Root bundles: Applications to F-theory Standard model

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Based on work with M. Cvetič, R. Donagi, M. Liu, M. Ong
2102.10115, 2104.08297, 2205.00008, 2303.08144 & **2307.02535**.

Desired vector-like spectra in the F-theory QSMs [Cvetič Halverson Lin Liu Tian '19]

Rep. $\mathbf{R}$ of $SU(3) \times SU(2) \times U(1)$	$n_{\mathbf{R}} = \#$ chiral superfields in rep $\mathbf{R}$	$n_{\overline{\mathbf{R}}} = \#$ chiral superfields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$(\mathbf{3}, \mathbf{2})_{1/6}$			
$(\mathbf{1}, \mathbf{2})_{-1/2}$			
$(\overline{\mathbf{3}}, \mathbf{1})_{-2/3}$			
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$(\mathbf{1}, \mathbf{1})_{1}$			
How to compute?			

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How to compute?			$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

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How to compute?	$h^0(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$ [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17] [M.B. '18] and references therein	$h^1(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$	$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

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How to compute?	$h^0(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$	$h^1(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$	$\chi = \deg(\mathcal{L}_{\mathbf{R}}) - g(C_{\mathbf{R}}) + 1$ $\chi = \int_{S_{\mathbf{R}}} G_4 = 3$
	[M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17]		[Cvetič Halverson Lin Liu Tian '19]
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Challenges for vector-like spectra in F-theory QSMs

General findings

- Matter in rep. **R** localized on smooth, irreducible matter curve $C_{\mathbf{R}}$ of genus g.
- It holds $\mathcal{L}_{\mathbf{R}} = \mathcal{S} \otimes \mathcal{F}_{\mathbf{R}}$ [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]
  - $\mathcal{S}$ is one of the 2^{2g} spin bundles on $C_{\mathbf{R}}$. [Atiyah '71], [Mumford '71]
  - $\mathcal{F}_{\mathbf{R}}$ a contribution from the G_4-flux.

$\Rightarrow$ Compute $n_{\mathbf{R}} = h^0(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$ and $n_{\overline{\mathbf{R}}} = h^1(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$.

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Challenges

- $n_{\mathbf{R}}$, $n_{\overline{\mathbf{R}}}$ depend on choice of spin bundle. Which $\mathcal{S}$ is compatible with the physics?
- In QSMs, cannot (yet) find unique $\mathcal{F}_{\mathbf{R}}$. [M.B. Cvetič Donagi Liu Ong '21]
- Both points fairly tricky to address.
  $\rightarrow$ Instead, formulate **necessary** conditions and study their solutions.

Root bundle condition and challenges [M.B. Cvetič Donagi Liu Ong '21]

For $(\mathbf{3}, \mathbf{2})_{1/6}$, find **necessary** condition: $\mathcal{L}_{(\mathbf{3},\mathbf{2})_{1/6}} = \sqrt[20]{K_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}^{\otimes 16}}$

(Constraint stated for base 3-folds B_3 with $K_{B_3}^3 = 10$. See [M.B. Cvetič Donagi Liu Ong '21] for constraints in bases B_3 with other $K_{B_3}^3$.)

- Has 20^{12} solutions $\leftrightarrow$ Infinitely many line bundles with $\chi = 3$.
  $\rightarrow$ **Highly non-trivial** constraint.
- Likely, not all solutions physical ($\leftrightarrow$ **necessary** condition).
  $\rightarrow$ To be addressed in future work.

$\Rightarrow$ Goal: Compute $h^0(\mathcal{C}_{(\mathbf{3},\mathbf{2})_{1/6}}, \mathcal{L}_{(\mathbf{3},\mathbf{2})_{1/6}})$ for all 20^{12} solutions.

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Challenges

- Hard to construct solutions $\mathcal{L}_{(\mathbf{3},\mathbf{2})_{1/6}}$ on **smooth, irreducible** curve.
- $h^0(\mathcal{C}_{(\mathbf{3},\mathbf{2})_{1/6}}, \mathcal{L}_{(\mathbf{3},\mathbf{2})_{1/6}})$ may depend on complex structure.

Approximation by limit roots [M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22]

- F-theory QSMs admit **canonical nodal** matter curve $C_{\mathbf{R}}^{\bullet}$ [M.B. Cvetič Liu '21]

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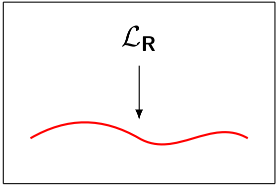

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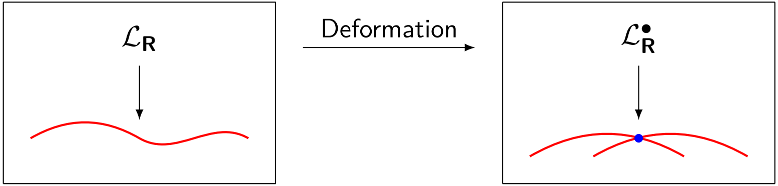

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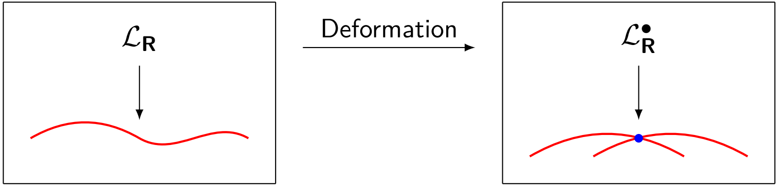

