Tensor products of finitely presented functors

Martin Bies[1] and Sebastian Posur[2]

[1] *Université Libre de Bruxelles, Belgium*
[2] *University of Siegen, Germany*

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ULB

1 Motivation

1.1 Sheaf cohomology in string model building

- Common task: Relate physical quantities to geometry
- Example: Zero mode counting ↔ sheaf cohomologies
- Also, deduce other geometric quantites from sheaf cohomology
- Example: Intersection number $D_1 \cdot D_2$ of divisors D_1, D_2 of surface S

$$D_1 \cdot D_2 = \deg(\mathcal{O}_S(D_1)|_{D_2}) = h^0(\mathcal{O}_S(D_1)|_{D_2}) - h^1(\mathcal{O}_S(D_1)|_{D_2}) + g(D_2) - 1$$

1.2 Architecture of computer implementation [1]

Coherent toric sheaves	↔	Objects in Freyd categories
Cohomology algorithms	↔	Induced by monoidal structures

2 Freyd categories

2.1 Freyd categories – generalities [2]

- Any additive category $\mathbf{A}$ admits a Freyd category $\mathcal{A}(\mathbf{A})$ with

$$\mathbf{A} \subseteq \mathcal{A}(\mathbf{A}) \quad \text{and} \quad \mathcal{A}(\mathbf{A}) \text{ has cokernels}$$

- Unified framework for f.p. (graded) modules and f.p. functors
- Completely constructive – see `CAP`-package [3]

2.2 Freyd categories – elementary constructions

- Objects: Be $a \xleftarrow{\rho_a} r_a \in \mathrm{Mor}(\mathbf{A})$, then $A \equiv (a \xleftarrow{\rho_a} r_a) \in \mathrm{Obj}(\mathcal{A}(\mathbf{A}))$
- Morphisms: Equiv. classes of commutative diagrams in $\mathbf{A}$
- Example: See *Defining data of protensor product T*

3 Promonoidal structures

3.1 A promonoidal structure on A consists (among others) of

- a functor $T \colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A})$ (*protensor product*),
- an additive functor $\underline{\mathrm{Hom}}(a, -) : \mathbf{A} \to \mathcal{A}(\mathbf{A})$ for every $a \in \mathrm{Obj}(\mathbf{A})$

subject to **restricted** pentagonal identity, hexagonal identities, ...

3.2 Towards extensions to monoidal structures

Given a bilinear functor $F : \mathbf{A} \times \mathbf{A} \longrightarrow \mathcal{A}(\mathbf{A})$, there exists a bilinear and right exact functor $\widehat{F} : \mathcal{A}(\mathbf{A}) \times \mathcal{A}(\mathbf{A}) \longrightarrow \mathcal{A}(\mathbf{A})$. For two objects $A_1 \equiv (a_1 \xleftarrow{\rho_1} r_1)$, $A_2 \equiv (a_2 \xleftarrow{\rho_2} r_2)$ of $\mathcal{A}(\mathbf{A})$ it holds

