Machine Learning and Algebraic Approaches towards Complete Matter Spectra in 4d F-theory

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With M. Cvetič, R. Donagi, L. Lin, M. Liu, F. Rühle – 2007.00009

Motivation

- Classical problem of string pheno: find realization of (MS)SM in string landscape.
- In particular: need (massless) vector-like pair(s) to accommodate the Higgs.
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  - describes strongly (in g_S) coupled IIB-string theory
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$\Rightarrow$ How can we control the vector-like spectrum in F-theory?

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4. Summary and conclusion

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Chiral excess

- Fields: (co)kernel of operator (e.g. $\Delta\phi = 0$)
- Chiral excess: $\chi = \mathrm{ind}(D)$ with D a Dirac operator:

$$\ker(D) : n \times \text{chiral fields } \phi\,, \qquad \mathrm{coker}(D) : \overline{n} \times \text{anti-chiral fields } \overline{\phi}$$

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String theory (MS)SM constructions with exact **chiral** spectrum

- $E_8 \times E_8$ [Candelas Horowitz Strominger Witten '85], [Greene Kirklin Miron Ross '86], [Braun He Ovrut Pantev '05], [Bouchard Donagi '05], [Anderson Gray He Lukas '10], ...
- Type II [Berkooz Douglas Leigh '96], [Aldazabal Franco Ibanez Rabadan Uranga '00], [Ibanez Marchesano Rabadan '00], [Blumenhagen Kors Lust Ott '01], [Cvetič Shiu Uranga '01], ...
- F-theory [Krause Mayrhofer Weigand '12], [Cvetič Klevers Mayorga Oehlmann Reuter '15], [Lin Weigand '16], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], [Taylor Turner '19], [Raghuram Taylor Turner '19], ...

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Chiral vs. vector-like spectrum

- Higgs doublet ϕ_H corresponds to pair $(\phi, \overline{\phi})$:

Irrep of G_{SM}	$(n, \overline{n})$	χ	Decomposition: Leptons + Higgs
$(\mathbf{1}, \mathbf{2})_{-1/2}$	$(3, 0)$	3	$(3, 0) = (3, 0) \oplus 0 \cdot (1, 1)$
$(\mathbf{1}, \mathbf{2})_{-1/2}$	$(4, 1)$	3	$(4, 1) = (3, 0) \oplus 1 \cdot (1, 1)$

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Elliptic 4-fold Y_4, gauge group G and irreps $\mathbf{R}$ of G

- IIB: Identify profile of axio-dilaton $\tau = C_0 + e^{i\phi}$ in presence of D7-branes
- Backreaction: Treat τ as complex structure modulus of elliptic curve

$\Rightarrow$ *Singular* 4-fold $\pi \colon Y_4 \twoheadrightarrow B_3$:

  - Gauge group G: Singularities of Y_4
  - Fields in irrep $\mathbf{R}$: Localize on curves $C_{\mathbf{R}} \subseteq B_3$
  - Irrep. $\mathbf{R}$ of G: $\mathbb{P}^1$-fibration over $C_{\mathbf{R}}$ – **matter surface** $S_{\mathbf{R}}$

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- Pick flux background $G_4 \in H^{2,2}(Y_4)$

$\Rightarrow$ $\chi = \int_{S_{\mathbf{R}}} G_4$. [Donagi/Wijnholt, 09],[Braun/Collinucci/Valandro, 11], [Marsano/Schaefer-Nameki, 11], [Krause/Mayrhofer/Weigand,11,12], [Grimm/Hayashi, 11], [Cvetič/Grimm/Klevers, 12], [Braun/Grimm/Keitel, 13], [Cvetič/Grassi/Klevers/Piragua,13], [Borchmann/Mayrhofer/Palti/Weigand, 13], [Lin/Mayrhofer/Till/Weigand, 15], ...