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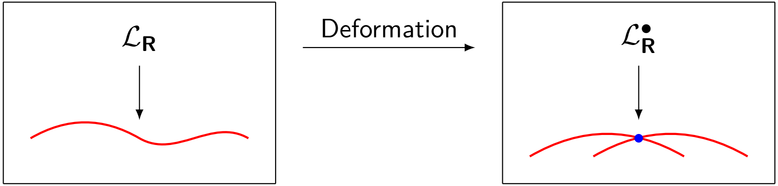

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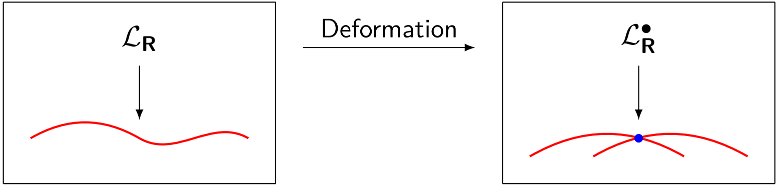

Enumerate all limit roots with computer and **try to** find h^0.

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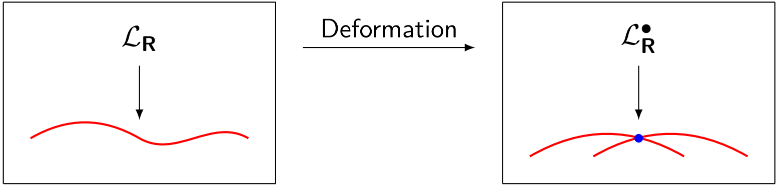

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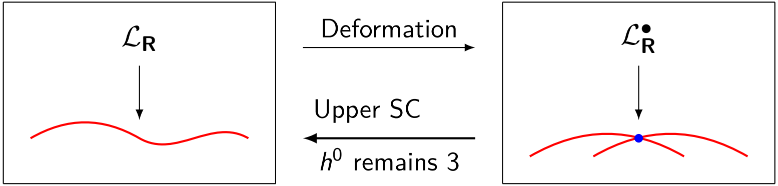

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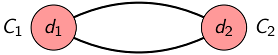

Line bundle L s.t. $L|_{C_i} = \mathcal{O}_{\mathbb{P}^1}(d_i)$

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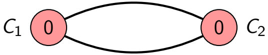

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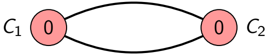

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↓

P s.t. $P^{\otimes}2 = L$

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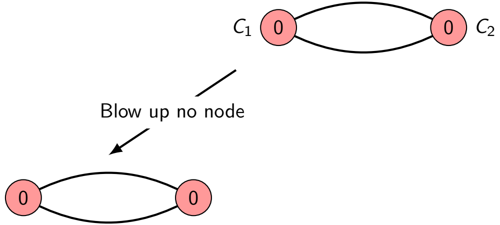

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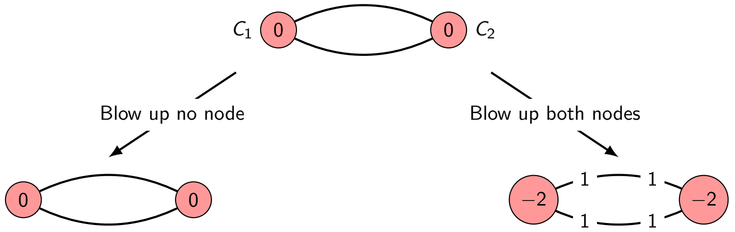

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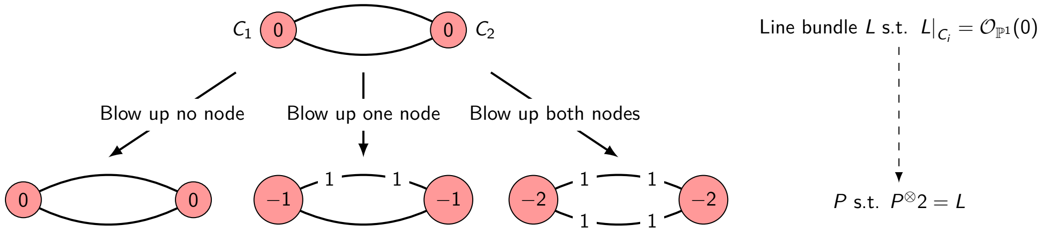

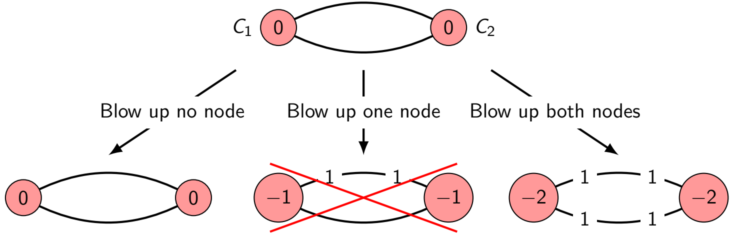

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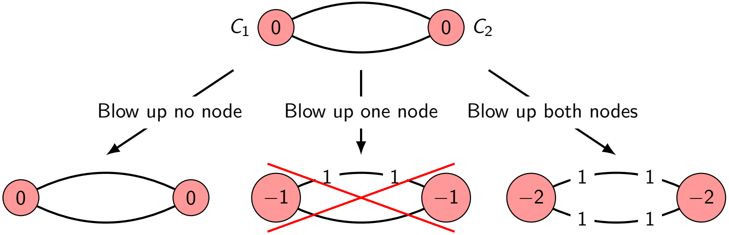

