_{2,1}) \subseteq \mathbb{P}^2_{\mathbb{Q}}$
- $a_{1,0} = c_1 x_1^4 + c_2 x_1^3 x_2 + c_3 x_1^2 x_2 x_3 + \ldots \in \mathbb{Q}[x_1, x_2, x_3]$
- $\deg a_{1,0} = 4$, $\deg a_{2,1} = 7$, $\deg a_{3,2} = 10$, $\deg a_{4,3} = 13$

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Strategy

- Moduli c_i enter definition of line bundle $L(S^a_{\mathbb{R}}(C_{\mathbf{R}}), A)$
- Smoothness of matter curve $C_{\mathbf{R}}$ **NOT** required for `CAP`

$\Rightarrow$ Can perform computation for **non-generic** values c_i

$\Rightarrow$ Probe moduli dependence of massless spectrum

Example: $SU(5)xU(1)$-Tate Model $(\mathbf{R} = \mathbf{5}_{-2})$ 1706.04616

	$\widetilde{a_{1,0}}$	$\widetilde{a_{2,1}}$	$\widetilde{a_{3,2}}$	$\widetilde{a_{4,3}}$	$h^0(C_{\mathbf{R}}, L_{\mathbf{R}})$
M_1	$(x_1 - x_2)^4$	x_1^7	x_2^{10}	x_3^{13}	22
M_2	$(x_1 - x_2)\, x_3^3$	x_1^7	x_2^{10}	x_3^{13}	21
M_3	x_3^4	x_1^7	$x_2^7 (x_1 + x_2)^3$	$x_3^{12} (x_1 - x_2)$	11
M_4	$(x_1 - x_2)^3\, x_3$	x_1^7	x_2^{10}	x_3^{13}	9
M_5	x_3^4	x_1^7	$x_2^8 (x_1 + x_2)^2$	$x_3^{11} (x_1 - x_2)^2$	7
M_6	x_3^4	x_1^7	x_2^{10}	$x_3^8 (x_1 - x_2)^5$	6
M_7	x_3^4	x_1^7	$x_2^9 (x_1 + x_2)$	$x_3^{10} (x_1 - x_2)^3$	5

$SU(5) \times U(1)_X$ GUT with $\Delta \cong dP_3$ 1802.08860

Summary

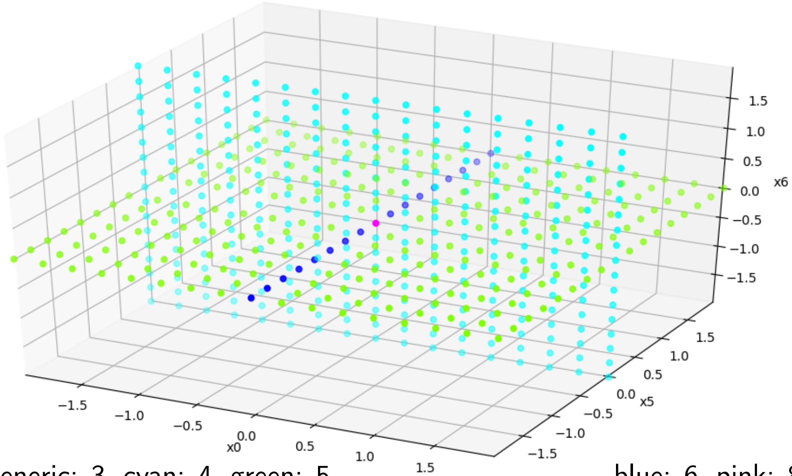
- Three matter curves $C_{\mathbf{10}_1}$, $C_{\mathbf{5}_3}$ and $C_{\mathbf{5}_{-2}}$
- Fix G_4-flux A and hypercharge flux A_Y

