Computational Frontiers in Singular Elliptic Fibrations and F-Theory Model Building

Martin Bies

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Tübingen, Germany
September 25, 2025

Based on work with M. E. Miķelsons, A. P. Turner, and the OSCAR team.
arXiv: 2506.13849

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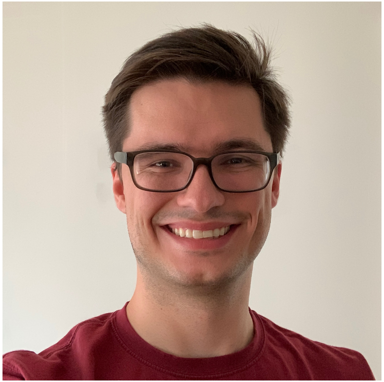
M. E. Mikelsons



A. P. Turner

https://www.oscar-system.org/

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(Review: Weigand 2018 – 1806.01854)

Motivation and Roadmap

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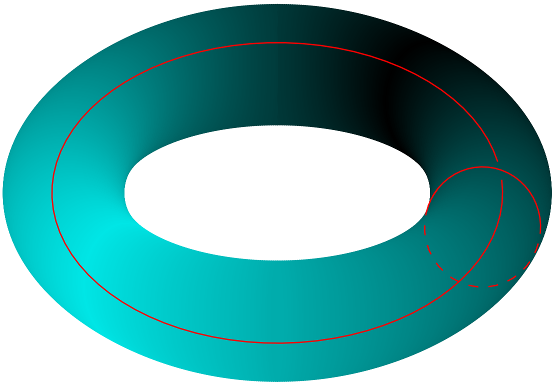
  Details in our latest preprint: arXiv – 2506.13849.

What is an Elliptic Curve?

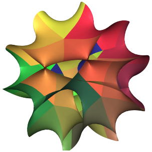


$$\mathbb{E}_{f,g} = \left\{ (x, y, z) \in \mathbb{P}^2_{(2,3,1)} \,\middle|\, x^3 - y^2 + fxz^4 + gz^6 = 0 \right\}$$

(non-singular iff $\Delta := 4f^3 + 27g^2 \neq 0$)

From Elliptic Curve to Elliptic Fibration

1



Base B, $f \in H^0(\overline{K}_B^4)$, $g \in H^0(\overline{K}_B^6)$.

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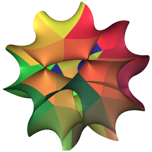

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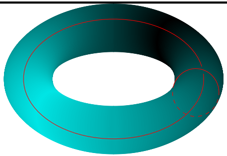

$\mathbb{V}\left(x^3 - y^2 + f(p)xz^4 + g(p)z^6\right) \subseteq \mathbb{P}^2_{(2,3,1)}$

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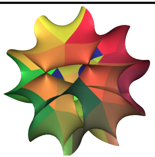

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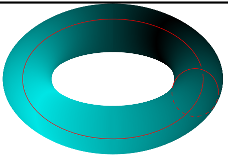

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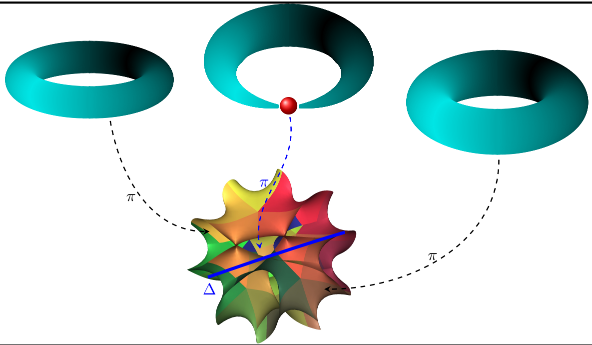
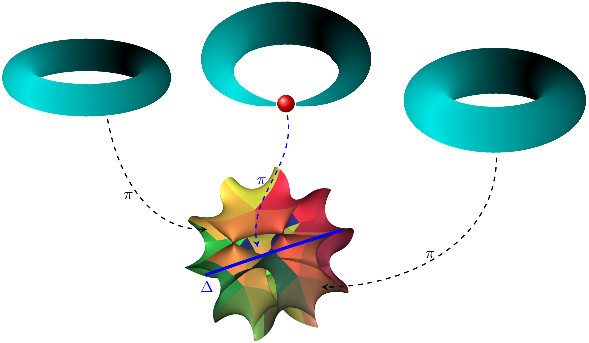


Singular Elliptic Fibrations

- *Elliptic fibration*: A morphism of varieties/schemes $\pi : Y \to B$ whose generic fiber is a smooth elliptic curve.



