Statistics of limit root bundles and towards F-Theory MSSMs

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Work with M. Cvetič, R. Donagi, M. Liu, M. Ong – 2102.10115, 2104.08297

What is this talk is about?

My goal: Answer one of the biggests open questions in theoretical physics

Find minimally supersymmetric standard model (MSSMs) solutions to string theory.
$\rightarrow$ Study high-dimensional geometries subject to physically inspired conditions.

Approaches and (technical) challenges

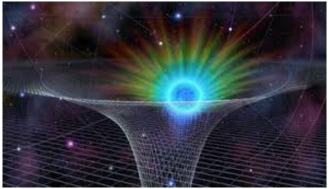
- Toric geometry: Model **many** geometries and investigate with **computers**
- Tools clustered over many different systems: `Calabi-Yau tools`, `cohomCalg`, homalg project, `Macaulay2`, `Sage`, `Singular`, `Topcom`, `Palp`, `Polymake`, ...

$\Rightarrow$ My vision: `OSCAR` has ALL functionality (for F-theory studies).

Content of todays talk

- Explain state-of-the-art of my big goal.
- Explain the "ALL" in my vision for toric and algebraic geometry in `OSCAR`.

String theory = General relativity + Standard Model?



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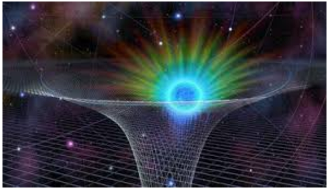
String theory = General relativity + Standard Model?



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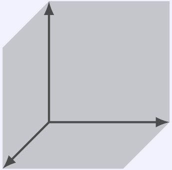

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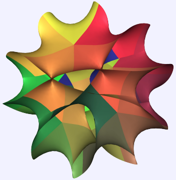
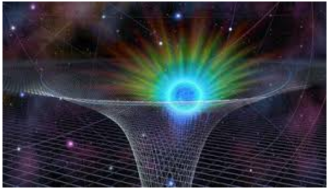


our 4-dim. world $\mathcal{W}$

'small' 3 complex-dim. manifold $\mathcal{B}_3$

Challenge: Find $\mathcal{B}_3$ s.t. ST reproduces Standard model

String theory = General relativity + Standard Model?



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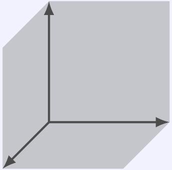

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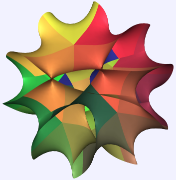
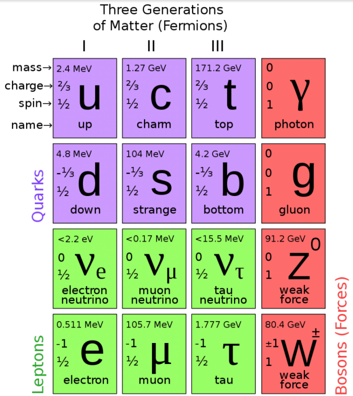


our 4-dim. world $\mathcal{W}$ ‘small’ 3 complex-dim. manifold $\mathcal{B}_3$

Challenge: Find $\mathcal{B}_3$ s.t. ST reproduces

Particles in the standard model



- Each particle given by representation of group

$$G_{\text{SM}} = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \,.$$

- **Leptons** e, μ, τ: $(\mathbf{1}, \mathbf{2})_{-1/2}$
- Charge conjugate leptons e_C, μ_C, τ_C: $(\overline{\mathbf{1}}, \overline{\mathbf{2}})_{+1/2}$
- (e, e_C) is leptonic vector-like pair – not observed!
- Famous vector-like pair: Higgs field.

Necessary criterion for String theory MSSM

Number of vector-like pairs must match the experimental findings.

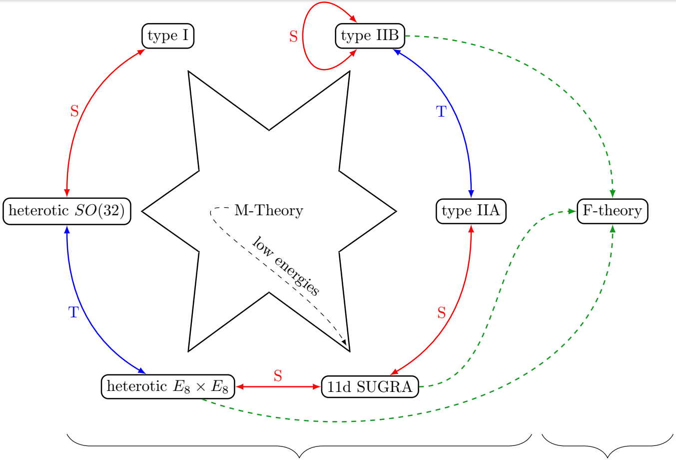
Experimentally observed vector-like spectra (LH/RH is for left/right handed)

Particle name	Rep. R	# Fields n_R	Conjugate rep. $\overline{R}$	# Fields $n_{\overline{R}}$
LH quarks	$(3,2)_{1/6}$	3	$(\overline{3},\overline{2})_{-1/6}$	0
RH up-quarks	$(3,1)_{2/3}$	3	$(\overline{3},\overline{1})_{-2/3}$	0
RH down-quarks	$(3,1)_{-1/3}$	3	$(\overline{3},\overline{1})_{1/3}$	0
LH leptons	$(1,2)_{-1/2}$	3	$(\overline{1},\overline{2})_{1/2}$	0
neutrinos	$(1,1)_{1}$	3	$(\overline{1},\overline{1})_{-1}$	0
Higgs	$(1,2)_{-1/2}$	1	$(\overline{1},\overline{2})_{1/2}$	1

Note

- Chiral index $n_R - n_{\overline{R}}$ is topological invariant – used as simple criterion.
- The "magic number" 3 follows from experiments. (→ Explain with string theory?)
- Crucial: We need one vector-like pair to accommodate the Higgs field.

Different formulations of string theory – the M-theory star



$g_s, \alpha' \ll 1$ ($\alpha' = \frac{l_s^2}{4\pi^2}$)

$\alpha' \ll 1$ but
g_s strongly coupled

Overview of SM constructions in string theory

Obtain (MS)SM from String theory construction ...

- $E_8 \times E_8$ [Candelas Horowitz Strominger Witten '85], [Greene Kirklin Miron Ross '86], [Braun He Ovrut Pantev '05], [Bouchard Donagi '05], [Anderson Gray He Lukas '10], ...
- Type II [Berkooz Douglas Leigh '96], [Aldazabal Franco Ibanez Rabadan Uranga '00], [Ibanez Marchesano Rabadan '00], [Blumenhagen Kors Lust Ott '01], [Cvetič Shiu Uranga '01],
- F-theory [Krause Mayrhofer Weigand '12], [Cvetič Klevers Pena Oehlmann Reuter '15], [Lin Weigand '16], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], ...