$\Rightarrow$ **Parametrisation** of moduli space by **208 parameters**

Strategy

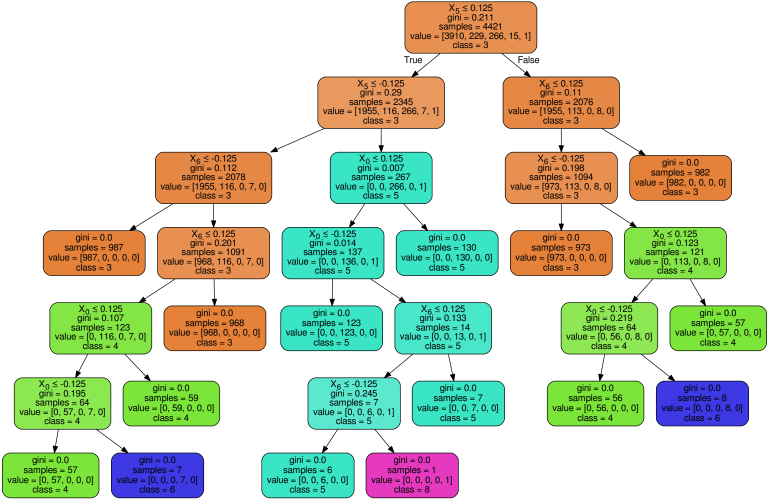
- Recently: Focus on **3-dim. patches** of parameter space
- For future work: Extend analysis to understand global structure

$h^0(C_{\mathbf{10}_1}, L(S^a_{\mathbf{R}}, A))$ with $\mathbf{R} = (\mathbf{3}, \mathbf{2})_{1_X, 1_Y}$



generic: 3, cyan: 4, green: 5, blue: 6, pink: 8

Result explained by decision tree



obtained from *scikit-learn*: 4913 data points, 3193 used for training, 1720 correctly predicted

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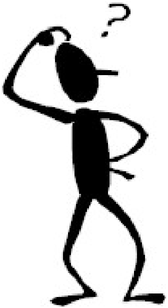

Step 1 – Physics analysis of massless matter in F-theory

Task

4 dim. F-theory compactification $\Longrightarrow$ Count (anti)-chiral massless matter fields in 4d effective theory

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Steps and results

- With T. Weigand, C. Mayrhofer, C. Pehle

  1402.5144, 1706.04616, 1706.08528, 1802.08860

- Use $A \in \mathrm{CH}^2(\hat{Y}_4)$ as parametrisation of **full** G_4-gauge data

$\Rightarrow$ Identified line bundles $L(S^i_{\mathbf{R}}, A)$ such that

$$\begin{array}{lcl} \text{Chiral } \mathcal{N} = 1 \text{ multiplets} & \leftrightarrow & H^0(C_{\mathbf{R}}, L(S^i_{\mathbf{R}}, A)) \\ \text{Anti-chiral } \mathcal{N} = 1 \text{ multiplets} & \leftrightarrow & H^1(C_{\mathbf{R}}, L(S^i_{\mathbf{R}}, A)) \end{array}$$

- Challenge: $L(S^i_{\mathbf{R}}, A)$ in general **not pullback**

Step 2 – Compute sheaf cohomologies on toric varieties

Building blocks

- Have combined two approaches:
  1. *cohomCalg* by R. Blumenhagen et al.
     1003.5217, 1006.0780, 1006.2392, 1010.3717
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Properties of algorithm

- In collaboration with M. Barakat et al. (*Siegen university*) implemented in `CAP` *https://github.com/homalg-project/CAP_project*
- Applies to smooth, complete toric varieties 1802.08860
- Has improved performance 1802.08860
- Available at *GitHub* *https://github.com/HereAround*

Step 3 – Applications to F-Theory GUTs

Challenges in F-theory GUT-models

- Hypercharge flux A_Y (used for GUT-breaking) **not pullback**
- Applicability of `CAP` requires thorough investigation of geometry (e. g. **explicit** isomorphism $dP_3 \cong \Delta$)

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Results

1. We have conjectured a construction for isomorphism $dP_3 \cong \Delta$

$\Rightarrow$ Passes lots on consistency checks

2. CAP can indeed handle A_Y

$\Rightarrow$ Given $dP_3 \cong \Delta$, computation of massless spectra feasible

Current and near future developments

Moduli dependence of massless spectra

- Matter curves **not** required to be **smooth** (nor complete intersections)

$\Rightarrow$ Can study moduli space dependence of massless spectrum

Several challenges

1. Moduli space (parametrisation) high dimensional

$\Rightarrow$ Dimensional reduction?

