Root Bundles and Towards Exact Matter Spectra of F-theory MSSMs

Martin Bies

University of Pennsylvania

Summer Series On String Phenomenology – September 21, 2021

Work with M. Cvetič, R. Donagi, M. Liu, M. Ong – 2102.10115, 2104.08297

Motivation

Obtain (MS)SM from String theory construction ...

- $E_8 \times E_8$ [Candelas Horowitz Strominger Witten '85], [Greene Kirklin Miron Ross '86], [Braun He Ovrut Pantev '05], [Bouchard Donagi '05], [Anderson Gray He Lukas '10], ...
- Type II [Berkooz Douglas Leigh '96], [Aldazabal Franco Ibanez Rabadan Uranga '00], [Ibanez Marchesano Rabadan '00], [Blumenhagen Kors Lust Ott '01], [Cvetič Shiu Uranga '01],
- F-theory [Krause Mayrhofer Weigand '12], [Cvetič Klevers Pena Oehlmann Reuter '15], [Lin Weigand '16], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], ...

... including vector-like spectra

- Why vector-like spectra? Higgs fields matter & characteristic feature of QFTs
- $E_8 \times E_8$: [Bouchard Donagi '05], [Bouchard Cvetič Donagi '06], [Anderson Gray Lukas Palti '10 & '11], ...
- F-theory: [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18], [M.B. Cvetič Donagi Lin Liu Ruehle '20], [M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21]

Outline

In this talk

- Focus on Quadrillion F-theory Standard Models (QSMs) [Cvetič Halverson Lin Liu Tian '19]

  globally-consistent, gauge coupling unification, no chiral exotics

- Spectra counted by cohomologies of special line bundles, namely **root bundles**.

  Are there roots with cohomologies of an MSSM vector-like spectrum?

Outline

1. The appearance of root bundles in the QSMs.
2. Proving existence.
3. Statistical study.

Vector-like spectra in 4d $\mathcal{N} = 1$ F-theory vacua

[M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]

- Defined by elliptic 4-fold $Y_4 \twoheadrightarrow B_3$ and flux $G_4 \in H_{\mathbb{Z}}^{(2,2)}(\widehat{Y}_4)$:
    - Gauge degrees **localized** on 7-branes $S \subset B_3$.
    - Zero modes **localized** on matter curves $C_{\mathrm{R}} \subset S$ and encoded by **matter surface** S_{R}.
- G_4 and S_{R} define line bundle L_{R} on C_{R} (details on next slide).
- Massless vector-like spectra:

$$\begin{aligned} \text{massless chiral supermultiplets in rep. } \mathrm{R} &\leftrightarrow h^0(C_{\mathrm{R}}, L_{\mathrm{R}}), \\ \text{massless chiral supermultiplets in rep. } \overline{\mathrm{R}} &\leftrightarrow h^1(C_{\mathrm{R}}, L_{\mathrm{R}})\,, \\ \text{chiral index} &\leftrightarrow h^0(C_{\mathrm{R}}, L_{\mathrm{R}}) - h^1(C_{\mathrm{R}}, L_{\mathrm{R}})\,. \end{aligned}$$

- Typically, $h^i(C_{\mathrm{R}}, L_{\mathrm{R}})$ non-topological and thus hard to determine.
    - Often, L_{R} not pullback from B_3 ($\mathrm{Pic}(C_{\mathrm{R}})$ typically continuous).
    - Deformation $C_{\mathrm{R}} \to C'_{\mathrm{R}}$ can lead to jumps [M.B. Cvetič Donagi Lin Liu Ruehle '20]

$$h^i(C_{\mathrm{R}}, L_{\mathrm{R}}) = (h^0, h^1) \to h^i(C'_{\mathrm{R}}, L'_{\mathrm{R}}) = (h^0 + a, h^1 + a)\,.$$

How to compute L_R from G_4, S_R? [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]