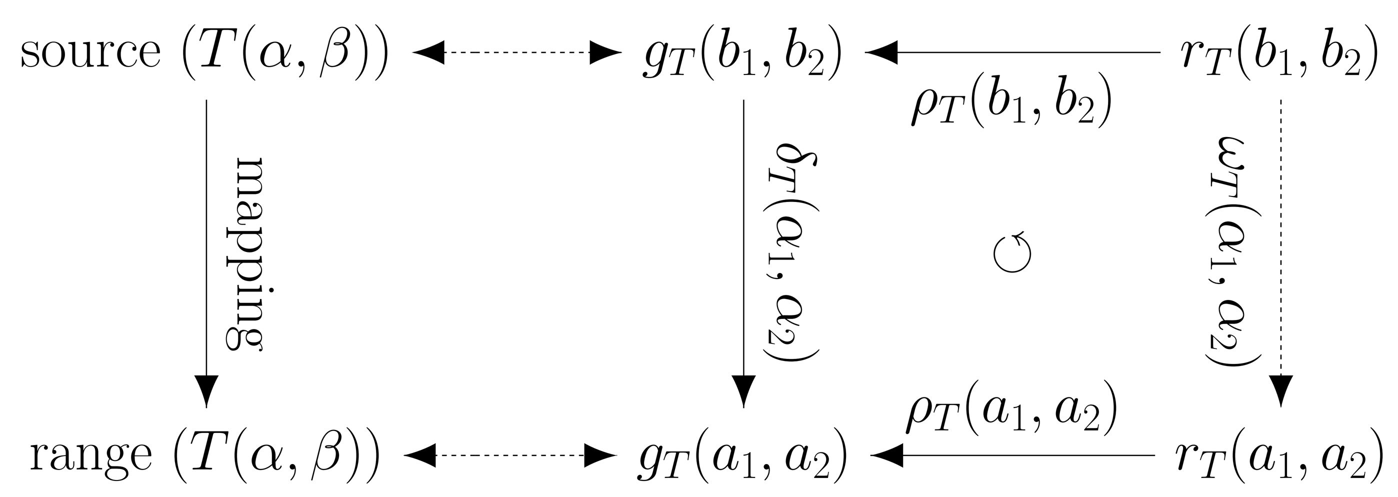
$$\widehat{F}(A_1, A_2) := \mathrm{cok}\left[F(a_1, a_2) \xleftarrow{\begin{pmatrix} F(\mathrm{id}_{a_1}, \rho_2) \\ F(\rho_1, \mathrm{id}_{a_2}) \end{pmatrix}} F(a_1, r_2) \oplus F(r_1, a_2) \right]$$

4 Defining data of protensor product T

- For $a_1, a_2 \in \mathrm{Obj}(\mathbf{A})$, denote $T(a_1, a_2) \in \mathrm{Obj}(\mathcal{A}(\mathbf{A}))$ by

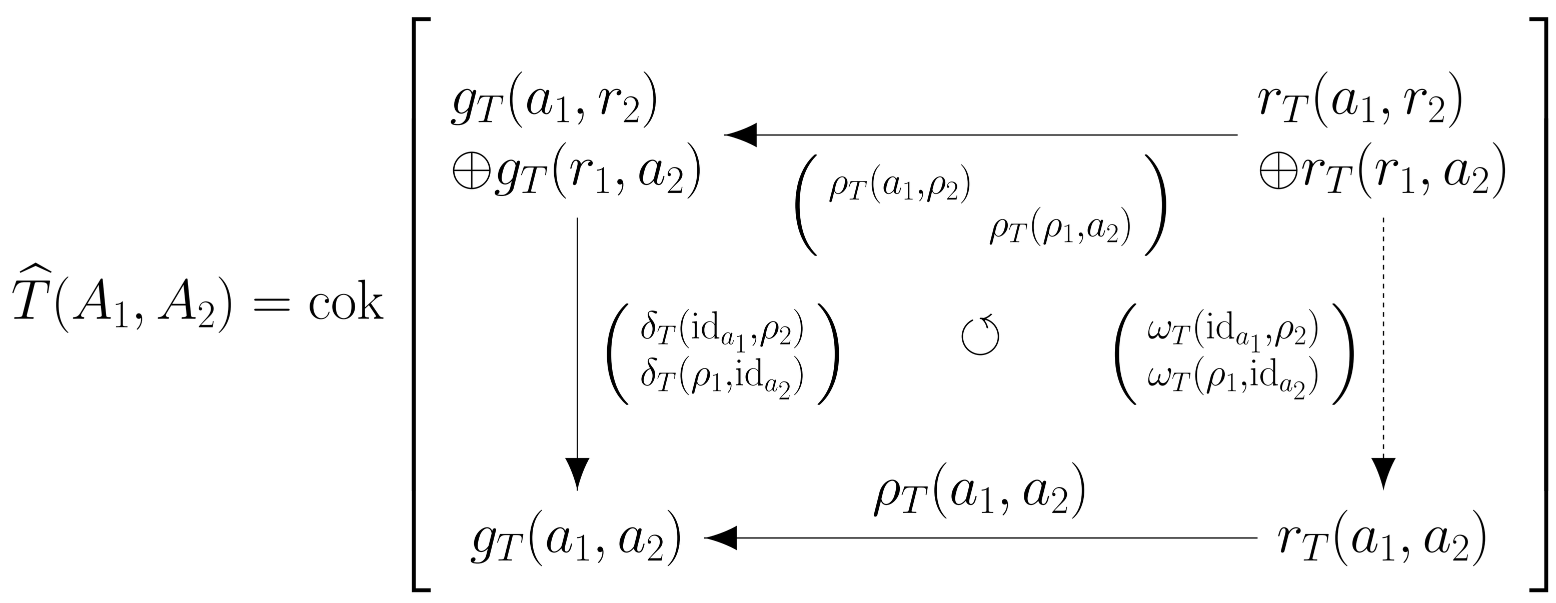
$$\left(g_T(a_1, a_2) \xleftarrow{\rho_T(a_1, a_2)} r_T(a_1, a_2) \right)$$