2. Machine learning suited for image processing (e. g. facial recognition, detection of cancer, . . .)

$\Rightarrow$ Make sense of our high-dimensional data?

3. Understand cohomology jumps as jumping lines along works of R. L. E. Schwarzenberger (1961) or M. Mulase (1979)

Other possible applications include ...

- Zero mode counting in topological string, IIB or heterotic compactifications 0403166, 0808.3621, 1106.4804, ...

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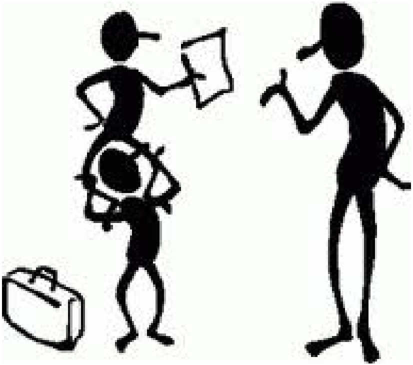
Thank you for your attention!



From Divisors to Modules

Input and Output

- $C = V(g_1, \ldots, g_k) \subseteq X_\Sigma$
- $D = V(f_1, \ldots, f_n) \in \mathrm{Div}(C)$

$\Longrightarrow$

M s.t. $\mathrm{supp}(\widetilde{M}) = C$
and $\widetilde{M}|_C \cong \mathcal{O}_C(-D)$

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Step 1: $S(C) := S/\langle g_1, \ldots, g_k \rangle$, $\pi \colon S \twoheadrightarrow S(C)$

$$\begin{tikzcd}
\bigoplus\limits_{j \in J} S(C)(j) \arrow[rr, "0"] \arrow[d, "\ker(m)"'] & & 0 \arrow[d, "0"] \\
\bigoplus\limits_{i \in I} S(C)(i) \arrow[rr, "m = (\pi(f_1), \ldots, \pi(f_n))"] \arrow[d, "\sim"] & & S(C) \arrow[d, "\sim"] \\
A_C \arrow[rr, hook, "\iota"] & & S(C)
\end{tikzcd}$$

From Divisors to Modules II

Step 2: Extend by zero to coherent sheaf on X_Σ

$$\boxed{\begin{tikzcd} \bigoplus_{j \in J} S(j) \arrow[d, "\ker(m)'"'] \\ \bigoplus_{i \in I} S(i) \end{tikzcd}} \otimes \boxed{\begin{tikzcd} \bigoplus_{k \in K} S(k) \arrow[d, "\begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}"'] \\ S(C) \end{tikzcd}}$$

$$\begin{tikzcd} \bigoplus_{i \in I} S(i) \arrow[d, "\sim"] & & S(C) \arrow[d, "\sim"] \\ A & & B \end{tikzcd}$$

$\Rightarrow M = A \otimes B$ satisfies $\mathrm{Supp}(\widetilde{M}) = C$ and $\widetilde{M}|_C \cong \mathcal{O}_C(-D)$

From Divisors to Modules III

Input and Output

- $C = V(g_1, \ldots, g_k) \subseteq X_\Sigma$
- $D = V(f_1, \ldots, f_n) \in \mathrm{Div}(C)$

$\Longrightarrow$

M s.t. $\mathrm{supp}(\widetilde{M}) = C$ and $\widetilde{M}|_C \cong \mathcal{O}_C(+D)$