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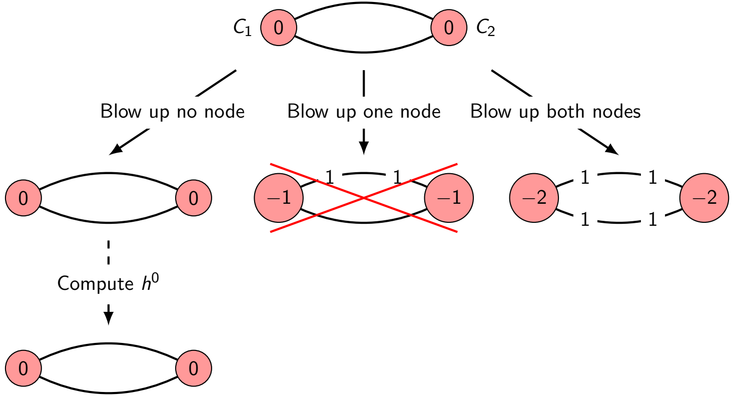

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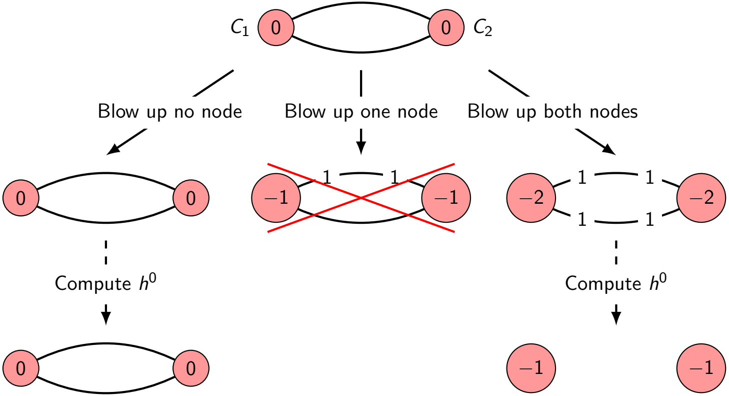

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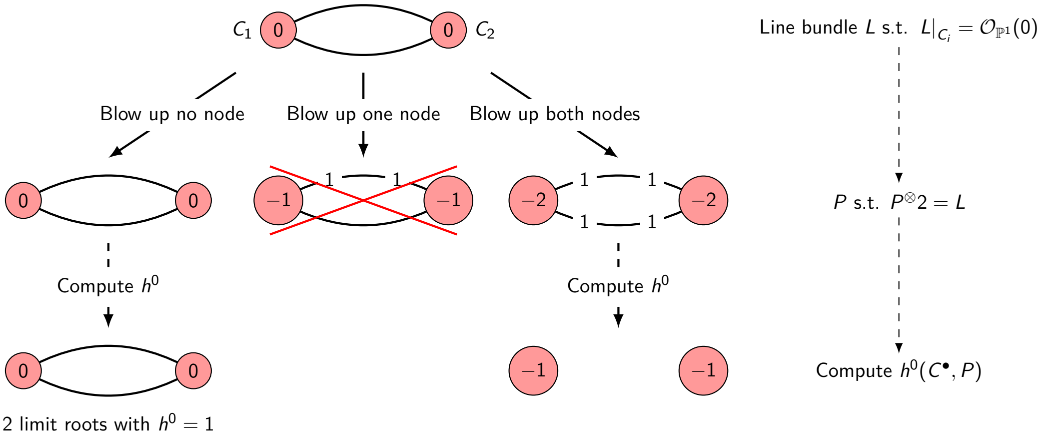
C_1
0
0
C_2
Blow up no node
Blow up one node
Blow up both nodes
0
0
-1
1
1
-1
-2
1
1
1
1
-2
Compute h^0
Compute h^0
0
0
-1
-1
2 limit roots with $h^0 = 1$
Line bundle L s.t. $L|_{C_i} = \mathcal{O}_{\mathbb{P}^1}(0)$
P s.t. $P^{\otimes}2 = L$
Compute $h^0(C^\bullet, P)$

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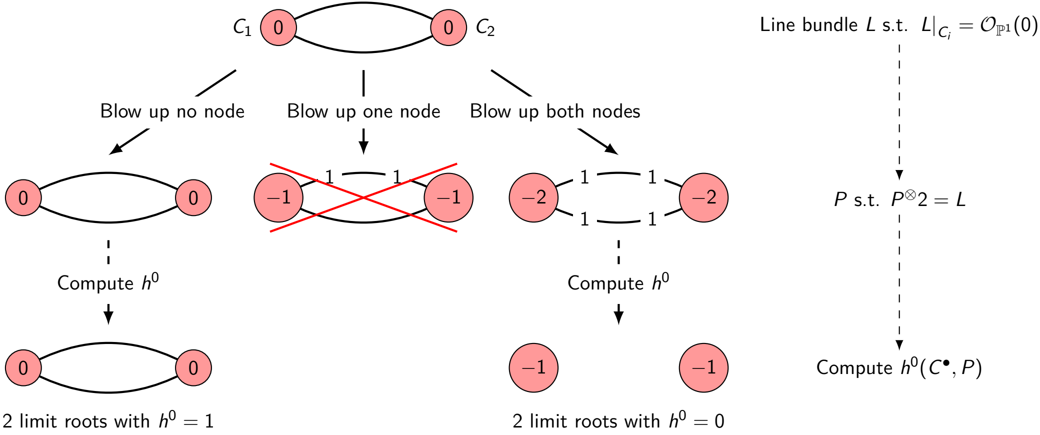