... including vector-like spectra

- $E_8 \times E_8$: [Bouchard Donagi '05], [Bouchard Cvetič Donagi '06], [Anderson Gray Lukas Palti '10 & '11], ...
- F-theory: [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18], [M.B. Cvetič Donagi Lin Liu Ruehle '20], [M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21]

Why F-theory

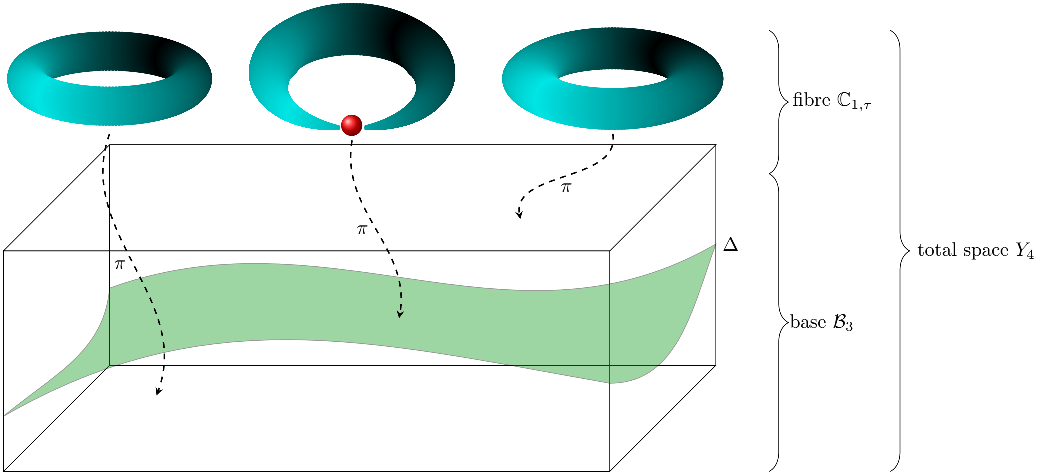
1. Conceptional reasons:

   [Vafa '96], [Morrison Vafa '96], [Beasley Heckman Vafa '08], [Apruzzi Heckman Morrison Tizzano '18], ...

   - More general physics setups (e.g. more general gauge groups).
   - Access to strongly-coupled regime of string theory.

2. Model building:
   - Consistency conditions: Geometry of **elliptic fibration** [Vafa '96], [Morrison Vafa '96], ...
   - Yukawa couplings: **Matter curve** intersection [Cecotti Cheng Heckman Vafa '10], [Donagi, Wijnholt '12], [Cvetic Lin Liu Zhang Zoccarato '19], ...
   - **Largest** (currently-known) family of string theory standard models from F-theory:
     - *A Quadrillion (10^{15}) F-theory Standard Models* [Cvetič Halverson Lin Liu Tian '19]
     - Involve toric 3-folds from triangulations of reflexive polytopes [Kreuzer Skarke '98].

F-theory from singuar fibration $Y_4 \twoheadrightarrow B_3$



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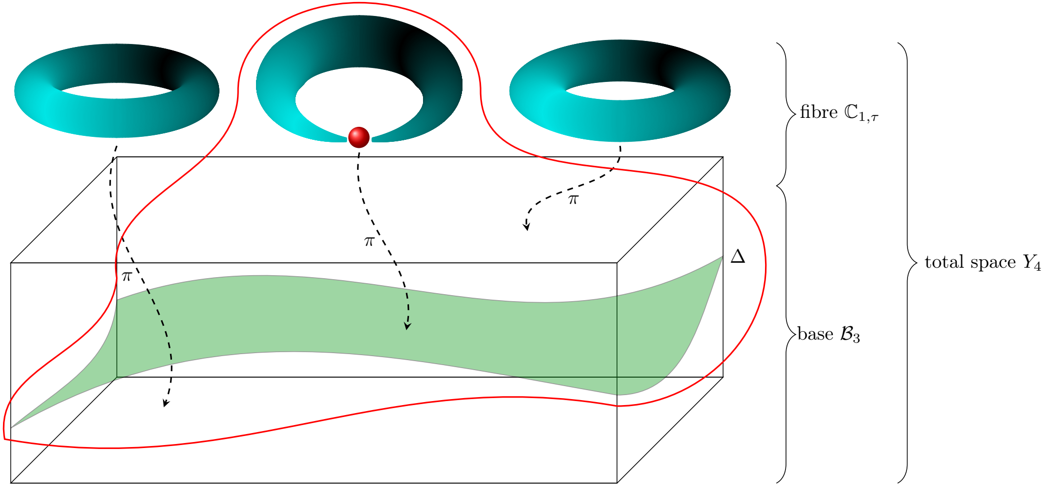
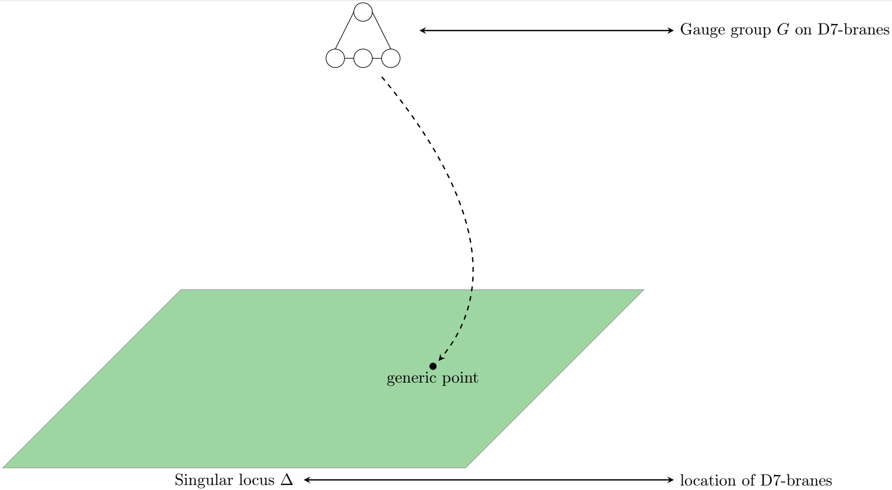