Strategy

1. Compute A_C
2. Dualise via $A_C^\vee := \mathrm{Hom}_{S(C)}\left(S\left(C\right), A_C\right)$
3. Extend by zero by considering $A^\vee \otimes B$

$\Rightarrow$ $M^\vee := A^\vee \otimes B$ satisfies $\mathrm{Supp}(\widetilde{M}) = C$ and $\widetilde{M}|_C \cong \mathcal{O}_C\left(+D\right)$

An idea of the sheafification functor

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Affine open cover

- Toric variety X_Σ with Cox ring S

$\Rightarrow$ Covered by affine opens $\left\{ U_\sigma = \text{Specm}(S_{(x^{\hat{\sigma}})}) \right\}_{\sigma \in \Sigma}$

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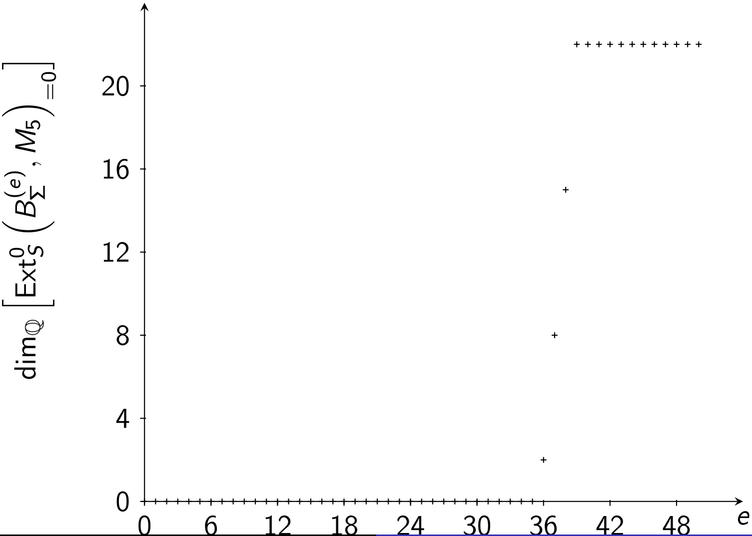
Consequence

- $M_{(x^{\hat{\sigma}})} \leftrightarrow$ coherent sheaf on $U_\sigma = \text{Specm}(S_{(x^{\hat{\sigma}})})$
- local sections: $\widetilde{M_{(x^{\hat{\sigma}})}}(D(f)) = M_{(x^{\hat{\sigma}})} \otimes_{S_{(x^{\hat{\sigma}})}} \left(S_{(x^{\hat{\sigma}})} \right)_f$

Module M_5 from 1706.04616: Quality Check I



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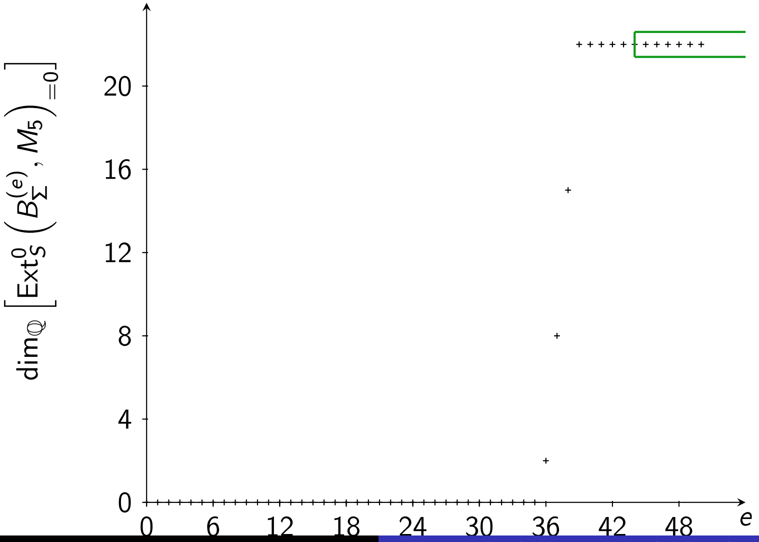