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Vector-like spectra in F-theory

Gauge potential for field strength G_4

- $G_4 \to A_3 \in \mathrm{CH}^2(Y_4) \subseteq H^2_D(Y_4, \mathbb{Z}(2))$ [Curio/Donagi, 98], [Donagi/Wijnholt,12,13], [Anderson/Heckman/Katz, 13], [Intriligator,Jockers,Mayr,Morrison,Plesser '12]
- Consider $\mathcal{L}_{\mathbf{R}} = \pi^*(A_3 \cdot S_{\mathbf{R}}) \otimes \mathcal{O}_{C_{\mathbf{R}},\text{spin}} \in \mathrm{Pic}(C_{\mathbf{R}})$

$\Rightarrow$ $\mathcal{L}_{\mathbf{R}}$ counts vector-like spectra [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]

$$\text{chiral fields } \leftrightarrow H^0(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})\,, \qquad \text{anti-chiral fields} \leftrightarrow H^1(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})\,.$$

Typically, $h^i(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$ hard to determine:

- Non-topological, i.e. deformation $C_{\mathbf{R}} \to C'_{\mathbf{R}}$ can lead to jumps

$$h^i(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}}) = (h^0, h^1) \to h^i(C'_{\mathbf{R}}, \mathcal{L}'_{\mathbf{R}}) = (h^0 + a, h^1 + a)$$

$\Rightarrow$ Higgs pairs/exotic matter

Example: Line bundles in F-theory (MS)SM

curve	g	$\mathcal{L}$	d	BN-theory
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	10	$\mathcal{L}_{(\mathbf{3},\mathbf{2})_{1/6}}^{\otimes 36} = K_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}^{\otimes 24}$	12	h^0 h^1 ρ: 3 0 10; 4 1 6; 5 2 0
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	82	$\mathcal{L}_{(\mathbf{1},\mathbf{2})_{-1/2}}^{\otimes 36} = K_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}^{\otimes 22} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30 \cdot Y_1)$	84	h^0 h^1 ρ: 3 0 82; 4 1 78; ⋮ ⋮ ⋮; 10 7 12
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	$\ldots$			
$\vdots$	$\ddots$			

F-theory and heterotic challenges with vector-like spectra

- In heterotic compactifications [Anderson Gray Lukas Palti '10 & '11 and subsequent works]
  - X is (favourable) CICY 3-fold with known $\mathrm{Pic}(X)$
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  6. In many (MS)SM constructions: $L_{\mathbf{R}}$ is root bundle ($\sim$ generalized spin-bundle).

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F-theory and heterotic challenges with vector-like spectra

- In heterotic compactifications [Anderson Gray Lukas Palti '10 & '11 and subsequent works]
  - X is (favourable) CICY 3-fold with known $\mathrm{Pic}(X)$
  - $V \in \mathfrak{Coh}(X)$ is a pullback of vector bundle from toric ambient space
- F-theory situation qualitatively different:
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$\Rightarrow$ Focus on simpler situation first, then apply these lessons to involved scenarios.

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- Approaches:
  1. Use software to compute $h^0(\mathbf{c})$ and interpret the results with machine learning. (Surge of similar works, but mostly suited for heterotic ST [Ruehle, 17], [Klaewer/Schlechter, 18], [Larfors/Schneider, 19,20], [Brodie/Constantin/Deen/Lukas, 19])
  2. Find $h^0(\mathbf{c})$ from Koszul resolutions and interpret it with Brill-Noether theory.

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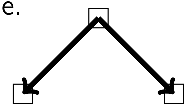
3. Interpret results with binary decision trees.

Decision trees

- Decision tree: directed, connected graph with unique root node.
- Binary tree: each node has either 0 or 2 sub-nodes.
- Nodes with no sub-nodes are 'leaves'.



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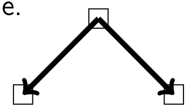

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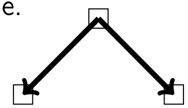

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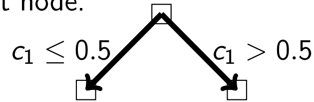

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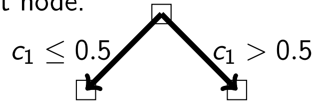

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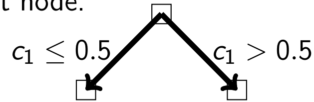

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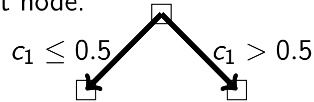

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- For training: minimize Gini impurity for given training data.