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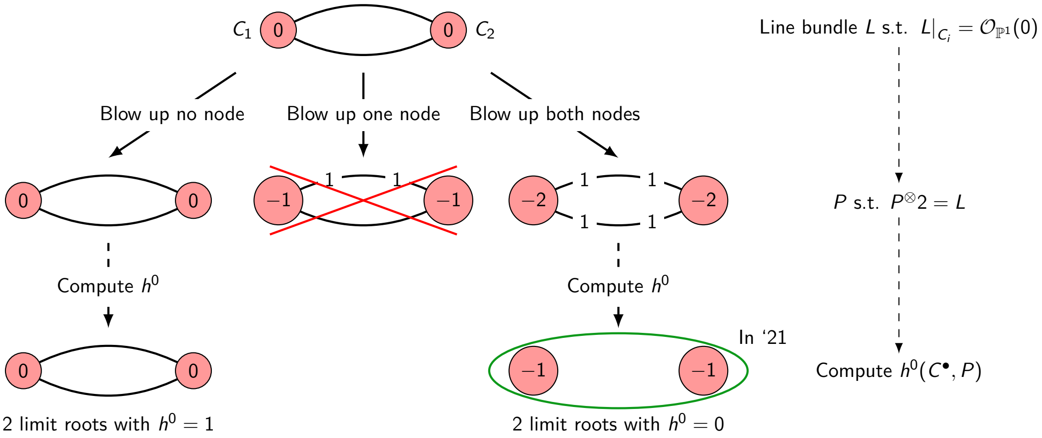

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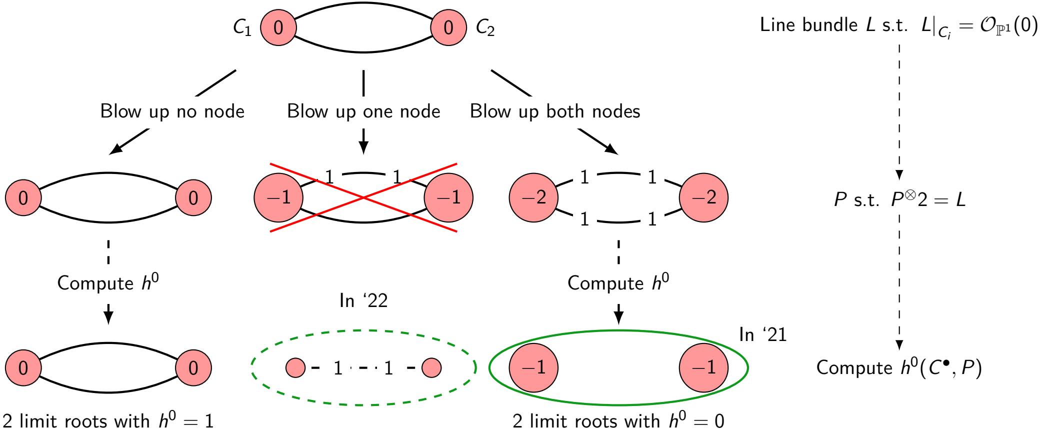

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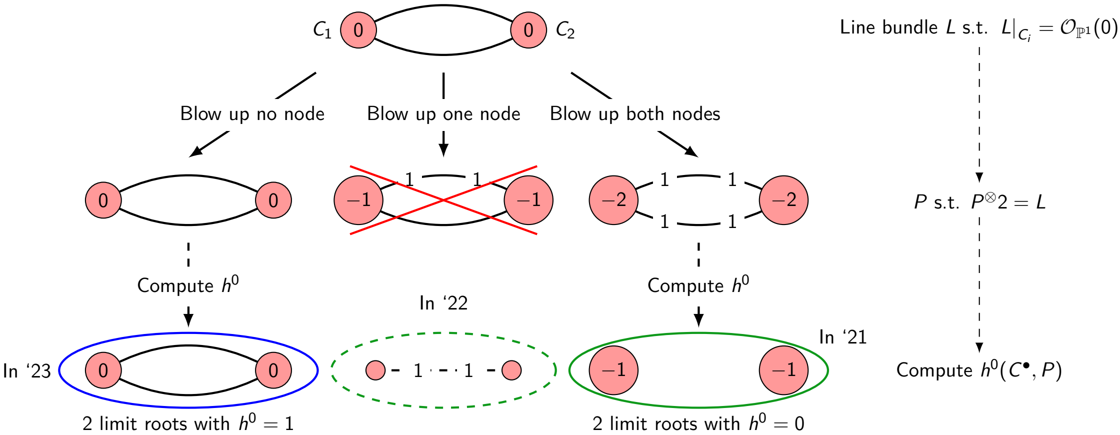