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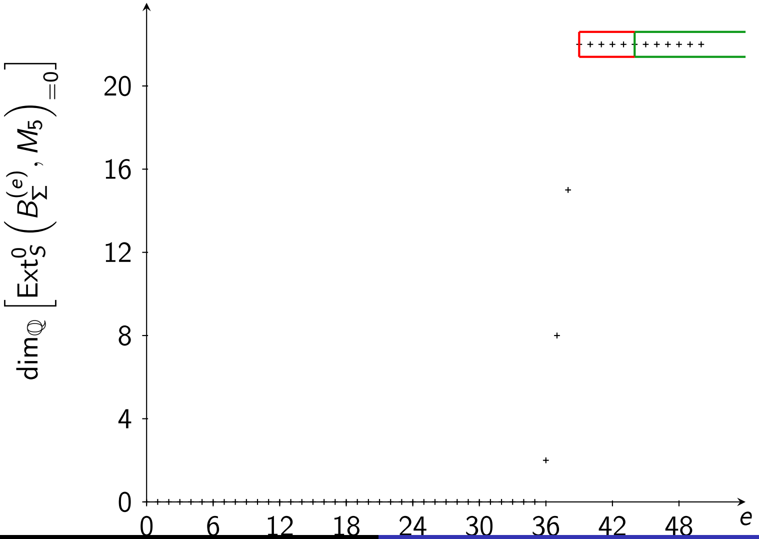
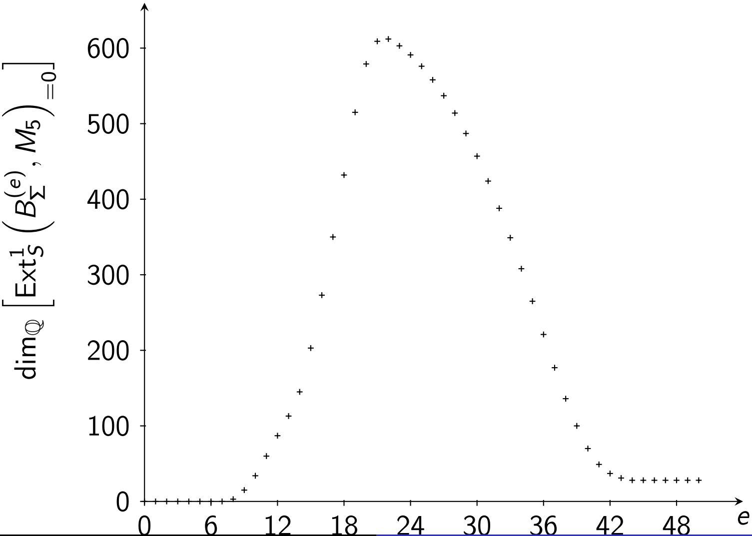


Module M_5 from 1706.04616: Quality Check II



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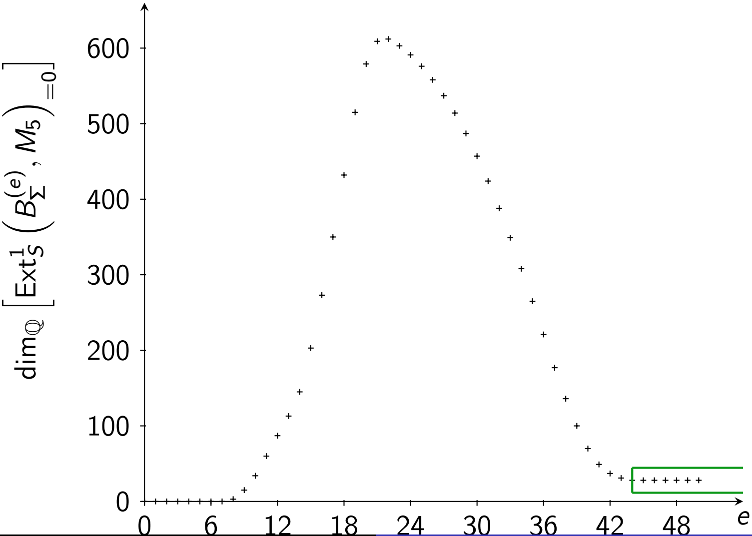

How to determine the ideal I in step 2 of algorithm?