The data, features and classes

- Data:
  - Hypersurface curves $C(\mathbf{c}) = V(P(\mathbf{c}))$ in dP_3 with $1 \leq g \leq 6$.
  - Coefficients $\mathbf{c} = \{c_k\}$ with $c_k \in \{0, 1\}$.
  - For each $C(\mathbf{c})$, consider 13 line bundles $L \in \mathrm{Pic}(dP_3)$ and compute $h^0(C(\mathbf{c}), L|_{C(\mathbf{c})})$
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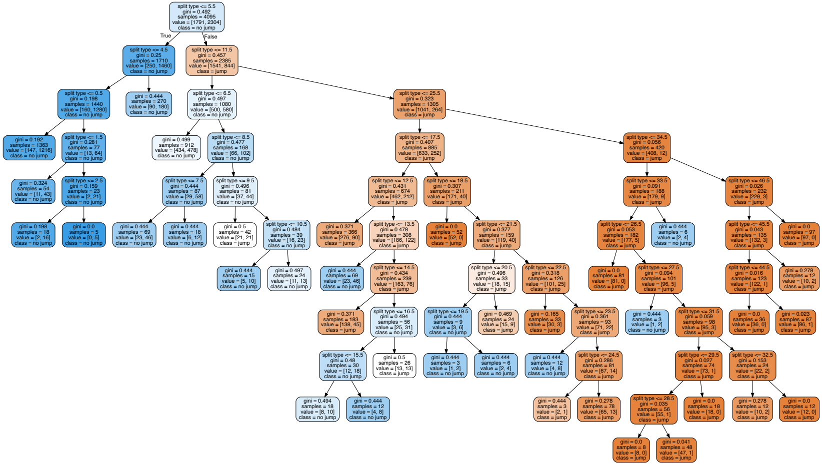
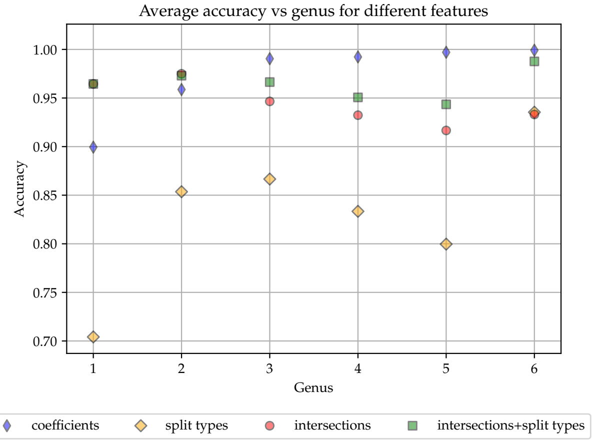
$\Rightarrow$ Train tree to make implication 'feature' $\Rightarrow$ 'class' (training-testing ratio: 90:10).

Example of tree trained on split-type ($g = 3$, $d = 3$)



Average accuracy



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- Failure of topological criteria:
  - Other sources/origins of jumps in cohomology.
  - Most likely under-represented due to bias in data set ($\leftrightarrow c_i \in \{0, 1\}$).

Application to F-theory GUT model

- Geometry of 4-fold:
  - $SU(5)$ supported on $S \cong dP_3 \subseteq B_3$ [Beasley Heckman Vafa I&II '09]
  - $U(1)$-restricted Tate model Grimm/Weigand, '10]
  - $\Rightarrow$ Explicit fourfold $Y_4 \twoheadrightarrow B_3$ with $SU(5) \times U(1)$ gauge symmetry in [M.B., '17]
- Chiral spectrum:

$$\chi(\mathbf{10}_1) = 3\,, \qquad \chi(\mathbf{5}_{-2}) = -18\,, \qquad \chi(\mathbf{5}_3) = 15\,.$$

- Focus on $C_{\mathbf{5}_3} \equiv C$:

$$g = 24\,, \qquad \deg(\mathcal{L}_{\mathbf{5}_3}) = 38\,, \qquad 44 \text{ coefficients } c_i\,.$$

- Study splittings $C \to \widetilde{C} \cup \mathbb{P}^1$ where $\mathbb{P}^1$ is one of the 6 rigid divisors in dP_3.
  - $E_{1,2}$ lead to jumps. They satisfy $L \cdot E_{1,2} < -1$.
  - Splitting off combinations of $E_{1,2}$ gives $h^0 \in \{15, 17, 18, 19, 20, 21\}$.
  - Cannot get $h^0 = 16$ in this way!