F-theory from resolved smooth fibration $\widehat{Y}_4 \twoheadrightarrow B_3$



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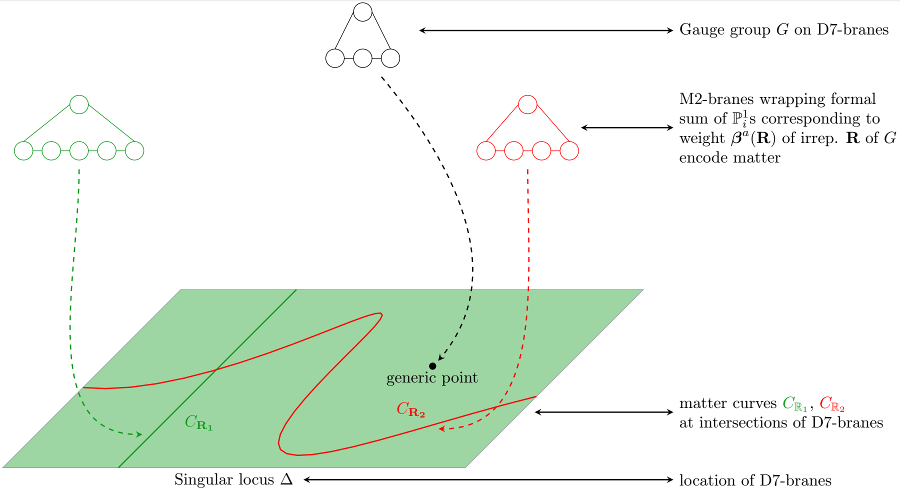

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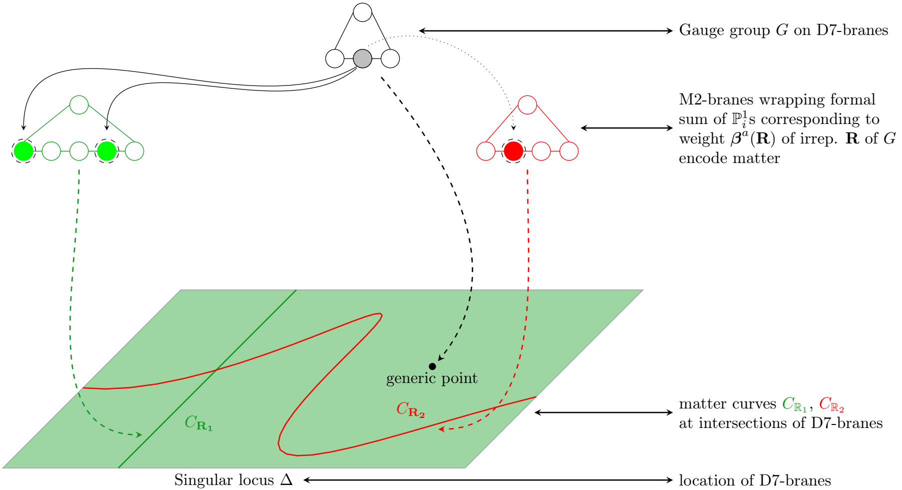


Vector-like spectra in 4d $\mathcal{N} = 1$ F-theory vacua

[M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]

- Defining data: Elliptic 4-fold $\widehat{Y}_4 \twoheadrightarrow B_3$ and flux $G_4 \in H^{(2,2)}_{\mathbb{Z}}(\widehat{Y}_4)$.

$\Rightarrow$ Can "read-off" the physics [Weigand '17]

  - Particles localize on **matter curves** $C_R \subset B_3$.
  - Representation encoded by **matter surface** S_R (a $\mathbb{P}^1$-fibration over C_R).
  - G_4 and S_R specify line bundle L_R on C_R (details on next slide):

$$\text{massless chiral supermultiplets in rep. R} \leftrightarrow n_R = h^0(C_R, L_R),$$

$$\text{massless chiral supermultiplets in rep. } \overline{R} \leftrightarrow n_{\overline{R}} = h^1(C_R, L_R).$$

- Challenges:
  - n_R, $n_{\overline{R}}$ **strongly** depend on the complex structure of $\widehat{Y}_4$.
  - Deformation $\widehat{Y}_4 \to \widehat{Y}_4'$ can lead to **jumps** [M.B. Cvetič Donagi Lin Liu Ruehle '20]

$$h^i(C_R, L_R) = (h^0, h^1) \to h^i(C_R', L_R') = (h^0 + a, h^1 + a).$$

  - Higgs field: Need **non-generic solution** $h^i(C_R, L_R) = (1, 1)$.

How to compute L_R from G_4, S_R? [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]

- Lift $G_4 \in H^{(2,2)}_{\mathbb{Z}}(\widehat{Y}_4)$ to a "gauge field" $A \in H^4_D(\widehat{Y}_4, \mathbb{Z}(2))$ or $\mathcal{A} \in \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}^2_{\mathrm{hom}}(\widehat{Y}_4, \mathbb{Z}) & \longrightarrow & \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z}) & \xrightarrow{\gamma} & H^{(2,2)}_{\mathrm{alg}}(\widehat{Y}_4) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \widehat{\gamma} & & \downarrow & & \\ 0 & \longrightarrow & J^2(\widehat{Y}_4) & \longrightarrow & H^4_D(\widehat{Y}_4, \mathbb{Z}(2)) & \xrightarrow{\widehat{c}} & H^{(2,2)}_{\mathbb{Z}}(\widehat{Y}_4) & \longrightarrow & 0 \end{array}$$

  Always exists, but is in general **not unique** since $J^2(\widehat{Y}_4) \neq 0$.

- For matter surface $S_R \in \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$ define $\iota_{S_R} \colon S_R \hookrightarrow \widehat{Y}_4$, $\pi_{S_R} \colon S_R \twoheadrightarrow C_R$. Then

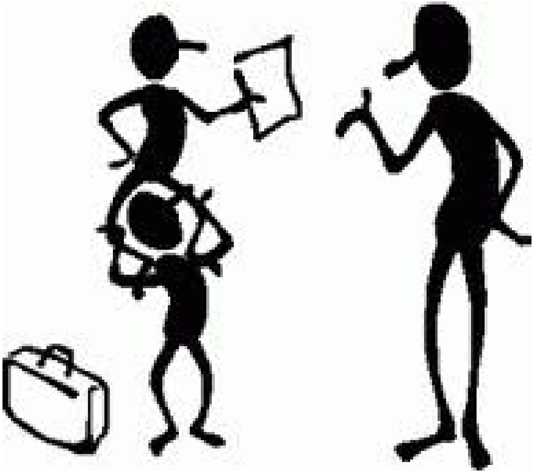
$$L_R(\mathcal{A}) = \mathcal{O}_{C_R}\left[\pi_{S_R *}\left(\iota^*_{S_R}(\mathcal{A})\right) + D_{\mathrm{spin},R}\right] \in \mathrm{Pic}(C_R) \, .$$