- Lift $G_4 \in H^{(2,2)}_{\mathbb{Z}}(\widehat{Y}_4)$ to a "gauge field" $A \in H^4_D(\widehat{Y}_4, \mathbb{Z}(2))$ or $\mathcal{A} \in \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}^2_{\text{hom}}(\widehat{Y}_4, \mathbb{Z}) & \longrightarrow & \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z}) & \xrightarrow{\gamma} & H^{(2,2)}_{\text{alg}}(\widehat{Y}_4) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \widehat{\gamma} & & \downarrow & & \\ 0 & \longrightarrow & J^2(\widehat{Y}_4) & \longrightarrow & H^4_D(\widehat{Y}_4, \mathbb{Z}(2)) & \xrightarrow{\widehat{c}} & H^{(2,2)}_{\mathbb{Z}}(\widehat{Y}_4) & \longrightarrow & 0 \end{array}$$

  Always exists, but it is in general **not unique** since $J^2(\widehat{Y}_4) \neq 0$.
- For $S_R \in \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$ define $\iota_{S_R} \colon S_R \hookrightarrow \widehat{Y}_4$, $\pi_{S_R} \colon S_R \twoheadrightarrow C_R$. Then

$$L_R(\mathcal{A}) = \mathcal{O}_{C_R}\left[\pi_{S_R *}\left(\iota^*_{S_R}(\mathcal{A})\right) + D_{\text{spin,R}}\right] \in \mathrm{Pic}(C_R) \, .$$

Lifting G_4 in the QSMs: Rational prefactor

- QSMs: $\mathcal{O}(10^{15})$ elliptic 4-folds $\widehat{Y}_4$ with choice of G_4 [Cvetič Halverson Lin Liu Tian '19]
  - Elliptic 4-folds $\widehat{Y}_4 \twoheadrightarrow B_3$:
    - Obtained from toric geometry.
    - Constraints: no chiral exotics, massless $U(1)$-gauge boson, cancel D_3-tadpole.

    $\Rightarrow$ B_3 from triangulations of 708 3-dim reflexive polytopes Kreuzer Skarke '98

$$\overline{K}_{B_3} \cdot \overline{K}_{B_3} \cdot \overline{K}_{B_3} \in \{6, 10, 18, 30\} \,.$$

  - G_4-flux candidate ($\leftrightarrow$ satisfies necessary conditions to be integral):

$$G_4 = \frac{-3}{\overline{K}_{B_3}^3} \cdot (5[e_1] \wedge [e_4] + \ldots) \in H^{(2,2)}_{\text{alg}}(\widehat{Y}_4) \,.$$

- Naive lift $\mathcal{A} = \frac{-3}{\overline{K}_{B_3}^3} \cdot (5V(e_1, e_4) + \ldots) \notin \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$ since $\frac{-3 \cdot 5}{\overline{K}_{B_3}^3} \notin \mathbb{Z}$.

Lifting G_4 in the QSMs: An easier multiple

- Lack of computational control over $J^2(\widehat{Y}_4)$.
  $\rightarrow$ Cannot directly write-down lift G_4 to $\mathcal{A} \in \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$.
- Circumvent this ignorance as follows:
  1. Consider $G_4' = \overline{K}_{B_3}^3 \cdot G_4$ instead of
  $$G_4 = \frac{-3}{\overline{K}_{B_3}^3} \cdot \left(5 [e_1] \wedge [e_4] + \ldots \right) \, .$$
  2. Lift G_4' to $\mathcal{A}' = -3 \cdot \left(5 V(e_1, e_4) + \ldots \right) \in \mathrm{CH}^2(\widehat{Y}_4, \mathbb{Z})$ and find
  $$D_{\mathrm{R}} \left(\mathcal{A}' \right) = \pi_{S_{\mathbf{R}}*} \left(\iota_{S_{\mathbf{R}}}^* \left(\mathcal{A}' \right) \right) \in \mathrm{Pic} \left(C_{\mathrm{R}} \right) \, .$$

$\Rightarrow$ Root bundle constraint in $\mathrm{Pic}\left(C_{\mathrm{R}} \right)$: $\overline{K}_{B_3}^3 \cdot D_{\mathrm{R}} \left(\mathcal{A} \right) \sim D_{\mathrm{R}} \left(\mathcal{A}' \right)$.