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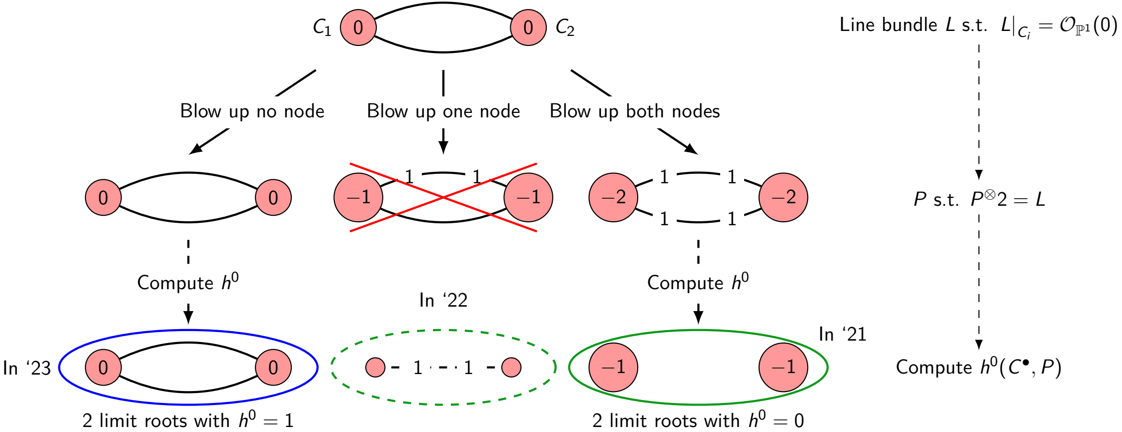
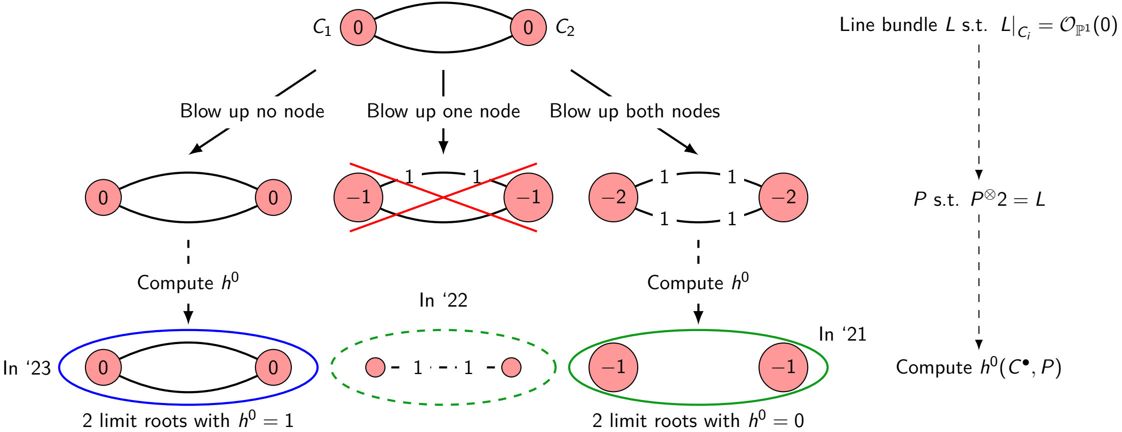


Smooth, irreducible curve		Nodal curve			
		$h^0 = 0$	$h^0 \geq 0$	$h^0 = 1$	$h^0 \geq 1$
		2		2	



Smooth, irreducible curve				Nodal curve			
$h^0 = 0$	$h^0 \geq 0$	$h^0 = 1$	$h^0 \geq 1$	$h^0 = 0$	$h^0 \geq 0$	$h^0 = 1$	$h^0 \geq 1$
3		1		2		2	

Brill-Noether numbers of $(\overline{\mathbf{3}}, \mathbf{2})_{1/6}$ in QSMs [M.B. Cvetič Liu '21]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ_8°	57.3	?	?	?	?	?	?	?

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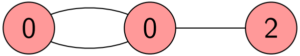
Δ°_{8}	57.3	?	?	?	?	?	?	?
Δ°_{4}	53.6	?	?	?	?	?	?	?
Δ°_{134}	48.7	?	?	?	?	?	?	?
Δ°_{128}, Δ°_{130}, Δ°_{136}, Δ°_{236}	42.0	?	?	?	?	?	?	?
Δ°_{88}	61.1	?	?	?	?	?	?	?
Δ°_{110}	57.8	?	?	?	?	?	?	?
Δ°_{272}, Δ°_{274}	57.5	?	?	?	?	?	?	?
Δ°_{387}	57.3	?	?	?	?	?	?	?
Δ°_{798}, Δ°_{808}, Δ°_{810}, Δ°_{812}	54.0	?	?	?	?	?	?	?
Δ°_{254}	54.7	?	?	?	?	?	?	?
Δ°_{52}	54.7	?	?	?	?	?	?	?
Δ°_{302}	54.7	?	?	?	?	?	?	?
Δ°_{786}	51.3	?	?	?	?	?	?	?
Δ°_{762}	51.3	?	?	?	?	?	?	?
Δ°_{417}	51.3	?	?	?	?	?	?	?
Δ°_{838}	51.3	?	?	?	?	?	?	?
Δ°_{782}	51.3	?	?	?	?	?	?	?
Δ°_{377}, Δ°_{499}, Δ°_{503}	48.2	?	?	?	?	?	?	?
Δ°_{1348}	48.2	?	?	?	?	?	?	?
Δ°_{882}, Δ°_{856}	48.2	?	?	?	?	?	?	?
Δ°_{1340}	45.2	?	?	?	?	?	?	?
Δ°_{1879}	45.2	?	?	?	?	?	?	?
Δ°_{1384}	42.5	?	?	?	?	?	?	?