Summary: Computing vector-like spectra in F-theory

1. Defining data: Elliptic 4-fold $\widehat{Y}_4 \twoheadrightarrow B_3$ and flux $G_4 \in H^{(2,2)}_{\mathbb{Z}}(\widehat{Y}_4)$.
2. Read-off matter curves $C_{\mathbf{R}} \subset B_3$.
3. Compute line bundles $L_{\mathbf{R}} \in \mathrm{Pic}(C_{\mathbf{R}})$ induced from G_4.
4. Compute cohomologies to determine vector-like spectra:

Particle	R	$n_{\mathbf{R}}$	desired $(n_{\mathbf{R}}, n_{\overline{\mathbf{R}}})$
LH quarks	$(3,2)_{1/6}$	$h^0(C_{(3,2)_{1/6}}, L_{(3,2)_{1/6}})$	(3,0)
RH up-quarks	$(3,1)_{2/3}$	$h^0(C_{(3,1)_{2/3}}, L_{(3,1)_{2/3}})$	(3,0)
RH down-quarks	$(3,1)_{-1/3}$	$h^0(C_{(3,1)_{-1/3}}, L_{(3,1)_{-1/3}})$	(3,0)
LH leptons	$(1,2)_{-1/2}$	$h^0(C_{(1,2)_{-1/2}}, L_{(1,2)_{-1/2}})$	(3,0)
neutrinos	$(1,1)_1$	$h^0(C_{(1,1)_1}, L_{(1,1)_1})$	(3,0)
Higgs	$(1,2)_{-1/2}$	$h^0(C_{(1,2)_{-1/2}}, L_{(1,2)_{-1/2}})$	(1,1)

Questions?



Generalities of the QSMs

- QSMs: $\mathcal{O}(10^{15})$ elliptic 4-folds $\widehat{Y}_4$ with choice of G_4 [Cvetič Halverson Lin Liu Tian '19]
- Elliptic 4-folds $\widehat{Y}_4 \twoheadrightarrow B_3$:
  - Obtained from toric geometry.
  - Constraints: no chiral exotics, massless $U(1)$-gauge boson, cancel D_3-tadpole.
  - $\Rightarrow$ B_3 from triangulations of 708 3-dim reflexive polytopes [Kreuzer Skarke '98]

$$\overline{K}_{B_3} \cdot \overline{K}_{B_3} \cdot \overline{K}_{B_3} \in \{\cancel{2}, 6, 10, 18, 30, \cancel{90}\} \, .$$

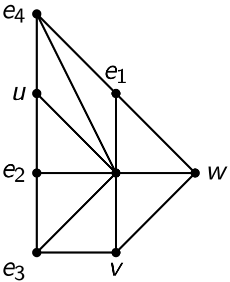
($\overline{K}_{B_3}^3 = 2, 90$ not realized by toric 3-folds.)

  - Largest polytope Δ_8°: 39 lattice points and $\sim 10^{15}$ triangulations [Halverson Tian '17].
- G_4-flux candidate: ($\leftrightarrow$ satisfies necessary conditions to be integral)

$$G_4 = \frac{-3}{\overline{K}_{B_3}^3} \cdot (5[e_1] \wedge [e_4] + \dots) \in H^{(2,2)}_{\text{alg}}(\widehat{Y}_4) \, .$$

Toric 5-fold ambient space for QSMs

- $\widehat{Y}_4$ is hypersurface in toric 5-fold space $X_5 = B_3 \times \mathbb{P}_{F_{11}}$.
- B_3 from triangulations of 708 3-dim reflexive polytopes [Kreuzer Skarke '98]
- $\mathbb{P}_{F_{11}}$ a particular toric surface [Klevers Pena Oehlmann Piragua Reuter '14]:



	u	v	w	e_1	e_2	e_3	e_4
H	1	1	1				
E_1	-1		-1	1			
E_2	-1	-1			1		
E_3		-1			-1	1	
E_4	-1			-1			1

$$I_{\text{SR}}\left(\mathbb{P}_{F_{11}}\right) = \langle e_4 w, e_4 e_2, e_4 e_3, e_4 v, e_1 u, e_1 e_2, e_1 e_3, e_1 v, wu, we_2, we_3, ve_2, uv, e_3 u \rangle \,.$$

$\widehat{Y}_4$ as hypersurface in X_5

- Consider sections $s_1, s_2, s_3, s_5, s_6, s_9 \in H^0(B_3, \overline{K}_{B_3})$.
- Define $\widehat{Y}_4 = V(p_{F_{11}}) \in X_5 = B_3 \times \mathbb{P}_{F_{11}}$ with

$$p_{F_{11}} = s_1 e_1^2 e_2^2 e_3 e_4^4 u^3 + s_2 e_1 e_2^2 e_3^2 e_4^2 u^2 v + s_3 e_2^2 e_3^3 u v^2 + s_5 e_1^2 e_2 e_4^3 u^2 w + s_6 e_1 e_2 e_3 e_4 u v w + s_9 e_1 v w^2 \,.$$

- $H^0(\mathbb{P}_{F_{11}}, \overline{K}_{\mathbb{P}_{F_{11}}})$ has basis

$$\left\{ e_1^2 e_2^2 e_3 e_4^4 u^3, e_1 e_2^2 e_3^2 e_4^2 u^2 v, e_2^2 e_3^3 u v^2, e_1^2 e_2 e_4^3 u^2 w, e_1 e_2 e_3 e_4 u v w, e_1 v w^2 \right\} \,.$$

$\Rightarrow$ $\widehat{Y}_4$ is Calabi-Yau.

Matter curves

- $\widehat{Y}_4$ is resolution of a singular elliptic fibration Y_4.

$\Rightarrow$ Can read-off the physics:

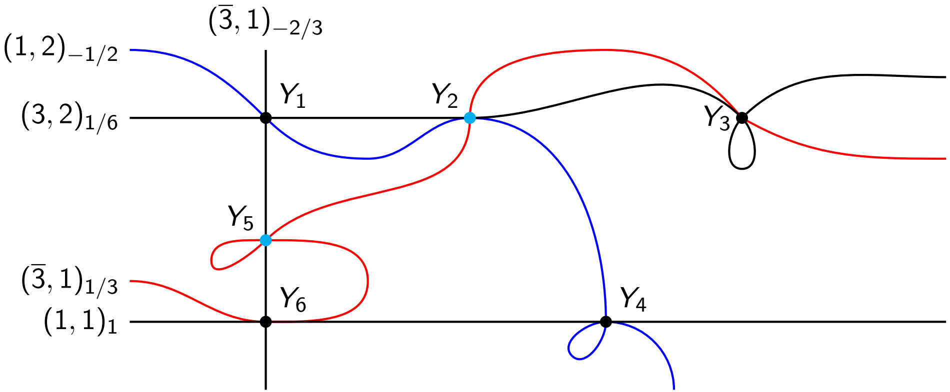
  - Codimension-1 loci over which the fiber of Y_4 is singular are *gauge surfaces*.
  - Codimension-2 loci over which the fiber of Y_4 is more singular than over the gauge surfaces are *matter curves*.