Summary of root bundle constraints in QSMs

curve	constraint
$C_{(3,2)_{1/6}} = V(s_3, s_9)$	$P_{(3,2)_{1/6}}^{\otimes 2\overline{K}_{B_3}^3} = K_{(3,2)_{1/6}}^{\otimes\left(6+\overline{K}_{B_3}^3\right)}$
$C_{(1,2)_{-1/2}} = V(s_3, P_H)$	$P_{(1,2)_{-1/2}}^{\otimes 2\overline{K}_{B_3}^3} = K_{(1,2)_{-1/2}}^{\otimes\left(4+\overline{K}_{B_3}^3\right)} \otimes \mathcal{O}_{C_{(1,2)_{-1/2}}}(-30 \cdot Y_1)$
$C_{(\overline{3},1)_{-2/3}} = V(s_5, s_9)$	$P_{(\overline{3},1)_{-2/3}}^{\otimes 2\overline{K}_{B_3}^3} = K_{(\overline{3},1)_{-2/3}}^{\otimes\left(6+\overline{K}_{B_3}^3\right)}$
$C_{(\overline{3},1)_{1/3}} = V(s_9, P_R)$	$P_{(\overline{3},1)_{1/3}}^{\otimes 2\overline{K}_{B_3}^3} = K_{(\overline{3},1)_{1/3}}^{\otimes\left(4+\overline{K}_{B_3}^3\right)} \otimes \mathcal{O}_{C_{(\overline{3},1)_{1/3}}}(-30 \cdot Y_3)$
$C_{(1,1)_1} = V(s_1, s_5)$	$P_{(1,1)_1}^{\otimes 2\overline{K}_{B_3}^3} = K_{(1,1)_1}^{\otimes\left(6+\overline{K}_{B_3}^3\right)}$

(P_H, P_R are complicated polynomials, Y_1, Y_3 are Yukawa points.)

Local bottom-up analysis

- Which root bundles are physical, i.e. induced from $A \in H^4_D(\widehat{Y}_4, \mathbb{Z}(2))$?

  (If $g > h^{21}(\widehat{Y}_4)$, then not all are physical.)

  $\rightarrow$ Interesting, but also very challenging question for future work.
- Necessary condition for existence of F-theory MSSMs within QSM:

  Existence of root bundles on C_{R} with MSSM-suitable cohomologies.

  Perform local bottom-up analysis:

  1. For a 3-fold B_3 with $\overline{K}_{B_3}^{\,3} = 18$ (from triangulation of Δ°_{40}), I will prove that the following has a solution:

     $$P_{(3,2)_{1/6}}^{\otimes 2 \overline{K}_{B_3}^{\,3}} = K_{(3,2)_{1/6}}^{\otimes \left(6 + \overline{K}_{B_3}^{\,3}\right)} \qquad \text{and} \qquad h^0(C_{\mathrm{R}}, P_{\mathrm{R}}) = 3\,.$$

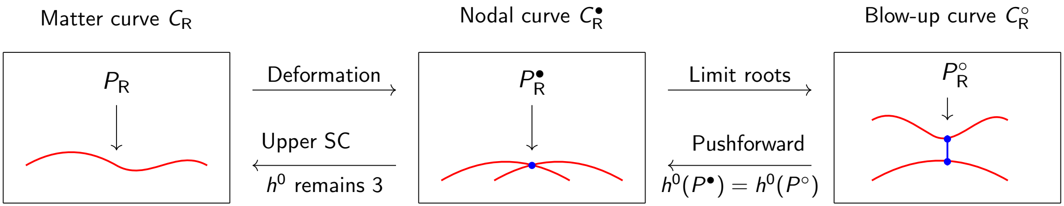
     Since $\overline{K}_{B_3}^{\,3} = 18$, it is sufficient to construct a solution to

     $$P_{(3,2)_{1/6}}^{\otimes 3} = K_{(3,2)_{1/6}}^{\otimes 2} \qquad \text{and} \qquad h^0(C_{\mathrm{R}}, P_{\mathrm{R}}) = 3\,.$$
  2. Extend to statistical study across all QSM bases.