Rational from machine learning approach:

- What we did learn:
  - Oftentimes, topological criteria sufficient to engineer jumps.
  - In particular: $C \to \widetilde{C} \cup \mathbb{P}^1$ with $\deg (L|_{\mathbb{P}^1}) < -1$ likely to give jump.
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$\Rightarrow$ Answers follow from Koszul resolution, h^0-stratifications and Brill-Noether theory.

How to find $h^0\left(C(\mathbf{c}), \mathcal{L}\right) \equiv h^0(\mathbf{c})$ in theory?

1. Pullback line bundle admits Koszul resolution:

$$0 \to \mathcal{O}_{dP_3}(D_L - D_C) \xrightarrow{P(\mathbf{c})} \mathcal{O}_{dP_3}(D_L) \to \mathcal{L} \to 0$$

2. Obtain long exact sequence in sheaf cohomology:

$$\begin{aligned} 0 \longrightarrow H^0\left(D_L - D_C\right) &\longrightarrow H^0\left(D_L\right) \longrightarrow H^0\left(\mathcal{L}\right) \\ \hookrightarrow H^1\left(D_L - D_C\right) &\longrightarrow H^1\left(D_L\right) \longrightarrow H^1\left(\mathcal{L}\right) \\ \hookrightarrow H^2\left(D_L - D_C\right) &\longrightarrow H^2\left(D_L\right) \longrightarrow 0 \longrightarrow 0 \end{aligned}$$

3. Sometimes: $0 \to H^0(\mathcal{L}) \to H^1(D_L - D_C) \xrightarrow{M_\varphi(\mathbf{c})} H^1(D_L) \to H^1(\mathcal{L}) \to 0$
4. By exactness: $h^0(\mathcal{L}) = \ker(M_\varphi(\mathbf{c}))$

$\Rightarrow$ Study $\ker\left(M_\varphi(\mathbf{c})\right)$ as function of complex structure $\mathbf{c}$

Example: $g = 3$, $\chi = 1$ $(d = 3)$

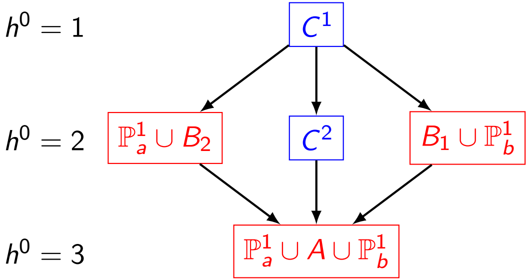
- $C(\mathbf{c}) = V(P(\mathbf{c}))$ and $P(\mathbf{c}) = c_1 x_1^3 x_2^3 x_3^2 x_4 + \cdots + c_{12} x_3^2 x_4 x_5^3 x_6^3$
- For $D_L = H + 2E_1 - 2E_2 - E_3$ find

$$0 \to H^0(\mathcal{L}) \to \mathbb{C}^3 \xrightarrow{M_\varphi(\mathbf{c})} \mathbb{C}^2 \to H^1(\mathcal{L}) \to 0\,, \qquad M_\varphi = \begin{pmatrix} c_3 & c_2 & c_1 \\ 0 & c_{12} & c_{11} \end{pmatrix}$$

- $h^0(\mathcal{L}) = 3 - \mathrm{rk}(M_\varphi(\mathbf{c}))$ & stratification of curve geometries:

$\mathrm{rk}(M_\varphi)$	explicit condition	curve splitting
2	$(c_3 c_{11}, c_3 c_{12}, c_2 c_{11} - c_1 c_{12}) \neq \mathbf{0}$	C^1
1	$c_3 = 0,\ c_2 c_{11} - c_1 c_{12} = 0$	C^2
1	$c_1 = c_2 = c_3 = 0$	$B_2 \cup \mathbb{P}^1_b$
1	$c_{11} = c_{12} = 0$	$\mathbb{P}^1_a \cup B_1$
0	$c_1 = c_2 = c_3 = c_{11} = c_{12} = 0$	$\mathbb{P}^1_a \cup A \cup \mathbb{P}^1_b$

