# *Monoidal structures in Freyd categories*

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GAP Singular Meeting

# Overview

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**Presentation based on work with *Sebastian Posur***

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## Outline

1. Introduction to
    - Freyd categories
    - Monoidal structures
2. Derivation: From promonoidal to monoidal structures

# Motivation

### Why are Freyd categories useful?

- Unified framework for f.p. modules, f.p. graded modules and f.p. functors
- Iterated Freyd categories yield approach to free Abelian categories
- Completely constructive – see `CAP`-package *Freyd categories*
- Application: Computer models for coherent (toric) sheaves

# Freyd categories in a nutshell

Any additive category $\mathbf{A}$ admits a Freyd category $\mathcal{A}(\mathbf{A})$ with

- $\mathbf{A} \subseteq \mathcal{A}(\mathbf{A})$,
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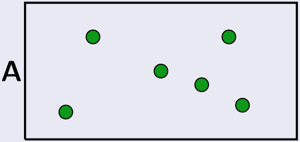

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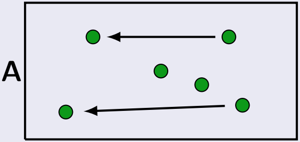

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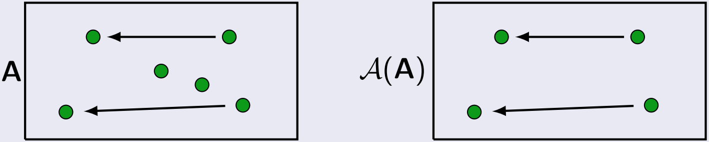

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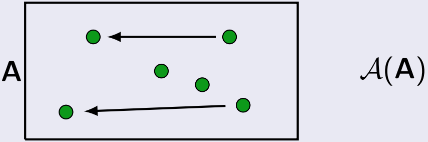

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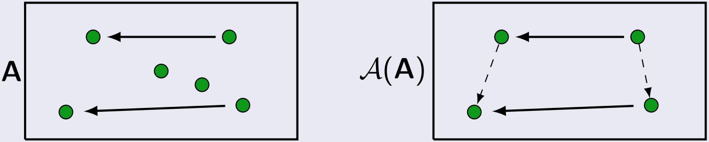

### Summary

- Be $a \xleftarrow{\rho_a} r_a \in \mathrm{Mor}(\mathbf{A})$, then $A \equiv (a \xleftarrow{\rho_a} r_a) \in \mathrm{Obj}(\mathcal{A}(\mathbf{A}))$.
- (Equivalence classes of) commutative diagrams in $\mathbf{A}$ form the morphisms in $\mathcal{A}(\mathbf{A})$.

# Motivation

### Why are monoidal structures (on Freyd categories) interesting?

- Simple approach to Day convolution in the f.p. context
- Allows to study monoidal structures on free Abelian categories (c.f. *Purity, Spectra and Localisation* by M. Prest)
- Completely constructive – implementation available in `CAP`-package *Freyd categories*
- Applications to coherent (toric) sheaves (via $\underline{\mathrm{Hom}}$): Indices, Hodge diamonds, intersection numbers, . . .

# (Pro)monoidal structures in a nutshell

### Monoidal structure

A monoidal structure on $\mathcal{A}(\mathbf{A})$ consists of

- a functor $\hat{T} \colon \mathcal{A}(\mathbf{A}) \times \mathcal{A}(\mathbf{A}) \to \mathcal{A}(\mathbf{A})$ (*tensor product*),
- an object $1 \in \mathcal{A}(\mathbf{A})$ (*tensor unit*),
- $\ldots$

subject to pentagonal identity, hexagonal identities, $\ldots$

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## Promonoidal structure

A promonoidal structure on **A** consists of

- a functor $T \colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A})$ (*protensor product*),
- an object $1 \in \mathcal{A}(\mathbf{A})$ (*protensor unit*),
- ...

subject to **restricted** pentagonal identity, hexagonal identities, ...

# From (Pro)monoidal to monoidal structures

**Task**

Extend promonoidal tensor product $T \colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A})$ to $\mathcal{A}(\mathbf{A})$.