Vector-like spectrum from deformation theory and limit root bundles

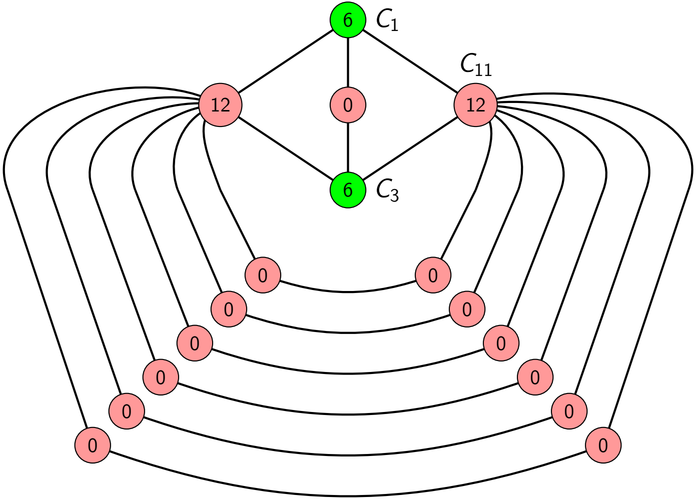
- Smooth, irreducible C_{R} with $g > 1$: Very hard to explicitly construct root bundles.
- Nodal curves $C_{\mathrm{R}}^{\bullet}$: well understood. [Jarvis '98], [Jarvis '99], [Caporaso Casagrande Cornalba '07]

$\Rightarrow$ Our approach is summarized as follows:



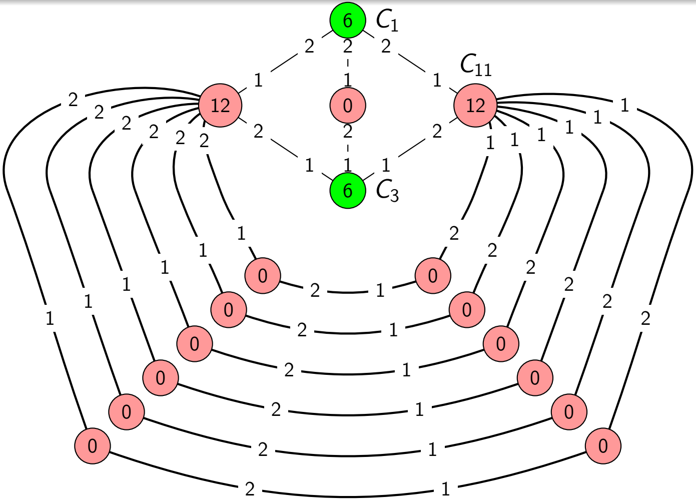
(To model *all* roots, must also consider partial blow-ups. This makes the section counting hard. Hence, we currently ignore this.)

Limit root bundle construction: Step 1 – dual graph of $C^{\bullet}_{(3,2)_{1/6}}$.



- Red bullet: $g = 0$ cpnt.
- Green bullet: $g = 1$ cpnt.
- Line: node
- Number:
  $2 \cdot \deg(K^{\bullet}_{C_{(3,2)_{1/6}}})$
- Task: Find 3rd roots with $h^0 = 3$!
  (Here fortunate case, as we can divide the local degrees by 3. This is not always true for QSM setups.)

Limit root bundle construction: Step 2 – shift degrees to blow-ups $E_j \cong \mathbb{P}^1$.

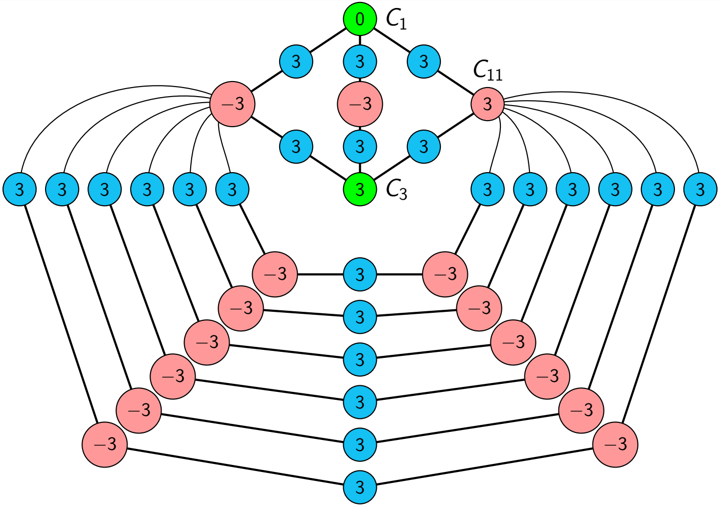


Rules for k-th roots:
(here $k = 3$):