  $\leftrightarrow$ depends on choice of s_i.

- For generic s_i have five matter curves: [Klevers Pena Oehlmann Piragua Reuter '14]

$$C_{(3,2)_{1/6}} = V(s_3, s_9)\,, \qquad C_{(1,2)_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)\,,$$
$$C_{(\overline{3},1)_{-2/3}} = V(s_5, s_9)\,, \qquad C_{(\overline{3},1)_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)\,,$$
$$C_{(1,1)_1} = V(s_1, s_5)\,.$$

A cartoon of the matter curves in the QSMs



The topological intersection number is $\overline{K}^3_{B_3}$ at Y_1, Y_3, Y_4, Y_6 and $2 \cdot \overline{K}^3_{B_3}$ at Y_2, Y_5.

Can OSCAR help?

My hope/vision: Automatically perform fully fledged F-theory construction

- OSCAR has ALL ingredients to perform a fully-fledged F-theory compactification.
- As a starting point, focus on QSMs [Klevers Pena Oehlmann Piragua Reuter '14].

Basic ingredients

- Toric varieties from triangulation of polytopes (cf. PR-848):
  - `Calabi-Yau tools`, `Polymake` [Jordan, Joswig, Kastner '18] and `Topcom`
  - Kreuzer-Skarke lists [Kreuzer Skarke '98], [Kreuzer Skarke '00], [Altman Gray He Jejjala Nelson '14], ...
  - QSM list (currently in `GAP-4`-package QSMExplorer)
- Complete intersection subvarieties:
  - Cohomologies: `cohomCalg`, `Calabi-Yau tools`, `ToricVarieties_project`, ...
  - Smoothness.

Can OSCAR help? II

Chiral and vector-like spectra

- **Topological intersection** numbers of algebraic cycles.

$\Rightarrow$ Construct (vertical) G_4-fluxes and compute chiral spectra.

- For $\mathcal{A}, S_R \in \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$ compute intersection in **Chow ring**.

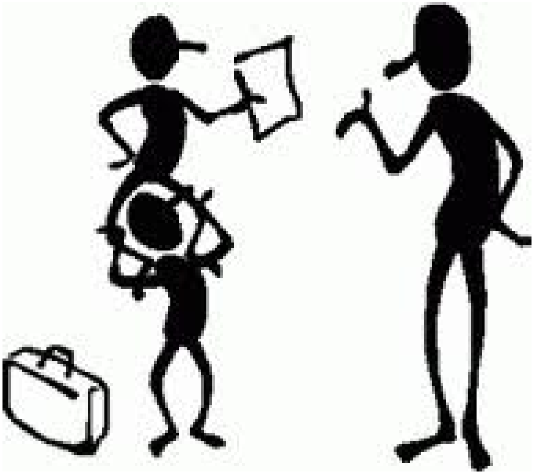
$\Rightarrow$ Automatically computate line bundle L_R.

Example for intersection computation in Chow ring (cf. PR-520)

In $\mathbb{P}^2_{x,y,z}$ compute (a) self-intersection locus of algebraic cycles $C_1 = V(x)$, $C_2 = V(x)$:

1. Move in general position: $C_2 = V(x) \sim C_2' = V(y)$ (since $x - y \in I_{\text{Linear Relations}}$).
2. Hence $C_1 \cdot C_2 \sim V(x) \cdot V(y) = V(x, y)$.

Questions?



Vector-like spectra in the QSMs: Lifting G_4 by easier multiple

- For the QSMs, the G_4-flux is

$$G_4 = \frac{-3}{\overline{K}_{B_3}^3} \cdot \left(5[e_1] \wedge [e_4] + \ldots \right) \, .$$

  - Naive lift $\mathcal{A} = \frac{-3}{\overline{K}_{B_3}^3} \cdot (5V(e_1, e_4) + \ldots) \notin \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$ since $\frac{-3 \cdot 5}{\overline{K}_{B_3}^3} \notin \mathbb{Z}$.

$\leftrightarrow$ Lack of computational control over $J^2(\widehat{Y}_4)$.

$\rightarrow$ Can (currently) not directly write-down lift of G_4 to $\mathcal{A} \in \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$.

- Circumvent ignorance:
  1. Consider $G_4' = \overline{K}_{B_3}^3 \cdot G_4$ instead.
  2. Lift G_4' to $\mathcal{A}' = -3 \cdot (5V(e_1, e_4) + \ldots) \in \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$ and find

$$D_{\mathrm{R}}\left(\mathcal{A}'\right) = \pi_{S_{\mathrm{R}}*}\left(\iota_{S_{\mathrm{R}}}^*\left(\mathcal{A}'\right)\right) \in \mathrm{Pic}\left(C_{\mathrm{R}}\right) \, .$$

$\Rightarrow$ Root bundle constraint in $\mathrm{Pic}\left(C_{\mathrm{R}}\right)$: $\overline{K}_{B_3}^3 \cdot D_{\mathrm{R}}\left(\mathcal{A}\right) \sim D_{\mathrm{R}}\left(\mathcal{A}'\right)$.

Summary of **necessary** root bundle constraints in the QSMs

curve	constraint
$C_{(3,2)_{1/6}} = V(s_3, s_9)$	$P_{(3,2)_{1/6}}^{\otimes 2\overline{K}_{B_3}^3} = K_{(3,2)_{1/6}}^{\otimes\left(6+\overline{K}_{B_3}^3\right)}$
$C_{(1,2)_{-1/2}} = V(s_3, P_H)$	$P_{(1,2)_{-1/2}}^{\otimes 2\overline{K}_{B_3}^3} = K_{(1,2)_{-1/2}}^{\otimes\left(4+\overline{K}_{B_3}^3\right)} \otimes \mathcal{O}_{C_{(1,2)_{-1/2}}}(-30 \cdot Y_1)$
$C_{(\overline{3},1)_{-2/3}} = V(s_5, s_9)$	$P_{(\overline{3},1)_{-2/3}}^{\otimes 2\overline{K}_{B_3}^3} = K_{(\overline{3},1)_{-2/3}}^{\otimes\left(6+\overline{K}_{B_3}^3\right)}$
$C_{(\overline{3},1)_{1/3}} = V(s_9, P_R)$	$P_{(\overline{3},1)_{1/3}}^{\otimes 2\overline{K}_{B_3}^3} = K_{(\overline{3},1)_{1/3}}^{\otimes\left(4+\overline{K}_{B_3}^3\right)} \otimes \mathcal{O}_{C_{(\overline{3},1)_{1/3}}}(-30 \cdot Y_3)$
$C_{(1,1)_1} = V(s_1, s_5)$	$P_{(1,1)_1}^{\otimes 2\overline{K}_{B_3}^3} = K_{(1,1)_1}^{\otimes\left(6+\overline{K}_{B_3}^3\right)}$

(P_H, P_R are complicated polynomials, Y_1, Y_3 are Yukawa points.)