Input

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Preparation

- $p \in \mathrm{Cl}(X_\Sigma)$ **ample**, $m(p) = \{m_1, \ldots, m_k\}$ all monomials of degree p and $I(p,e) = \left\langle m_1^e, \ldots, m_k^e \right\rangle$
- Pick $e = 0$ and increase it until subsequent conditions are met

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$\Rightarrow$ Increase e further until $H^m\left(\mathbf{C}^0\right) \cong \mathrm{Ext}^m_S(I(p,e), M)_0$

The Hom-Embedding

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\rho"'] & G_M^\vee \otimes R_N \arrow[r, "\rho_M^\vee \otimes \mathrm{id}_{R_N}"] \arrow[d, "\mathrm{id}_{G_M^\vee} \otimes \rho_N"] & R_M^\vee \otimes R_N \arrow[d, "\mathrm{id}_{R_M^\vee} \otimes \rho_N"] \\
G \arrow[r, "\alpha"] \arrow[d] & G_M^\vee \otimes G_N \arrow[r, "\rho_M^\vee \otimes \mathrm{id}_{G_N}"] \arrow[d] & R_M^\vee \otimes G_N \arrow[d] \\
\mathrm{Hom}_S(M, N) \arrow[r, hook, red, "\iota"] & G_M^\vee \otimes N \arrow[r, red, "\rho_M^\vee \otimes \mathrm{id}_N"] & R_M^\vee \otimes N
\end{tikzcd}$$

S-fpgrmod 1 – Category of projective graded S-modules

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Morphisms:

morphisms of **graded** modules

Example: S the Cox ring of $\mathbb{P}^2_{\mathbb{Q}}$

$\varphi \colon S(-1) \xrightarrow{(x_1)} S(0)$ is morphism in this category since

$$\underbrace{S\left(-1\right) \ni 1}_{\text{degree1}} \mapsto \varphi\left(1\right) = \underbrace{x_1 \in S\left(0\right)}_{\text{degree1}}$$

S-fpgrmod 2: Objects

General rule:

Objects in $S\text{-fpgrmod}$ $\widehat{=}$ morphisms of projective graded S-modules

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Example on $\mathbb{P}^2_{\mathbb{Q}}$: $S = \mathbb{Q}[x_1, x_2, x_3]$, $\deg(x_i) = 1$

$M_\varphi \equiv \operatorname{coker}(\varphi)$ and $M_\psi \equiv \operatorname{coker}(\psi)$ are abstractly described by

$$\psi \colon S(-2)^{\oplus 3} \xrightarrow{R} S(-1)^{\oplus 3}, \quad R = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \quad \varphi \colon 0 \to S(0)$$

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Notation

$$\boxed{\begin{tikzcd} S(-2)^{\oplus 3} \arrow[d, "R"'] \\ S(-1)^{\oplus 3} \end{tikzcd}} \qquad \boxed{\begin{tikzcd} 0 \arrow[d, "0"] \\ S(0) \end{tikzcd}}$$

S-fpgrmod 3: Morphisms

Definition: Morphism $M_\psi \to M_\varphi$ is commutative diagram

$$\begin{tikzcd}
S(-2)^{\oplus 3} \arrow[rr, "A"] \arrow[d, "R"'] & & 0 \arrow[d, "0"] \\
S(-1)^{\oplus 3} \arrow[rr, "B"] & & S(0)
\end{tikzcd}
\qquad
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Implementation for CAP at *https://github.com/HereAround:*