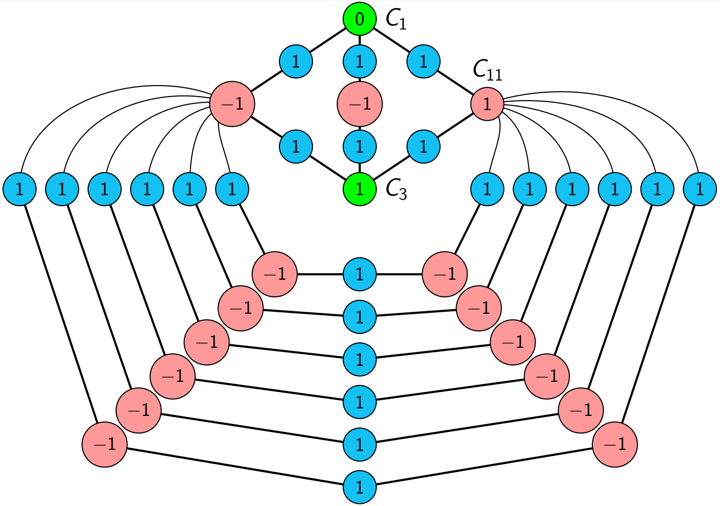
- $w_i \in \{1, \ldots, k-1\}$,
- $w_1 + w_2 = k$,
- On each component, the resulting degree is divisible by k.

$\Rightarrow$ Many possibilities!

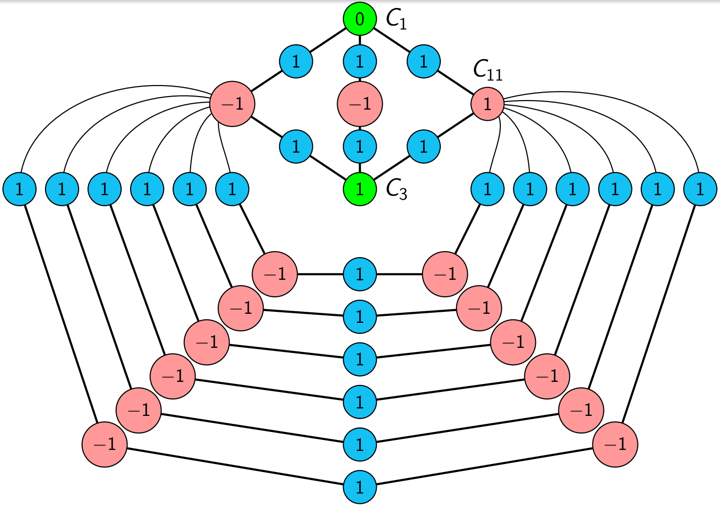
Limit root bundle construction: Step 3 – divide by $k = 3$.



Limit root bundle construction: $P^{\bullet}_{(3,2)_{1/6}}$ with $\left(P^{\bullet}_{(3,2)_{1/6}}\right)^{\otimes 3} = \left(K^{\bullet}_{(3,2)_{1/6}}\right)^{\otimes 2}$.



Counting h^0 of limit root bundle.



Observation:

- $h^0(E_j \cong \mathbb{P}^1, \mathcal{O}_{E_j}) = 2$

$\Rightarrow$ Uniquely fixed by boundary conditions.

$\Rightarrow$ $h^0(P^\bullet_R) = \sum_{C_i \neq E_j} h^0\left(C_i, P^\bullet_R|_{C_i}\right)$

- $h^0(C_3) = 1$, $h^0(C_{11}) = 2$
- $h^0(C_1) = 0$ for at least 8 of 9 local roots.

$\Rightarrow$ $\exists P^\bullet_R$ s.t. $h^0(C^\bullet_R, P^\bullet_R) = 3$.