Stratification diagram



Types of jumps

- Brill-Noether theory: C^2 smooth, irreducible but line bundle divisor **special**
- Curve splittings: Factoring off $\mathbb{P}^1_a$, $\mathbb{P}^1_b$ leads to jump

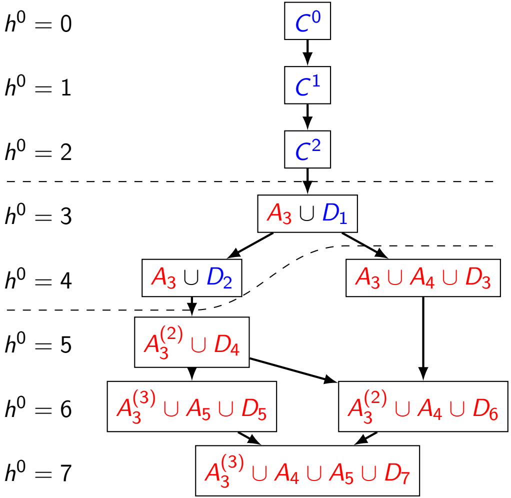
Example 2: $g = 5$, $\chi = 0$ $(d = 4)$

- $P(\mathbf{c}) = c_1 x_1^3 x_2^4 x_3^2 x_4^2 + \cdots + c_{16} x_3^3 x_4 x_5^4 x_6^3$
- $D_L = H + E_1 - 4E_2 + E_3$
- Koszul resolution gives

$$h^0(\mathcal{L}) = 7 - \mathrm{rk}(M_\varphi(\mathbf{c}))$$

$$M_\varphi = \begin{pmatrix} c_{15} & c_{11} & c_7 & 0 & 0 & 0 & 0 \\ 0 & c_{10} & c_6 & c_3 & c_{11} & c_7 & 0 \\ c_{12} & c_6 & c_3 & 0 & c_7 & 0 & 0 \\ 0 & c_5 & c_2 & 0 & c_6 & c_3 & c_7 \\ c_8 & c_2 & 0 & 0 & c_3 & 0 & 0 \\ 0 & c_{14} & c_{11} & c_7 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & c_2 & 0 & c_3 \end{pmatrix}$$

$\Rightarrow$ Study $\mathrm{rk}(M_\varphi(\mathbf{c}))$ as function of $\mathbf{c}$



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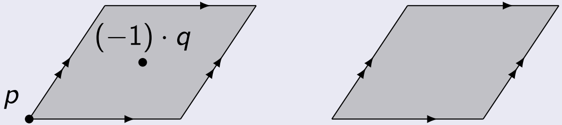

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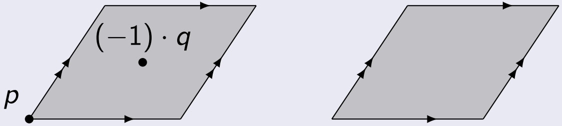

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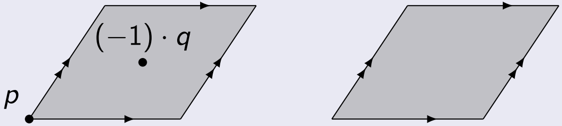

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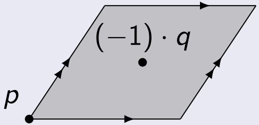

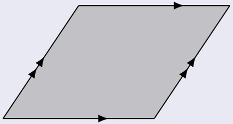

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General picture

- Abel-Jacobi map gives $\varphi_d \colon \mathrm{Div}_d(C) \rightarrow \mathrm{Jac}(C) \cong \mathbb{C}^g/\Lambda$
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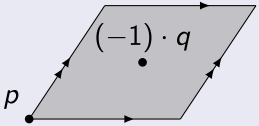

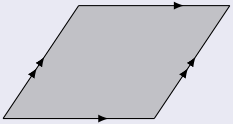

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h^0	h^1	ρ
0	0	1
1	1	0
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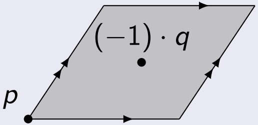