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ_8°	76.4	23.6						
Δ_4°	99.0	1.0						
Δ_{134}°	99.8	0.2						
Δ_{128}°, Δ_{130}°, Δ_{136}°, Δ_{236}°	99.9	0.1						
Δ_{88}°	74.9	22.1	2.5	0.5	0.0	0.0		
Δ_{110}°	82.4	14.1	3.1	0.4	0.0			
Δ_{272}°, Δ_{274}°	78.1	18.0	3.4	0.5	0.0	0.0		
Δ_{387}°	73.8	21.9	3.5	0.7	0.0	0.0		
Δ_{798}°, Δ_{808}°, Δ_{810}°, Δ_{812}°	77.0	17.9	4.4	0.7	0.0	0.0		
Δ_{254}°	95.9	0.5	3.5	0.0	0.0	0.0		
Δ_{52}°	95.3	0.7	3.9	0.0	0.0	0.0		
Δ_{302}°	95.9	0.5	3.5	0.0	0.0			
Δ_{786}°	94.8	0.3	4.8	0.0	0.0	0.0		
Δ_{762}°	94.8	0.3	4.9	0.0	0.0	0.0		
Δ_{417}°	94.8	0.3	4.8	0.0	0.0	0.0	0.0	
Δ_{838}°	94.7	0.3	5.0	0.0	0.0	0.0		
Δ_{782}°	94.6	0.3	5.0	0.0	0.0	0.0		
Δ_{377}°, Δ_{499}°, Δ_{503}°	93.4	0.2	6.2	0.0	0.1	0.0		
Δ_{1348}°	93.7	0.0	6.2	0.0	0.1		0.0	
Δ_{882}°, Δ_{856}°	93.4	0.3	6.2	0.0	0.1	0.0	0.0	
Δ_{1340}°	92.3	0.0	7.6	0.0	0.1		0.0	
Δ_{1879}°	92.3	0.0	7.5	0.0	0.1		0.0	
Δ_{1384}°	90.9	0.0	8.9	0.0	0.2		0.0	

Brill-Noether numbers of $(\overline{3}, 2)_{1/6}$ in QSMs [M.B. Cvetič Donagi Ong '23]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ_8°	99.9421		0.0579					
Δ_4°	99.9952		0.0048					
Δ_{134}°	99.9952		0.0048					
Δ_{128}°, Δ_{130}°, Δ_{136}°, Δ_{236}°	99.9952		0.0048					
Δ_{88}°	96.6700	0.3361	2.9850		0.0089			
Δ_{110}°	95.6268	0.8372	3.5179	0.0050	0.0131			
Δ_{272}°, Δ_{274}°	95.5097	0.5155	3.9552	0.0016	0.0180			
Δ_{387}°	95.1923	0.4981	4.2773		0.0323			
Δ_{798}°, Δ_{808}°, Δ_{810}°, Δ_{812}°	93.8268	0.8795	5.2390	0.0029	0.0518			
Δ_{254}°	96.3942	0.0687	3.5193	0.0003	0.0175			
Δ_{52}°	96.0587	0.0171	3.9066	0.0000	0.0176			
Δ_{302}°	96.3960	0.0636	3.5222	0.0001	0.0181			
Δ_{786}°	95.0714	0.0393	4.8466	0.0002	0.0425			
Δ_{762}°	95.0167	0.0369	4.9052	0.0005	0.0407			
Δ_{417}°	95.0745	0.0433	4.8389	0.0003	0.0429		0.0001	
Δ_{838}°	94.9092	0.0215	5.0216	0.0000	0.0477			
Δ_{782}°	94.9019	0.0161	5.0359	0.0000	0.0461			
Δ_{377}°, Δ_{499}°, Δ_{503}°	93.6500	0.0347	6.2312	0.0005	0.0836			
Δ_{1348}°	93.7075	0.0112	6.1978	0.0001	0.0833		0.0001	
Δ_{882}°, Δ_{856}°	93.6546	0.0425	6.2190	0.0009	0.0825		0.0005	
Δ_{1340}°	92.2989	0.0064	7.5515	0.0001	0.1427		0.0004	
Δ_{1879}°	92.3015	0.0108	7.5447	0.0002	0.1421		0.0007	
Δ_{1384}°	90.8524	0.0031	8.9219	0.0001	0.2213		0.0012	

Line bundle cohomologies on rational circuits I



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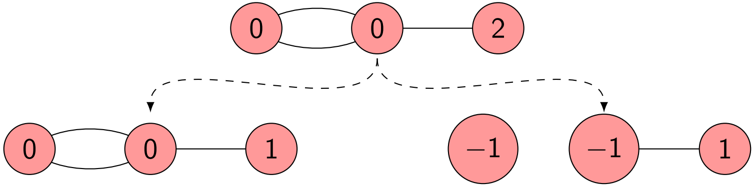

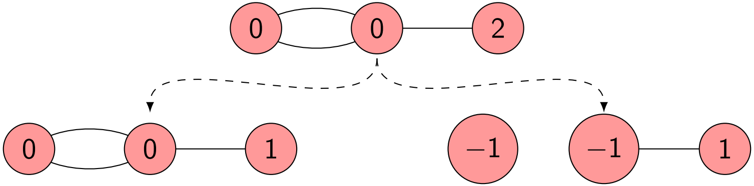

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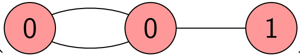

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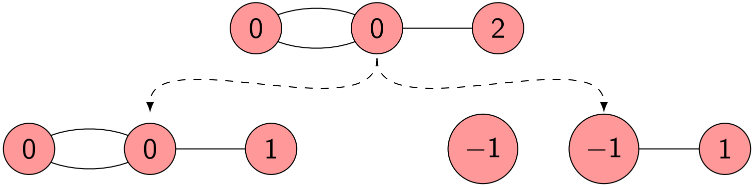

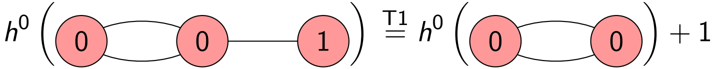

$T1$: Prune a leaf.