- Singularities arise when fibers degenerate (e.g. nodal or cuspidal curves).
- Fibers degenerate over **discriminant locus** $\Delta = \mathbb{V}\left(4f^3 + 27g^2\right) \subseteq B$.
- Classification of singularities of elliptic **surfaces** by Kodaira in 1963.

(https://doi.org/10.2307/1970131, https://doi.org/10.2307/1970500 – see also Kodaira/Weierstrass table)

What is FTheoryTools?

- FTheoryTools is a module of OSCAR (https://www.oscar-system.org).
- Key Features:
  1. Construct singular elliptic **Calabi-Yau** fibrations.
  2. Database of classical/famous F-theory constructions
     - Future-proof cross-platform standard by MaRDI: https://www.mardi4nfdi.de.
     - "Interactive" paper, to corrects typos, redos computations & extends them at ease.
  3. Tailormade algorithms for blowups & resolutions of singularities, cohomologies, Chern classes, Hodge numbers, **intersection numbers**, ...
- Hooked? More information available!
  - Docs: https://docs.oscar-system.org/stable/Experimental/FTheoryTools/introduction/.
  - Tutorials: https://www.oscar-system.org/tutorials/FTheoryTools/.
  - M. Bies, and A. Turner, F-Theory Applications – chapter in the OSCAR book,
  - M. Bies, M. E. Miķelsons, A. P. Turner, FTheoryTools: Advancing Computational Capabilities for F-Theory Research.

Documentation: https://docs.oscar-system.org/stable/Experimental/FTheoryTools/introduction/

Welcome to FTheoryTools

docs.oscar-system.org/stable/Experimental/FTheoryTools/introduction/

OSCAR
SYMBOLIC TOOLS
Oscar.jl

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F-Theory Tools

Welcome to `FTheoryTools`

- Overview
- Why Use `FTheoryTools`?
- Key Features
- Tutorials
- Getting Started
- Project Status
- Contact & Community
- Acknowledgements

Weierstrass Models

Experimental / F-Theory Tools / Welcome to `FTheoryTools`

GitHub

Welcome to `FTheoryTools`

Overview

`FTheoryTools` is a computational toolkit within the OSCAR computer algebra system, designed to assist researchers in working with F-theory models. It focuses on automating and simplifying calculations involving **singular elliptic fibrations**—key geometric objects in F-theory phenomenology.

While the module is tailored for string theorists, it is equally accessible to mathematicians interested in the rich geometry of singular fibrations, even if they are not familiar with F-theory itself.

This page is meant for *end users* of OSCAR, including students and researchers in mathematics and the natural sciences. No background in string theory or theoretical physics is assumed beyond what is needed to understand the geometry of elliptic fibrations. We encourage interested readers to consult the exposition in Weigand 2018 for more background information.

Why Use `FTheoryTools`?

Tutorials: https://www.oscar-system.org/tutorials/FTheoryTools

Tutorials · OSCAR Computer

oscar-system.org/tutorials/

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Tutorials

Explore Jupyter notebooks demonstrating OSCAR's capabilities. These tutorials are for **OSCAR v1.5.0**, the latest stable release. To avoid compatibility issues, make sure you're using this version (upgrade instructions).

To download and run a tutorial locally:

1. Open the tutorial in *nbviewer* (or view it directly on GitHub).
2. Right-click (or Ctrl-click on macOS) the download icon at the top-right.
3. Select *Save As...* and save the file to your computer.
4. If OSCAR is not yet installed, follow the installation guide. If it's already installed, ensure you're using version **OSCAR v1.5.0** by checking the upgrade instructions.
5. Complete Step 5 of the installation guide to ensure your system is set up to run the tutorials.

Looking for more examples? Check out the polymake examples and Hecke examples.

Want to improve or add a tutorial? See our contribution guidelines.

Preprint: https://arxiv.org/abs/2506.13849

[2506.13849] FTheoryTools

arxiv.org/abs/2506.13849

Cornell University

We gratefully acknowledge support from the Simons Foundation, member institutions, and all contributors.

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High Energy Physics - Theory

[Submitted on 16 Jun 2025 (v1), last revised 17 Sep 2025 (this version, v2)]

FTheoryTools: Advancing Computational Capabilities for F-Theory Research

Martin Bies, Miķelis E. Miķelsons, Andrew P. Turner

A primary goal of string phenomenology is to identify realistic four-dimensional physics within the landscape of string theory solutions. In F-theory, such solutions are encoded in the geometry of singular elliptic fibrations, whose study often requires particularly challenging and cumbersome computations. In this work, we introduce FTheoryTools, a novel software module integrated into the OSCAR computer algebra system, designed to automate the complex and tedious tasks involved in F-theory model building. Key features of FTheoryTools include the enumeration of G4-fluxes, the capability to perform blowups on arbitrary (including non-toric) loci, and a literature database of existing F-theory constructions employing a MaRDI-based data format for enhanced collaboration and reproducibility. As a demonstration of its power, we present a stress test by applying FTheoryTools to the challenging F-theory geometry with most flux vacua (arXiv:1511.03209). Our results illustrate the potential of FTheoryTools to streamline F-theory research and pave the way for future developments in the computational study of string phenomenology.