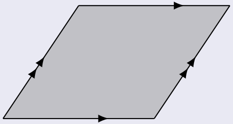

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$\Rightarrow$ Upper bound for h^0 on generic curves [Watari, 16]

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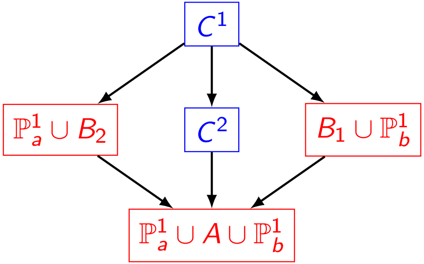
Gluing *local* sections

$h^0 = 1$

$h^0 = 2$

$h^0 = 3$



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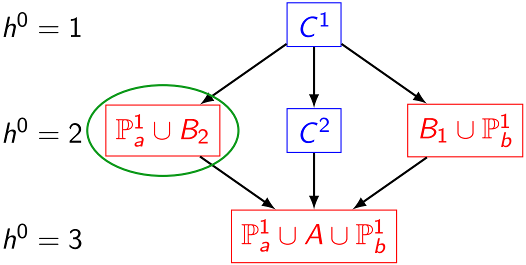

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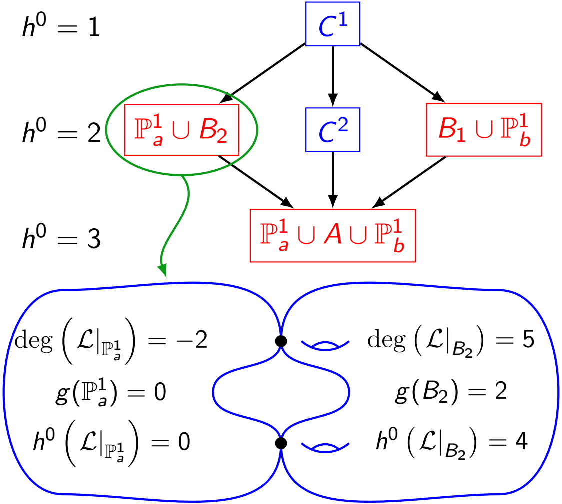
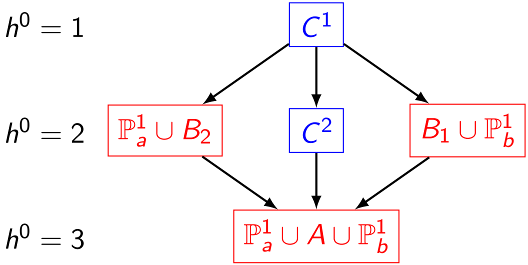


Gluing *local* sections II



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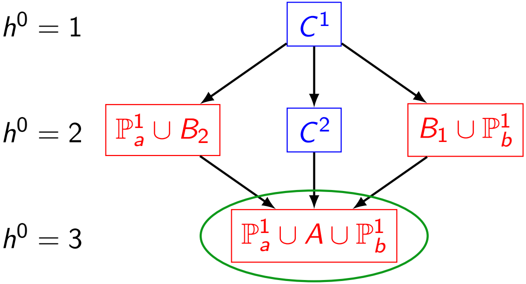

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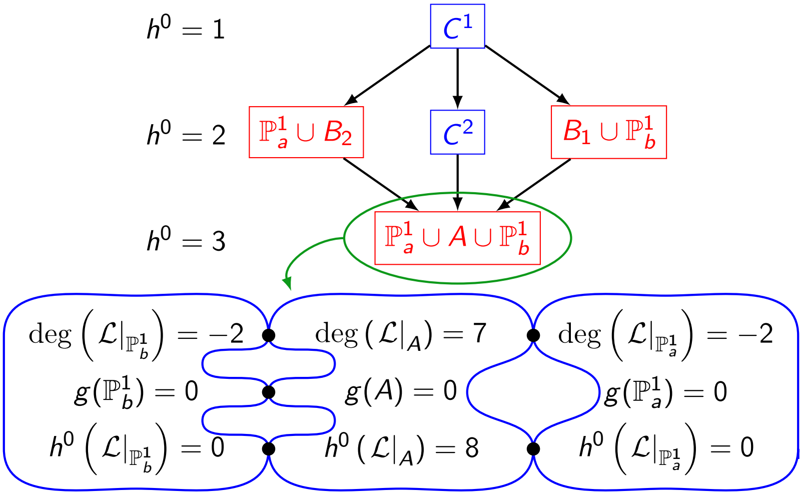