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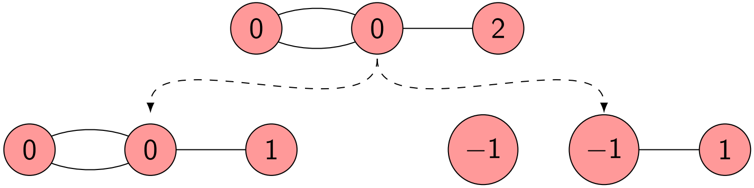
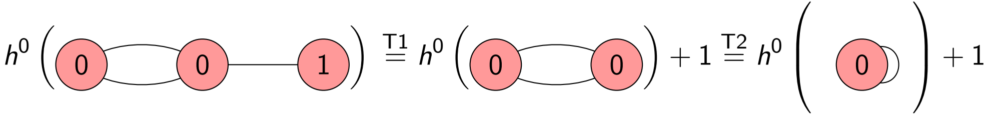
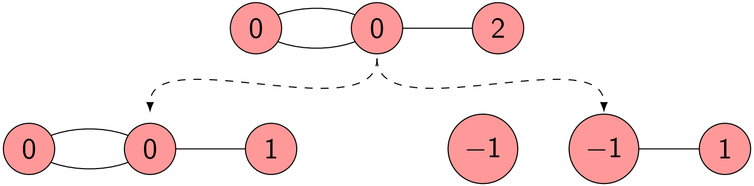
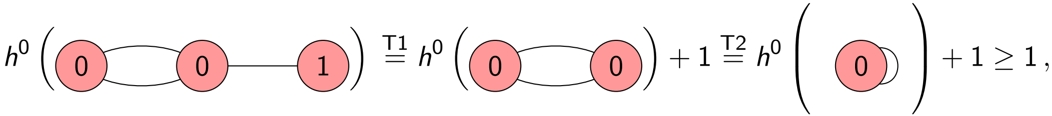




$T1$: Prune a leaf. $T2$: Remove an interior edge.





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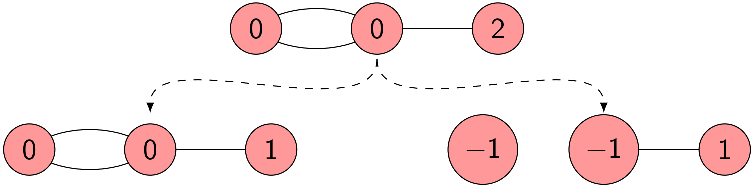

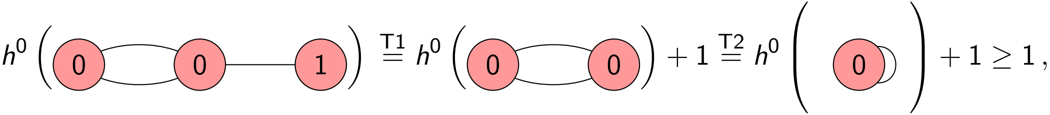

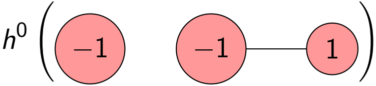

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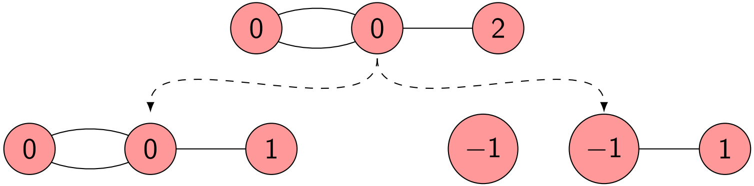

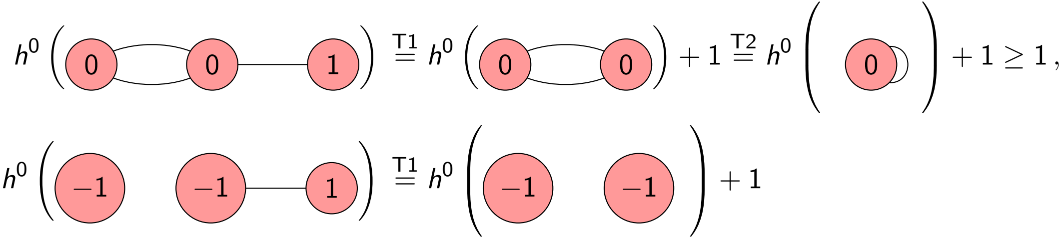

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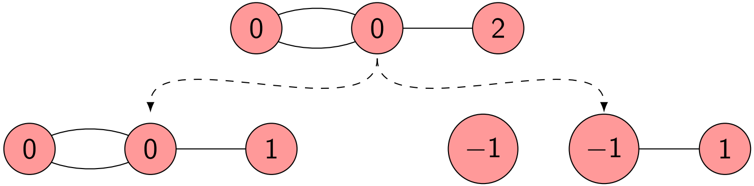

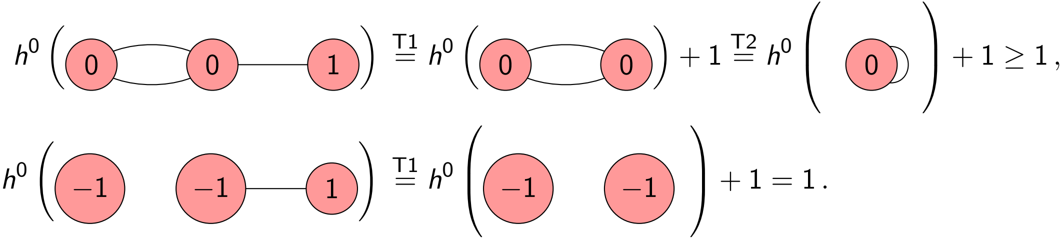

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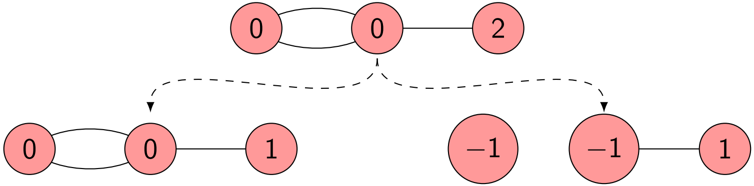