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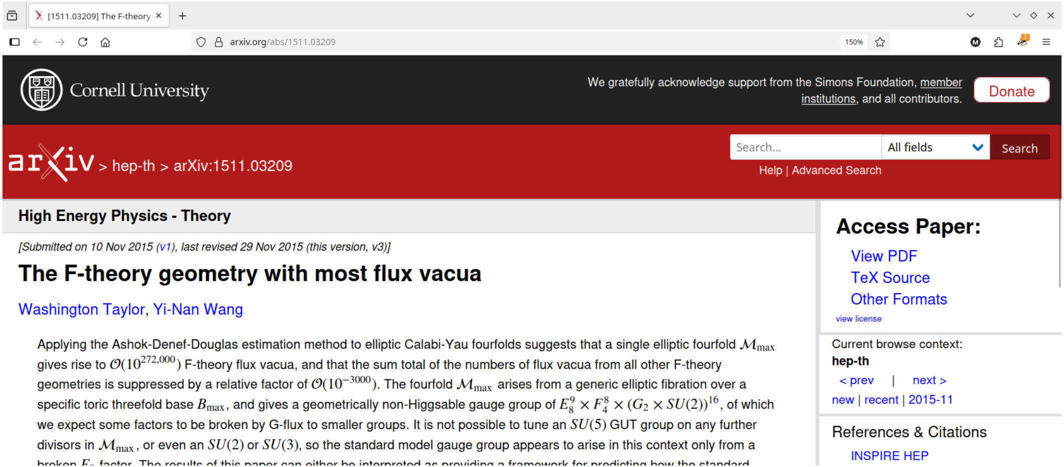
Questions?

Stress Test for FTheoryTools

- Challenge: How far can FTheoryTools be pushed?
- Case study: Taylor, Wang 2015 – arxiv: 1511.03209.


[1511.03209] The F-theory

arxiv.org/abs/1511.03209

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We gratefully acknowledge support from the Simons Foundation, member institutions, and all contributors.

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arXiv > hep-th > arXiv:1511.03209

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High Energy Physics - Theory

[Submitted on 10 Nov 2015 (v1), last revised 29 Nov 2015 (this version, v3)]

The F-theory geometry with most flux vacua

Washington Taylor, Yi-Nan Wang

Applying the Ashok-Denef-Douglas estimation method to elliptic Calabi-Yau fourfolds suggests that a single elliptic fourfold $\mathcal{M}_{\rm max}$ gives rise to $\mathcal{O}(10^{272,000})$ F-theory flux vacua, and that the sum total of the numbers of flux vacua from all other F-theory geometries is suppressed by a relative factor of $\mathcal{O}(10^{-3000})$. The fourfold $\mathcal{M}_{\rm max}$ arises from a generic elliptic fibration over a specific toric threefold base $B_{\rm max}$, and gives a geometrically non-Higgsable gauge group of $E_8^9 \times F_4^8 \times (G_2 \times SU(2))^{16}$, of which we expect some factors to be broken by G-flux to smaller groups. It is not possible to tune an $SU(5)$ GUT group on any further divisors in $\mathcal{M}_{\rm max}$, or even an $SU(2)$ or $SU(3)$, so the standard model gauge group appears to arise in this context only from a broken E_8 factor. The results of this paper can either be interpreted as providing a framework for predicting how the standard

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- Singular elliptic Calabi-Yau 4-fold as hypersurface in **toric** ambient space X_Σ.

Computational Challenge: Taylor, Wang 2015 – 1511.03209

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Computational Challenge: Taylor, Wang 2015 – 1511.03209

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- Hypersurface: non-generic Tate polynomial, **355,785 monomials**.

Computational Challenge: Taylor, Wang 2015 – 1511.03209

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- $+3$ extra blowups to smooth the ambient space.