Example: Absence of exotic vector-like quark-doublets

- Study QSM geometry defined by 3-fold B_3 with $\overline{K}_{B_3}^3 \in \{6, 10, 18, 30\}$.
- Pick (generic) sections $s_3, s_9 \in H^0(B_3, \overline{K}_{B_3})$.
- $C_{(3,2)_{1/6}} = V(s_3, s_9)$ and $g = \frac{\overline{K}_{B_3}^3 + 2}{2}$ $((4, 6, 10, 16)$ for $\overline{K}_{B_3}^3 \in \{6, 10, 18, 30\})$
- Number of fields: $h^0(C_{(3,2)_{1/6}}, P_{(3,2)_{1/6}})$, $h^1(C_{(3,2)_{1/6}}, P_{(3,2)_{1/6}})$ where

$$P_{(3,2)_{1/6}}^{\otimes 2\overline{K}_{B_3}{}^3} = K_{(3,2)_{1/6}}^{\otimes(6+\overline{K}_{B_3}{}^3)} .$$

$\Rightarrow$ Necessary condition: Prove existence of $P_{(3,2)_{1/6}}$ with

$$h^0(C_{(3,2)_{1/6}}, P_{(3,2)_{1/6}}) = 3\,, \qquad h^1(C_{(3,2)_{1/6}}, P_{(3,2)_{1/6}}) = 0\,. \tag{1}$$

Price of ignorance

- Which root bundles are physical, i.e. induced from $A \in H^4_D(\widehat{Y}_4, \mathbb{Z}(2))$?

  (If $g > h^{21}(\widehat{Y}_4)$, then not all are physical.)

→ Interesting, but also very challenging question for future work.

- For the time being: Ignore this issue.

Local bottom-up analysis

1. For B_3 from triangulation of Δ°_{40}, I will prove existence of solution to

$$P_{(3,2)_{1/6}}^{\otimes 2\overline{K}_{B_3}{}^3} = K_{(3,2)_{1/6}}^{\otimes\left(6+\overline{K}_{B_3}{}^3\right)} \qquad \text{and} \qquad h^0(C_{\text{R}}, P_{\text{R}}) = 3\,.$$

   Since $\overline{K}_{B_3}{}^3 = 18$, it is sufficient to solve

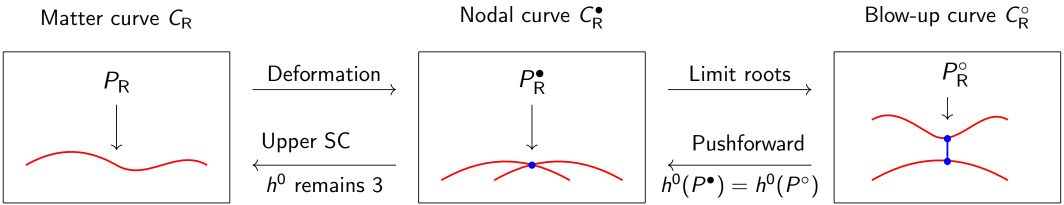
$$P_{(3,2)_{1/6}}^{\otimes 3} = K_{(3,2)_{1/6}}^{\otimes 2} \qquad \text{and} \qquad h^0(C_{\text{R}}, P_{\text{R}}) = 3\,.$$

2. Extend to statistical study across $\sim 10^{15}$ QSM bases.

Existence of roots from deformation theory and limit root bundles

- Smooth, irreducible C_{R} with $g > 1$: Very hard to explicitly construct root bundles.
- Nodal curves $C_{\mathrm{R}}^{\bullet}$: well understood. [Jarvis '98], [Jarvis '99], [Caporaso Casagrande Cornalba '07]

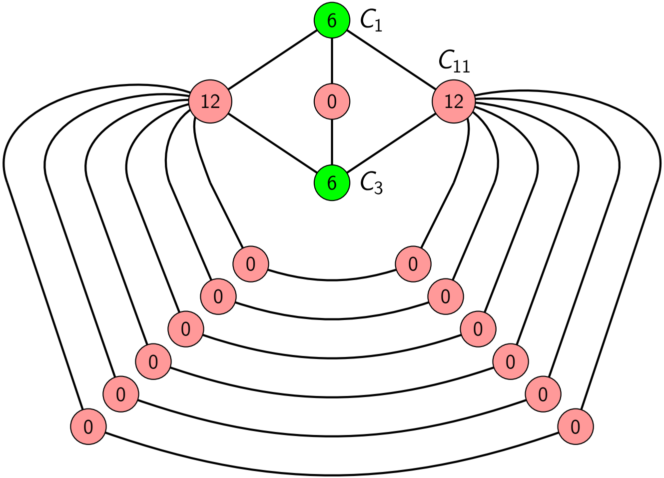
$\Rightarrow$ Our approach is summarized as follows:



(To model *all* roots, must also consider partial blow-ups. This makes the section counting hard. Hence, we currently ignore this.)

- Deformation: $C_{(3,2)_{1/6}} = V(s_3, s_9) \to C^{\bullet}_{(3,2)_{1/6}} = V(s_3, \prod_i x_i)$ ($\leftrightarrow$ Picard lattice of K3-surface)

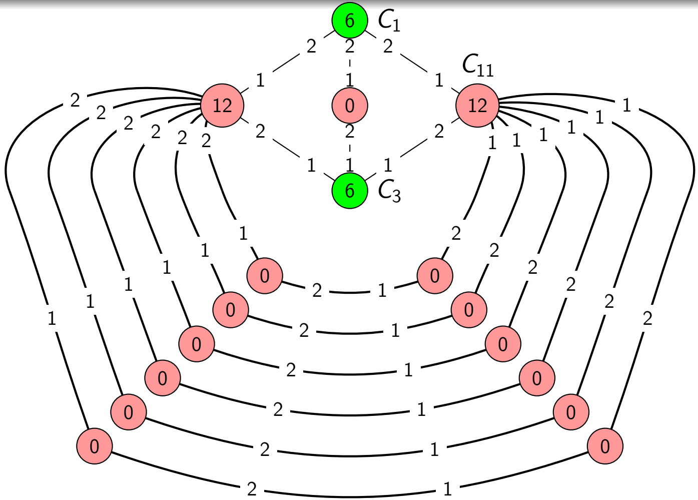
Limit root bundle construction: Step 1 – dual graph of $C^{\bullet}_{(3,2)_{1/6}}$.