- For $a_1 \xleftarrow{\alpha_1} b_1$, $a_2 \xleftarrow{\alpha_2} b_2$, denote $T(\alpha, \beta) \in \mathrm{Mor}(\mathcal{A}(\mathbf{A}))$ by



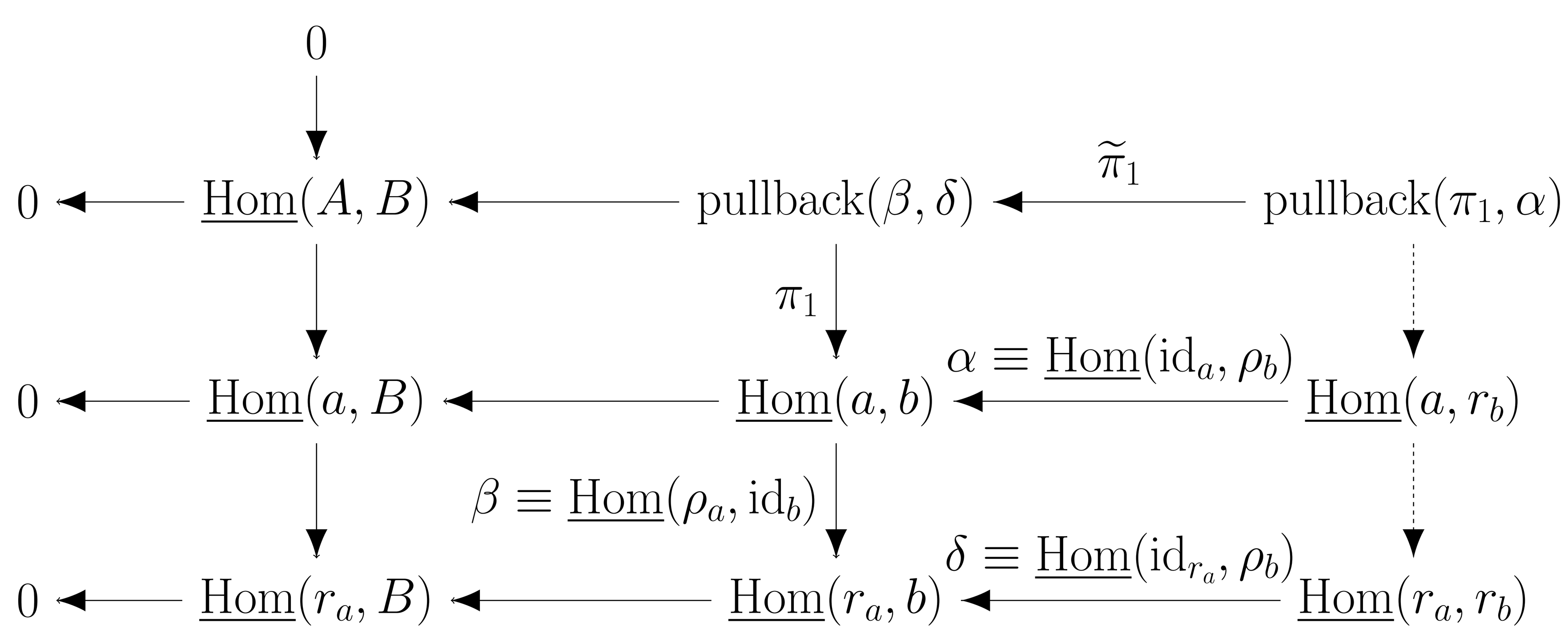
5 Examples: Promonoidal ⇒ monoidal

5.1 Application of section 3.2 to protensor product T



$$= \left(g_T(a_1, a_2) \xleftarrow{\begin{pmatrix} \rho_T(a_1,a_2) \\ \delta_T(\mathrm{id}_{a_1}, \rho_2) \\ \delta_T(\rho_1, \mathrm{id}_{a_2}) \end{pmatrix}} \begin{matrix} r_T(a_1, a_2) \\ \oplus g_T(a_1, r_2) \\ \oplus g_T(r_1, a_2) \end{matrix} \right)$$

5.2 internal homs in $\mathcal{A}(\mathbf{A})$ from internal homs in $\mathbf{A}$



6 Outlook: Zero modes in F-theory

- Elliptic fibration $\pi \colon \widehat{Y}_4 \twoheadrightarrow \mathcal{B}_3$ and G_4-flux $G_4 \in H^{2,2}(\widehat{Y}_4, \mathbb{Z})$
- $D_{\mathbf{R}} \equiv \pi_*(G_4 \cdot S_{\mathbf{R}}) \in \mathrm{Pic}(\Sigma_{\mathbf{R}})$ with $\Sigma_{\mathbf{R}}$ the matter curve of rep. $\mathbf{R}$

$\Rightarrow h^i(\Sigma_{\mathbf{R}}, \mathcal{O}_{\mathbf{R}}(D_{\mathbf{R}}))$ count zero modes in rep. $\mathbf{R}$ [4]