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**Notation**

- For $a_1, a_2 \in \mathrm{Obj}(\mathbf{A})$, denote $T(a_1, a_2) \in \mathrm{Obj}(\mathcal{A}(\mathbf{A}))$ by

$$\left( g_T(a_1, a_2) \xleftarrow{\rho_T(a_1,a_2)} r_T(a_1, a_2) \right) .$$

- For $a_1 \xleftarrow{\alpha_1} b_1$, $a_2 \xleftarrow{\alpha_2} b_2$, denote $T(\alpha, \beta) \in \mathrm{Mor}(\mathcal{A}(\mathbf{A}))$ by

$$\begin{tikzcd}
g_T(b_1, b_2) \arrow[d, "\delta_T(\alpha_1, \alpha_2)"] & & r_T(b_1, b_2) \arrow[ll, "\rho_T(b_1, b_2)"'] \arrow[d, dashed, "\omega_T(\alpha_1, \alpha_2)"'] \\
g_T(a_1, a_2) & \circlearrowleft & r_T(a_1, a_2) \arrow[ll, "\rho_T(a_1, a_2)"]
\end{tikzcd}$$

# From (Pro)monoidal to monoidal structures II

**Task**

Extend $T \colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A})$ to $\mathcal{A}(\mathbf{A})$ by use of the notation above.

# From (Pro)monoidal to monoidal structures II

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Extend $T \colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A})$ to $\mathcal{A}(\mathbf{A})$ by use of the notation above.

**Lemma (proof in *"tensor products of finitely presented functors"*)**

Given a bilinear functor $T : \mathbf{A} \times \mathbf{A} \longrightarrow \mathcal{A}(\mathbf{A})$, there exists a multilinear functor $\hat{T} : \mathcal{A}(\mathbf{A}) \times \mathcal{A}(\mathbf{A}) \longrightarrow \mathcal{A}(\mathbf{A})$ and for objects $A_1 \equiv (a_1 \xleftarrow{\rho_1} r_1)$, $A_2 \equiv (a_2 \xleftarrow{\rho_2} r_2)$ it holds

$$\hat{T}\left(A_1, A_2\right) := \operatorname{cok}\left( T\left(a_1, a_2\right) \xleftarrow{\begin{pmatrix} F(\mathrm{id}_{a_1}, \rho_2) \\ T(\rho_1, \mathrm{id}_{a_2}) \end{pmatrix}} \begin{matrix} T\left(a_1, r_2\right) \\ \oplus T\left(r_1, a_2\right) \end{matrix} \right)$$

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## Lemma (proof in *"tensor products of finitely presented functors"*)

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## New task

Evaluate this expression for promonoidal tensor product $T$.

# From (Pro)monoidal to monoidal structures III

$$\hat{T}(A_1, A_2) = \mathrm{cok} \left[ \begin{tikzcd}
\begin{array}{l} g_T(a_1, r_2) \\ \oplus g_T(r_1, a_2) \end{array} \arrow[d, "\begin{pmatrix} \delta_T(\mathrm{id}_{a_1}, \rho_2) \\ \delta_T(\rho_1, \mathrm{id}_{a_2}) \end{pmatrix}"] & & \begin{array}{l} r_T(a_1, r_2) \\ \oplus r_T(r_1, a_2) \end{array} \arrow[ll, swap, "\begin{pmatrix} \rho_T(a_1, \rho_2) & \\ & \rho_T(\rho_1, a_2) \end{pmatrix}"] \arrow[d, dashed, swap, "\begin{pmatrix} \omega_T(\mathrm{id}_{a_1}, \rho_2) \\ \omega_T(\rho_1, \mathrm{id}_{a_2}) \end{pmatrix}"] \arrow[dll, phantom, "\circlearrowleft"] \\
g_T(a_1, a_2) & & r_T(a_1, a_2) \arrow[ll, swap, "\rho_T(a_1, a_2)"]
\end{tikzcd} \right]$$

## Consequence

$$\hat{T}(A_1, A_2) = \left( g_T(a_1, a_2) \xleftarrow{\begin{pmatrix} \rho_T(a_1, a_2) \\ \delta_T(\mathrm{id}_{a_1}, \rho_2) \\ \delta_T(\rho_1, \mathrm{id}_{a_2}) \end{pmatrix}} \begin{array}{c} r_T(a_1, a_2) \\ \oplus g_T(a_1, r_2) \\ \oplus g_T(r_1, a_2) \end{array} \right)$$

# Summary and outlook

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- Any additive category $\mathbf{A}$ admits a Freyd category $\mathcal{A}(\mathbf{A})$.
- Promonidal structures on $\mathbf{A}$ $\Rightarrow$ monoidal structures on $\mathcal{A}(\mathbf{A})$.
- Example: $T \colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A}) \Rightarrow \hat{T} \colon \mathcal{A}(\mathbf{A}) \times \mathcal{A}(\mathbf{A}) \to \mathcal{A}(\mathbf{A})$
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## Outlook

- Tensor products on free Abelian categories

# Thank you for your attention!