Quality assessment of counting procedure

- Quick: Uses only topological data (genus, chiral index)
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- Lead-offs:
  1. Sufficient criteria for jumps
  2. Algorithmic h^0-spectrum estimate

Sufficient criteria for jumps

Let S be a smooth surface, $L \in \mathrm{Pic}(S)$ a line bundle, and $|C|$ a linear system of curves on S with smooth general member C. Consider a special member $C_1 \cup C_2$ s.t. C_1, C_2 meet transversely in $C_1 \cdot C_2 > 0$ distinct points.

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$$h^0\left(C_1 \cup C_2, L|_{C_1 \cup C_2}\right) \geq N_1 + N_2 - C_1 \cdot C_2 \,.$$

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$$h^0\left(C_1 \cup C_2, L|_{C_1 \cup C_2}\right) - h^0\left(C, L|_C\right) \geq g_1 - 1 - \deg\left(L|_{C_1}\right) \,.$$

Algorithmic estimate for h^0-spectrum

C

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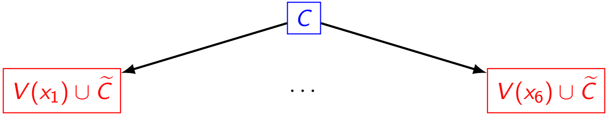

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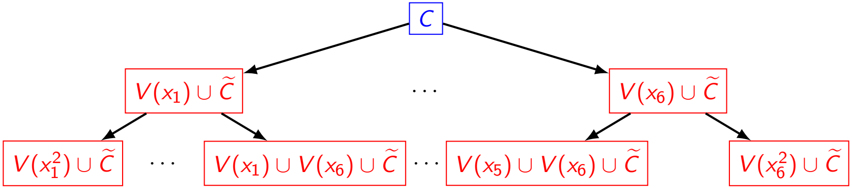

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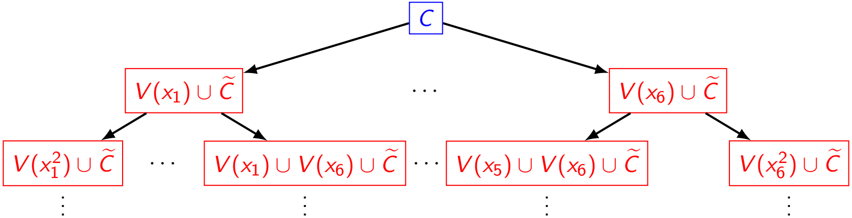

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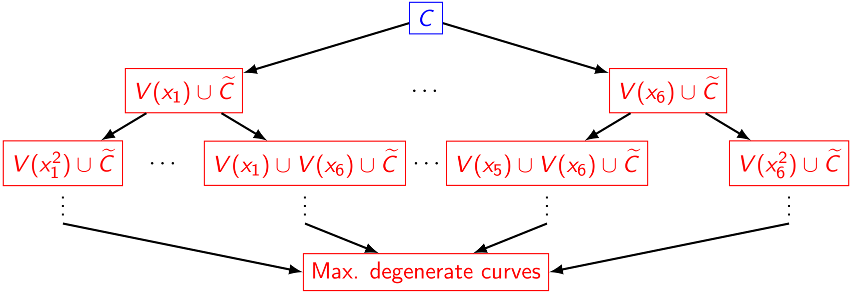

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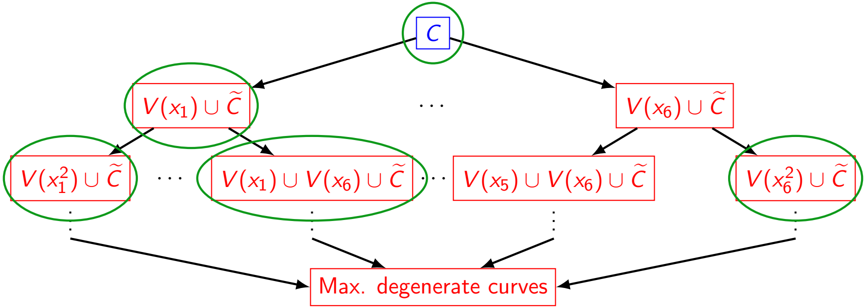