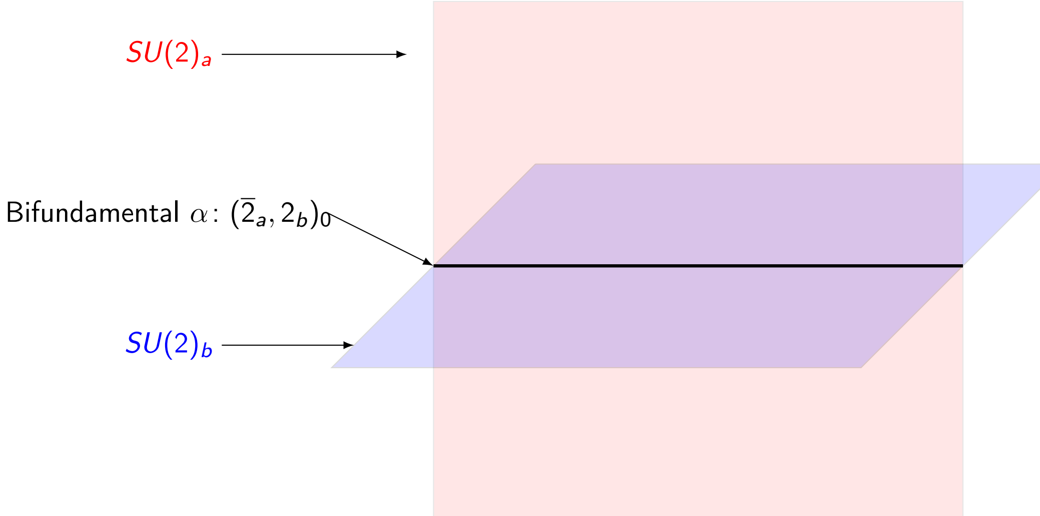
Jumps on rational curves

Let $C^\bullet = \bigcup_{i \in I} C_i$ be a connected, **rational**, nodal curve and $L^\bullet$ a line bundle of $\deg(L) \geq 0$ on $C^\bullet$. (This means that I is a connected tree-like graph and $C_i \cong \mathbb{P}^1$.) Let $k \in \mathbb{N}_{\geq 2}$ with $k | \deg(L)$ and P° a k-th limit root bundle on the full blow-up curve $C^\circ = \bigcup_{i \in I} C_i \cup \bigcup_{j \in J} E_j$.
Then, as we deform C° to a smooth rational curve C, the following are equivalent:

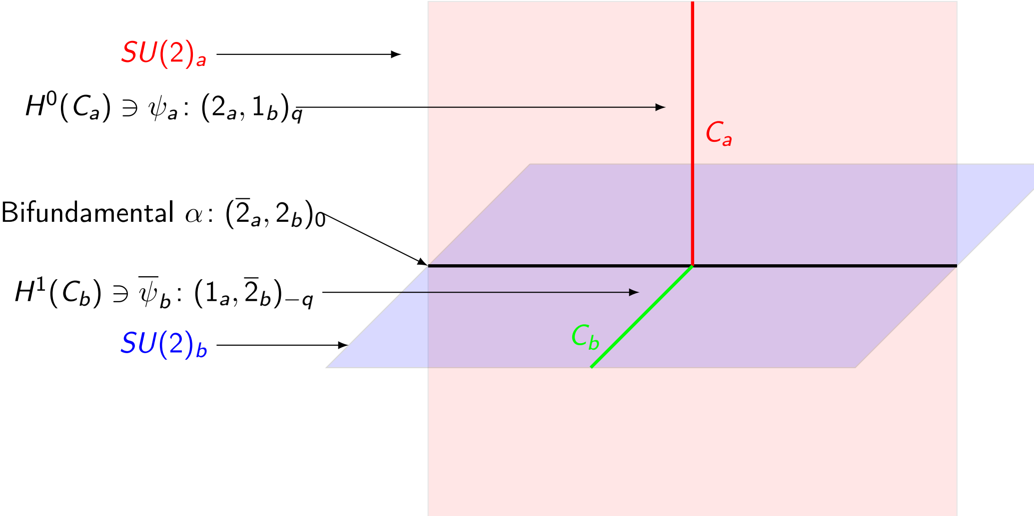
$$\sum_{i \in I} h^0\left(C_i, P^\circ|_{C_i}\right) = h^0(C^\circ, P^\circ) > h^0(C, P)$$

$$\Leftrightarrow \quad \exists i_1 \neq i_2 \, : \; h^0\left(C_{i_1}, P^\circ|_{C_{i_1}}\right) \cdot h^1\left(C_{i_2}, P^\circ|_{C_{i_2}}\right) \neq 0 \, .$$