- Latest toric algorithms use $H^0(X_\Sigma, \widetilde{M}) \cong \underline{\mathrm{Hom}}(I, M)_0$ (with ideal $I \leq S(X_\Sigma)$ of Cox ring of X_Σ) and monoidal derivations [1]
- Applications to F-theory setups in [5] currently on their way
  Example: 4-family Pati-Salam model with

$$h^i(\mathcal{L}_{1,2,2}) = (4,4), \qquad h^i(\mathcal{L}_{4,1,2}) = (1,5),$$
$$h^i(\mathcal{L}_{6,1,1}) = (4,4), \qquad h^i(\mathcal{L}_{4,2,1}) = (5,1).$$

References

[1] M. Bies, "SheafCohomologyOnToricVarieties." `https://github.com/homalg-project/SheafCohomologyOnToricVarieties`, 2019.

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[3] M. Bies and S. Posur, "FreydCategories." `https://github.com/homalg-project/CAP_project/tree/master/FreydCategoriesForCAP`, 2019.

[4] M. Bies, C. Mayrhofer and T. Weigand, *Gauge Backgrounds and Zero-Mode Counting in F-Theory, JHEP* **11** (2017) 081 [`1706.04616`].

[5] M. Cvetič, J. Halverson, L. Lin, M. Liu and J. Tian, *A Quadrillion Standard Models from F-theory*, `1903.00009`.