- Red bullet: $g = 0$ cpnt.
- Green bullet: $g = 1$ cpnt.
- Line: node
- Numbers: $2 \cdot \deg(K_{C^{\bullet}_{(3,2)_{1/6}}})$

$\Rightarrow$ Find 3rd roots with $h^0 = 3$!
(Here fortunate case, as we can divide the local degrees by 3. This is not always true for QSM setups.)

Limit root bundle construction: Step 2 – shift degrees to blow-ups $E_j \cong \mathbb{P}^1$.

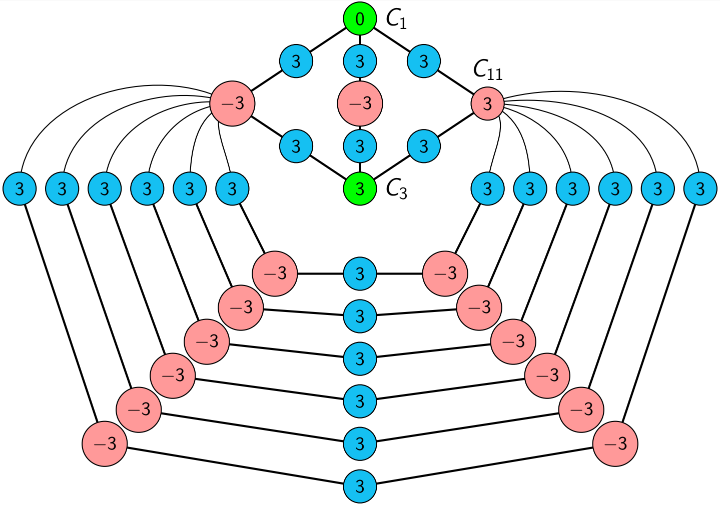


Rules for k-th roots:
(here $k = 3$):

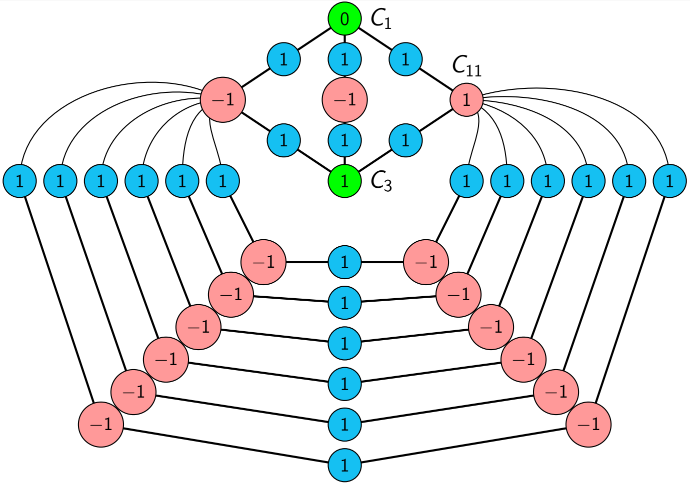
- $w_i \in \{1, \ldots, k-1\}$,
- $w_1 + w_2 = k$,
- On each component, the resulting degree is divisible by k.

$\Rightarrow$ Many possibilities!

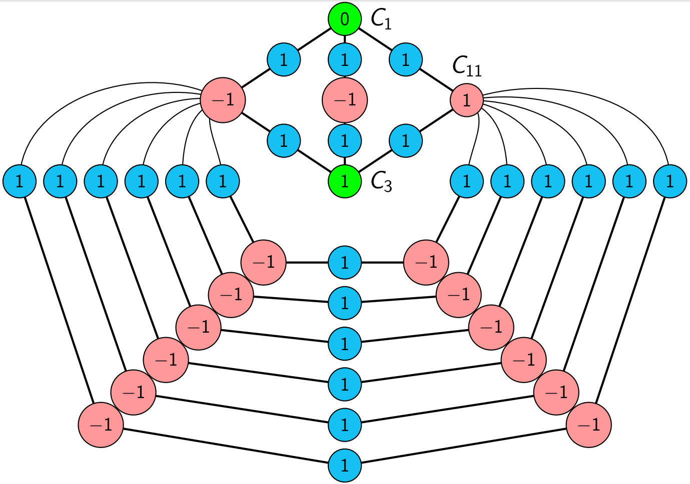
Limit root bundle construction: Step 3 – divide by $k = 3$.



Limit root bundle construction: $P^\bullet_{(3,2)_{1/6}}$ with $\left(P^\bullet_{(3,2)_{1/6}}\right)^{\otimes 3} = \left(K^\bullet_{(3,2)_{1/6}}\right)^{\otimes 2}$.



Counting h^0 of limit root bundle.



Observation:

- $h^0(E_j \cong \mathbb{P}^1, \mathcal{O}_{E_j}) = 2$

$\Rightarrow$ Uniquely fixed by boundary conditions.

$\Rightarrow$ $h^0(P_{\mathrm{R}}^\bullet) = \sum_{C_i \neq E_j} h^0\left(C_i, P_{\mathrm{R}}^\bullet|_{C_i}\right)$

- $h^0(C_3) = 1$, $h^0(C_{11}) = 2$
- $h^0(C_1) = 0$ for at least 8 of 9 local roots.

$\Rightarrow$ $\exists P_{\mathrm{R}}^\bullet$ s.t. $h^0(C_{\mathrm{R}}^\bullet, P_{\mathrm{R}}^\bullet) = 3$.

Towards promising F-theory base spaces

- Assume that $C_{\mathrm{R}}^{\bullet}$ consists only of $g = 0$ and $g = 1$ components. Then automate:
  - Given the dual graph of $C_{\mathrm{R}}^{\bullet}$, find all allowed weight assignments.
  - For the encoded limit root line bundles compute $h^0(C_{\mathrm{R}}^{\bullet}, P_{\mathrm{R}}^{\bullet})$.

$\rightarrow$ Implemented in `GAP-4` package *QSMExplorer*

https://github.com/homalg-project/ToricVarieties_project/tree/master/QSMExplorer

- Scan over selected QSM geometries:
  1. $h^{(2,1)}(\widehat{Y}_4) \geq g(C_{(3,2)_{1/6}})$.

     Necessary condition for many roots on $C_{(\mathbf{3},\mathbf{2})_{\mathbf{1}/\mathbf{6}}}$ to be physical.
  2. Components of $C_{(3,2)_{1/6}}$ must have at most $g = 1$.