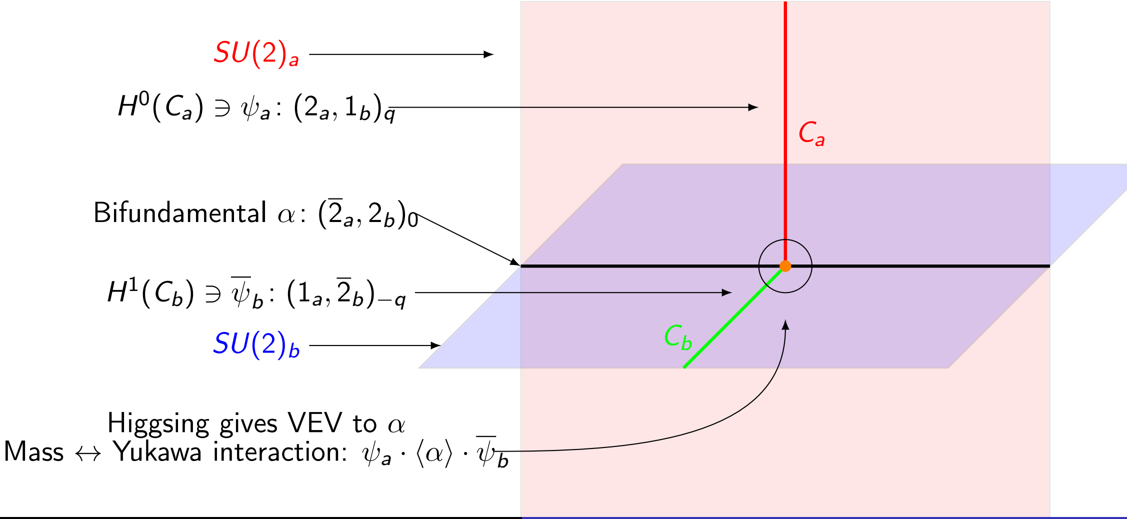
Attempt of a physics interpretation: mass term from Yukawa interaction



Attempt of a physics interpretation: mass term from Yukawa interaction



Attempt of a physics interpretation: mass term from Yukawa interaction



Towards promising F-theory base spaces

- Assume that $C_{\mathrm{R}}^{\bullet}$ consists only of $g = 0$ and $g = 1$ components. Then automate:
  - Given the dual graph of $C_{\mathrm{R}}^{\bullet}$, find all allowed weight assignments.
  - For the encoded limit root line bundles compute $h^0(C_{\mathrm{R}}^{\bullet}, P_{\mathrm{R}}^{\bullet})$.

$\rightarrow$ Implemented in `GAP-4` package *QSMExplorer*

https://github.com/homalg-project/ToricVarieties_project/tree/master/QSMExplorer

- Scan over selected QSM geometries:
  1. $h^{(2,1)}(\widehat{Y}_4) \geq g(C_{(3,2)_{1/6}})$.
     Necessary condition for many roots on $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ to be physical.
  2. Components of $C_{(3,2)_{1/6}}$ must have at most $g = 1$.

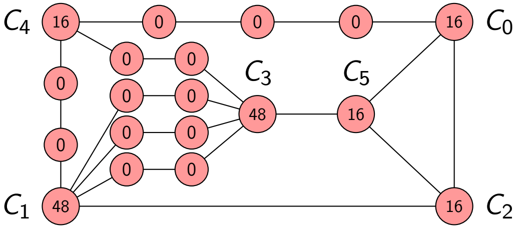
$\Rightarrow$ Base 3-folds B_3 obtained from triangulations of 33 3-dim. reflexive polytopes:

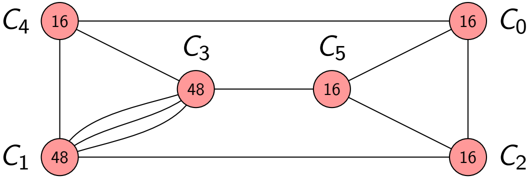
- All $\overline{K}_{B_3}^{\,3} = 6$ bases (7 polytopes),
- Some $\overline{K}_{B_3}^{\,3} = 10$ bases (26 polytopes).