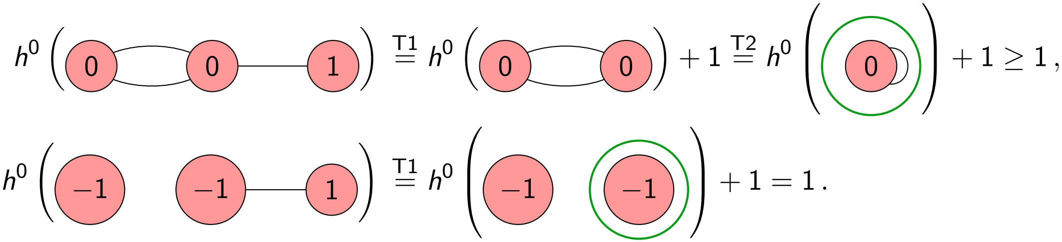

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Line bundle cohomologies on rational circuits II

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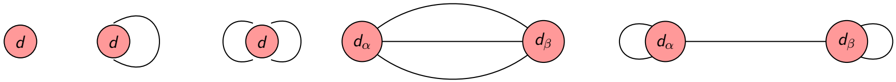

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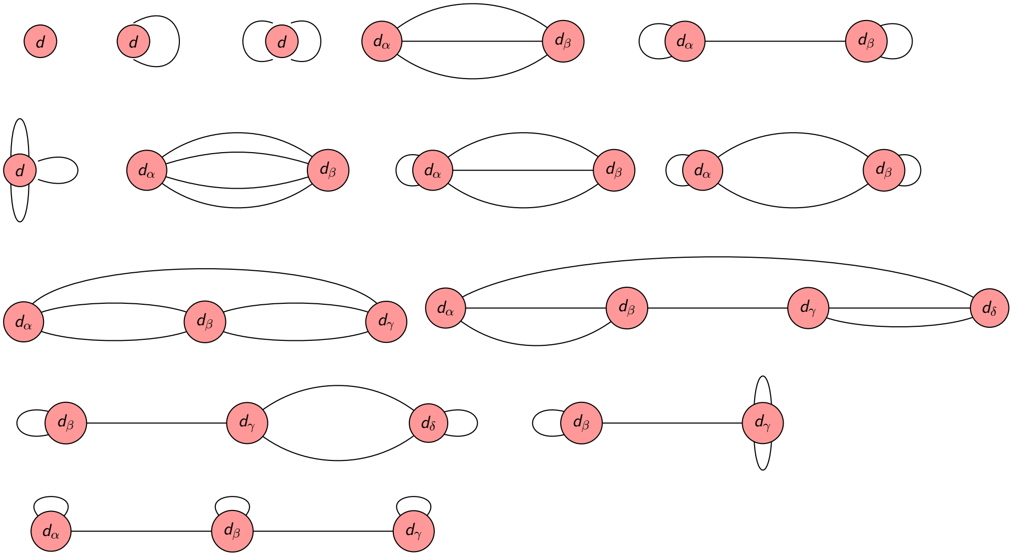

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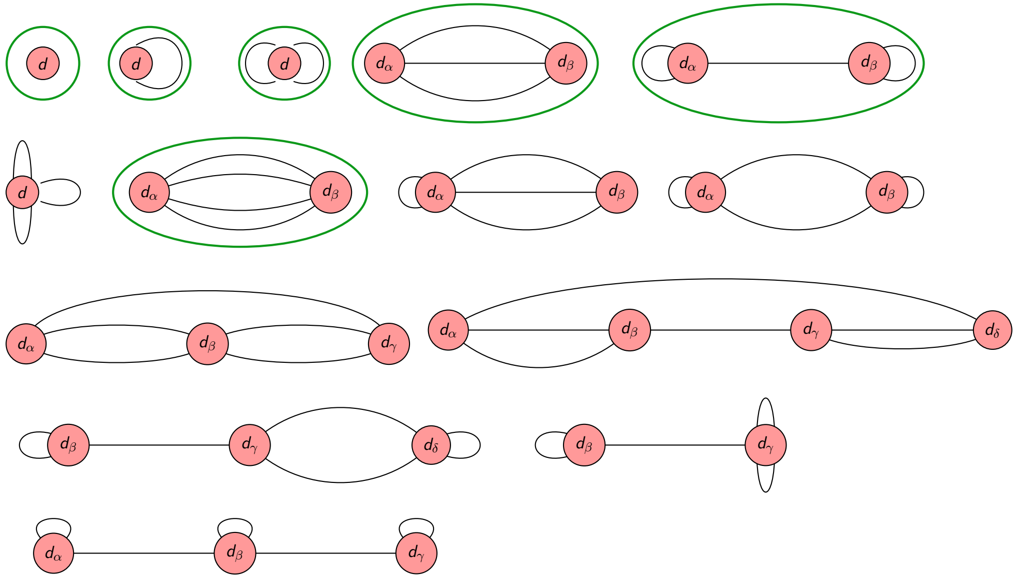

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Thank you for your attention!

Improvements – more details

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3. Yet more refinements [M.B. Cvetič Donagi Ong '23]:
   - h^0-computation on rational and elliptic **circuits**.
   - Achieved by 3-step procedure:
     1. Prune trees,
     2. Remove internal edges,
     3. Classification of terminal circuits and their line bundle cohomologies.

$\Rightarrow$ Optimal results: Refinements require geometric data that is currently not available.

(Required refined data: Descent data of line bundles, divisor on elliptic components.)