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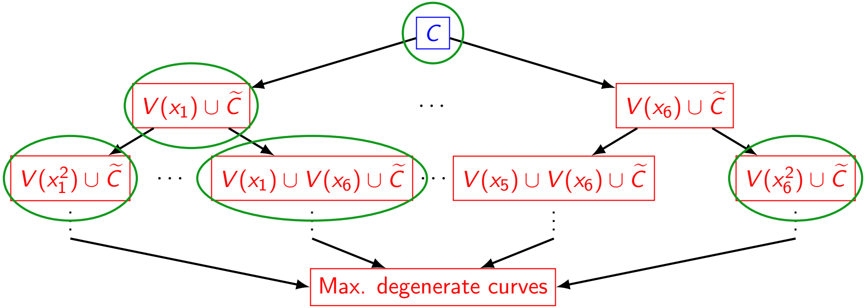

https://github.com/homalg-project/ToricVarieties_project

- Estimate h^0-spectrum from lower bounds at **subset of nodes**.
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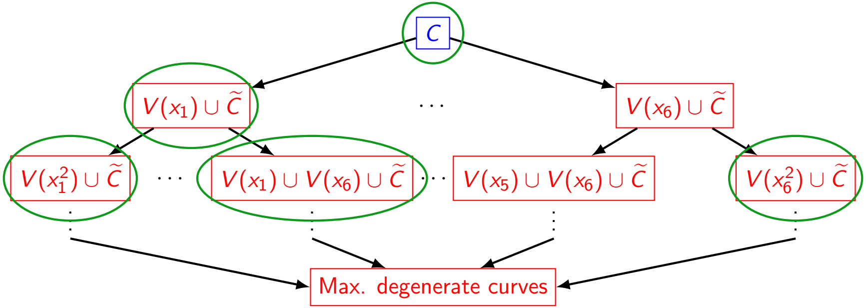

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- Caveat: Check that $\widetilde{C}$ is irreducible.

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- Take away message:
  Recipe for additional vector-like pair: Factor $C \to \widetilde{C} \cup \mathbb{P}^1$ with $\deg(L|_{\mathbb{P}^1}) < -1$.

Outlook: Back to F-theory (MS)SM constructions

curve	g	$\mathcal{L}$	d	BN-theory		
				h^0	h^1	ρ
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	10	$\mathcal{L}^{\otimes 36}_{(\mathbf{3},\mathbf{2})_{1/6}} = K^{\otimes 24}_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}$	12	3	0	10
				4	1	6
				5	2	0
				h^0	h^1	ρ
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	82	$\mathcal{L}^{\otimes 36}_{(\mathbf{1},\mathbf{2})_{-1/2}} = K^{\otimes 22}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30 \cdot Y_1)$	84	3	0	82
				4	1	78
				$\vdots$	$\vdots$	$\vdots$
				10	7	12
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	$\dots$					
$\vdots$	$\ddots$					

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- Origin of root bundles:
  - $G_4 \in H^{2,2}_{\mathbb{Q}}(Y_4)$: Associated 'gauge field' $A_{\mathbb{Q}} \in \mathrm{CH}^2_{\mathbb{Q}}(Y_4)$.

  $\Rightarrow$ $A_{\mathbb{Q}}$ does not uniquely fix vector-like spectrum.

  $\Rightarrow$ Wilson line(s) in intermediate Jacobian of Y_4 as additional datum?

Broader outlook

- Current technical extensions for (MS)SM model building:
  - non-pullback/root bundles
  - stratification for several curves in one global F-theory model
- Conceptual:
  - Vector-like spectra for pseudo-real representations
  - Non-vertical G_4 (flux moduli dependence!)
  - (Geometric) symmetries protecting vector-like pairs
- Further applications:
  - (S)CFTs
  - swampland program

Thank you for your attention!