- 'CAPCategoryOfProjectiveGradedModules'
- 'CAPPresentationCategory'
- 'PresentationByProjectiveGradedModules'

Computing H^0 – general idea

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$$H^0\left(X_\Sigma, \mathcal{F}\right) := \Gamma\left(\mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{O}_X, \mathcal{F}\right)\right)$$

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Idea

- M such that $\widetilde{M} \cong \mathcal{O}_X$
- F such that $\widetilde{F} \cong \mathcal{F}$

$\Rightarrow \Gamma\left(\mathscr{H}om_{\mathcal{O}_X}\left(\mathcal{O}_X, \mathcal{F}\right)\right) \stackrel{?}{=} \mathrm{Hom}_S\left(M, F\right)_0$

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Careful!

In general wrong – have to choose M carefully

Computing H^0 – different models for the structure sheaf

Example: $B_\Sigma = \langle x_1, x_2, x_3 \rangle$ and S are models for $\mathcal{O}_{\mathbb{P}^2_{\mathbb{Q}}}$

$$
\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"'] & S(-2)^{\oplus 3} \arrow[r, "0"] \arrow[d, "R"'] & 0 \arrow[r, "0"] \arrow[d, "0"] & S(-1)^{\oplus 3} \arrow[r, "0"] \arrow[d, "\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}"'] & 0 \arrow[d, "0"'] \\
0 \arrow[r, "0"] \arrow[d] & S(-1)^{\oplus 3} \arrow[r, "\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}"] \arrow[d] & S(0) \arrow[r, "\mathrm{id}"] \arrow[d] & S(0) \arrow[r, "0"] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r, hook, "\ker(\iota)"] \arrow[d, "\sim"] & B_\Sigma \arrow[r, hook, "\iota"] \arrow[d, "\sim"] & S(\mathbb{P}^2_{\mathbb{Q}}) \arrow[r, two heads, "\operatorname{coker}(\iota)"] \arrow[d, "\sim"] & S(\mathbb{P}^2_{\mathbb{Q}})/B_\Sigma \arrow[r, "0"] \arrow[d, "\sim"] & 0 \arrow[d, "\sim"'] \\
0 \arrow[r, hook, "\ker(\widetilde{\iota})"] & \widetilde{B(\Sigma)} \arrow[r, hook, "\iota"] & \widetilde{S(\mathbb{P}^2_{\mathbb{Q}})} \arrow[r, two heads, "\operatorname{coker}(\widetilde{\iota})"] & \widetilde{S(\mathbb{P}^2_{\mathbb{Q}})}/B_\Sigma \arrow[r, "0"] & 0
\end{tikzcd}
$$

Computing H^0 – is B_Σ or S better?

Task

- On $\mathbb{P}^2_{\mathbb{Q}}$, $F = B_\Sigma = \langle x_1, x_2, x_3 \rangle$ satisfies $\widetilde{F} \cong \mathcal{O}_{\mathbb{P}^2_{\mathbb{Q}}}$

$\Rightarrow$ $H^0(\mathbb{P}^2_{\mathbb{Q}}, \widetilde{F}) \cong \mathbb{Q}^1$

$\Rightarrow$ Task: Reproduce this from $\mathrm{Hom}_S(X, F)_0$ with $X \in \{S, B_\Sigma\}$

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Try 1: $X = S$

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Try 2: $X = B_\Sigma$

$\mathrm{Hom}_S(B_\Sigma, F)_0 \cong \mathbb{Q}^1$ – correct result!

Sketch of Algorithm in CAP

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Input and Output

- (smooth, complete) or (simplicial, projective) toric variety X_Σ
- $M \in S\text{-fpgrmod}$

$$\Longrightarrow \quad h^i\left(X_\Sigma, \widetilde{M}\right)$$

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Step-by-step

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