$\Rightarrow$ Base 3-folds B_3 obtained from triangulations of 33 3-dim. reflexive polytopes:

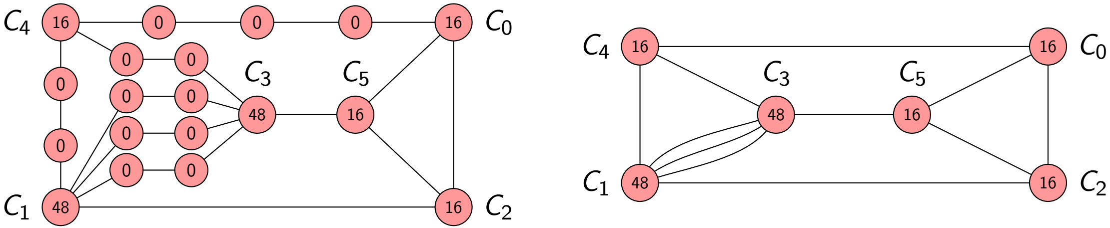
- All $\overline{K}_{B_3}^{\,3} = 6$ bases (7 polytopes),
- Some $\overline{K}_{B_3}^{\,3} = 10$ bases (26 polytopes).

Crucial observations which facilitate scan over $\sim 10^{15}$ different 3-folds B_3

1. The number of limit root bundles on C_R° with $h^0 = 3$ is independent of the triangulation! [Batyrev '93] [Cox Katz '99] [Kreuzer '06]
2. Remove all tree-like subgraphs

$\Rightarrow$ Dual graph simplifies, number of limit root bundles and cohomologies unchanged!

Example: Δ°_{52} ($\overline{K}_{B_3}{}^3 = 10$):



Results for bases with $\overline{K}_{B_3}^3 = 6$

- $N_P = 12^8 = \left(2\overline{K}_{B_3}\right)^{2g}$: total number of root bundles on $C_{(3,2)_{1/6}}$
- $\check{N}_P^{(3)}$: number of limit roots on $C^\circ_{(3,2)_{1/6}}$ with $h^0 = 3$
- Computer scan finds:

	$\check{N}_P^{(3)}$	$N_P / \check{N}_P^{(3)}$		$\check{N}_P^{(3)}$	$N_P / \check{N}_P^{(3)}$
Δ°_8	142560	$3.0 \cdot 10^3$	Δ°_{130}	8910	$4.8 \cdot 10^4$
Δ°_4	11110	$3.8 \cdot 10^4$	Δ°_{136}	8910	$4.8 \cdot 10^4$
Δ°_{134}	10100	$4.3 \cdot 10^4$	Δ°_{236}	8910	$4.8 \cdot 10^4$
Δ°_{128}	8910	$4.8 \cdot 10^4$			

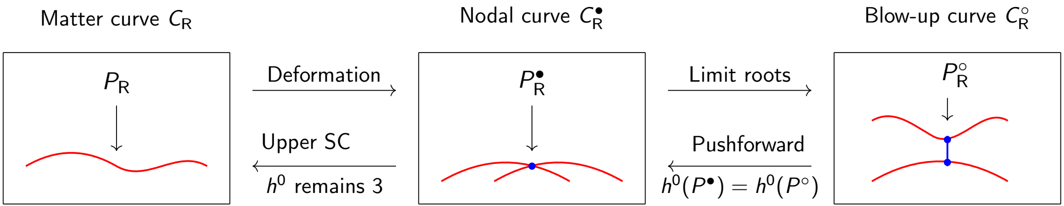
- Our scan is limited to a **subset** of all roots on $C^\bullet_{(3,2)_{1/6}}$.
- $\check{N}_P^{(3)}(C_{(3,2)_{1/6}}) \geq \check{N}_P^{(3)}(C^\bullet_{(3,2)_{1/6}})$ due to jumps along $C^\bullet_{(3,2)_{1/6}} \to C_{(3,2)_{1/6}}$.

$\Rightarrow$ Current techniques: $B_3(\Delta^\circ_8)$ most promising for F-theory MSSM (with $\overline{K}_{B_3}^3 = 6$).

Summary

- Root bundles arise naturally in the QSMs. [Cvetič Halverson Lin Liu Tian '19]
- On smooth, irreducible C_{R} hard, but easy for nodal $C_{\mathrm{R}}^{\bullet}$:

[Jarvis '98], [Jarvis '99], [Caporaso Casagrande Cornalba '07]



- Find roots on $C_{(3,2)_{1/6}}$, $C_{(\overline{3},1)_{-2/3}}$, $C_{(\overline{3},1)_{1/3}}$, $C_{(1,1)_1}$ without vector-like exotics.
- Extend systematically to all $\mathcal{O}(10^{15})$ QSM spaces.
- With current techniques: Absence of vector-like exotics on $C_{(3,2)_{1/6}}$ most likely for base 3-folds from triangulations of Δ_8°.

Theory outlook

- Goal: F-theory MSSM construction

$\Rightarrow$ Identify roots on Higgs curve with $h^i = (4, 1)$, i.e. exactly one Higgs pair.

- Technical extensions:
  - Perform limit root counting on Higgs curve.
  - Extend limit root counting beyond limit roots on full-blowup curve C_{R}°.
- Conceptional questions/obstructions:
  - Does the topology of the dual graph encode the root bundle distribution?

  $\Rightarrow$ Possibly a machine learning/data science project for the summer.
  - What conditions prevent/detect jumps in vector-like spectrum along $C_{\mathrm{R}}^{\bullet} \to C_{\mathrm{R}}$?
  - More ambitious: What is the defect, i.e. by how much does h^0 jump?
  - Which root bundles are realized top-down, i.e. from an F-theory gauge potential?

Outlook for toric (and algebraic) geometry in OSCAR

- My hope/vision: Automatically perform fully fledged F-theory construction.
- As a starting point, focus on QSMs [Klevers Pena Oehlmann Piragua Reuter '14].
- Toric varieties from triangulation of polytopes (cf. PR-848):
  - `Calabi-Yau tools`, `Polymake` [Jordan, Joswig, Kastner '18] and `Topcom`
  - Kreuzer-Skarke lists [Kreuzer Skarke '98], [Kreuzer Skarke '00], [Altman Gray He Jejjala Nelson '14], ...
  - QSM list (currently in `GAP-4`-package QSMExplorer)
- Complete intersection subvarieties:
  - Cohomologies: `cohomCalg`, `Calabi-Yau tools`, `ToricVarieties_project`, ...
  - Smoothness.
- Chiral and vector-like spectra:
  - **Topological intersection** numbers of algebraic cycles.
  - **Chow ring and intersection theory**.
- Root bundle counter (currently in `GAP-4`-package QSMExplorer)

Thank you for your attention!