Crucial observations

1. The number of limit root bundles on C_R° with $h^0 = 3$ is independent of the triangulation! [Batyrev '93] [Cox Katz '99] [Kreuzer '06]
2. Remove all tree-like subgraphs

$\Rightarrow$ Dual graph simplifies, number of limit root bundles and cohomologies unchanged! Example: Δ_{52}° $(\overline{K}_{B_3}{}^3 = 10)$:





Results for bases with $\overline{K}_{B_3}^3 = 6$

- $N_P = 12^8 = (2\overline{K}_{B_3})^{2g}$: total number of root bundles on $C_{(3,2)_{1/6}}$
- $\check{N}_P^{(3)}$: number of limit roots on $C^\circ_{(3,2)_{1/6}}$ with $h^0 = 3$
- Computer scan finds:

	$\check{N}_P^{(3)}$	$N_P/\check{N}_P^{(3)}$		$\check{N}_P^{(3)}$	$N_P/\check{N}_P^{(3)}$
Δ_8°	142560	$3.0 \cdot 10^3$	Δ_{130}°	8910	$4.8 \cdot 10^4$
Δ_4°	11110	$3.8 \cdot 10^4$	Δ_{136}°	8910	$4.8 \cdot 10^4$
Δ_{134}°	10100	$4.3 \cdot 10^4$	Δ_{236}°	8910	$4.8 \cdot 10^4$
Δ_{128}°	8910	$4.8 \cdot 10^4$			

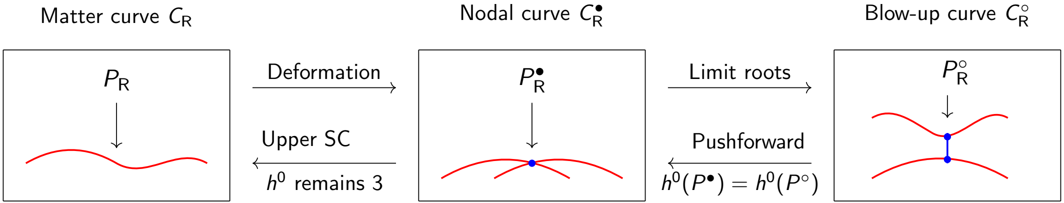
- Note:
  - Our scan is limited to subset of all roots on $C^\bullet_{(3,2)_{1/6}}$.
  - $\check{N}_P^{(3)}(C_{(3,2)_{1/6}}) \geq \check{N}_P^{(3)}(C^\bullet_{(3,2)_{1/6}})$ due to jumps along $C^\bullet_{(3,2)_{1/6}} \to C_{(3,2)_{1/6}}$.

$\Rightarrow$ Current techniques: $B_3(\Delta_8^\circ)$ most promising bases for F-theory MSSMs.

Summary

- Root bundles arise naturally in the QSMs. [Cvetič Halverson Lin Liu Tian '19]
- On smooth, irreducible C_{R} hard, but easy for nodal $C_{\mathrm{R}}^{\bullet}$:

[Jarvis '98], [Jarvis '99], [Caporaso Casagrande Cornalba '07]



- Find roots on $C_{(3,2)_{1/6}}$, $C_{(\overline{3},1)_{-2/3}}$, $C_{(\overline{3},1)_{1/3}}$, $C_{(1,1)_1}$ without vector-like exotics.
- Extend systematically to all $\mathcal{O}(10^{15})$ QSM spaces.
- With current techniques: Absence of vector-like exotics on $C_{(3,2)_{1/6}}$ most likely for base 3-folds from triangulations of Δ_8°.

Outlook

- Goal: F-theory MSSM construction

$\Rightarrow$ Identify roots on Higgs curve with $h^i = (4, 1)$, i.e. exactly one Higgs pair.

- Technical extensions:
    - Perform limit root counting on Higgs curve.
      Currently, this is combinatorially too challenging for our algorithms.
    - Extend limit root counting beyond limit roots on full-blowup curve C_{R}°.
- Conceptional questions/obstructions:
    - Does the topology of the dual graph encode the root bundle distribution?
    - What conditions prevent/detect jumps in vector-like spectrum along $C_{\mathrm{R}}^{\bullet} \to C_{\mathrm{R}}$?
    - More ambitious: What is the defect, i.e. by how much does h^0 jump?
    - Which root bundles are realized top-down, i.e. from an F-theory gauge potential?

Thank you for your attention!