FTheoryTools meets Taylor, Wang 2015 – 1511.03209

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 / _ \ / __\ / __\  / \  |  _ \   | Combining and extending ANTIC, GAP,
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  S Y M B O L I C   T O O L S     | Version 1.5.0

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  (_)     | (_) (_)    |
   _ _   _| |_  __ _   |  Type "?" for help, "]?" for Pkg help.
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  | | |_| | | | (_| |  |  Version 1.11.6 (2025-07-09)
 _/ |\__'_|_|_|\__'_|  |  Official https://julialang.org/ release
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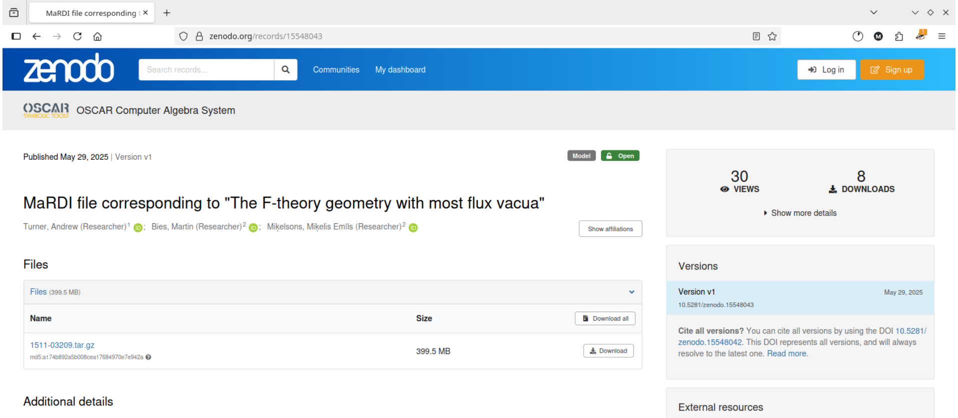
julia> t = literature_model(arxiv_id = "1511.03209")
Global Tate model over a concrete base -- The F-theory geometry with most flux vacua based on arXiv paper 1511
.03209 Eq. (2.11)

julia> t_res = resolve(t, 1)
Partially resolved global Tate model over a concrete base -- The F-theory geometry with most flux vacua based
on arXiv paper 1511.03209 Eq. (2.11)
```

MaRDI-File on Zenodo: https://zenodo.org/records/15548043



- Future-proof cross-platform standard by MaRDI: https://www.mardi4nfdi.de.
- Zenodo arxiv: One file for singular (461MB) and resolved geometry (1.3GB).

Physics vs. Mathematics of Singular Elliptic 4-Fold $\widehat{Y}_4$

(Review: Weigand 2018 – 1806.01854)

Physics	**Mathematics**
Nonabelian gauge algebras, matter curves, Yukawa points	Crepant resolution and intersection theory
Global gauge group structure & $U(1)$s	Mordell–Weil group
Discrete gauge group factors	Weil–Châtelet group
G_4-**fluxes** and chiral matter	**Middle cohomology** $H^{(2,2)}$
Vector-like matter	Deligne cohomology, root bundles

Focus for the rest of this talk:
G_4-fluxes of "The F-theory geometry with most flux vacua".

G_4-fluxes are important

- Krause, Mayrhofer, Weigand 2011 – 1109.3454
- Grimm, Hayashi 2012 – 1111.1232
- Krause, Mayrhofer, Weigand 2012 – 1202.3138
- Braun, Grimm, Keitel 2013 – 1306.0577
- Cvetič, Grassi, Klevers, Piragua 2013 – 1306.3987
- Cvetič, Klevers, Peña, Oehlmann, Reuter 2015 – 1503.02068
- Lin, Mayrhofer, Till, Weigand 2015 – 1508.00162
- Lin, Weigand 2016 – 1604.04292
- Cvetič, Lin, Liu, Oehlmann 2018 – 1807.01320
- Cvetič, Halverson, Lin, Liu, Tian 2019 – 1903.00009
- Bies 2023 – 2303.08144 (Overview of "root bundle" program)
- Li, Taylor 2024 – 2401.00040
- And many, many more.

(Review: Weigand 2018 – 1806.01854)

G_4-flux: An Element of the Middle Cohomology

- Singular elliptically fibered Calabi–Yau 4-fold: $\pi\colon Y_4 \to B_3$.
- Crepant resolution: $\widehat{\pi}\colon \widehat{Y}_4 \to B_3$.

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$$G_4 \in H^{2,2}(\widehat{Y}_4, \mathbb{R}) := H^{2,2}(\widehat{Y}_4, \mathbb{C}) \cap H^4(\widehat{Y}_4, \mathbb{R}) \tag{1}$$

satisfying the quantization condition

$$G_4 + \tfrac{1}{2} c_2(\widehat{Y}_4) \in H^4(\widehat{Y}_4, \mathbb{Z})\,. \tag{2}$$

and a set of additional physical consistency conditions (transversality, flux breaking, etc.), not discussed in this talk for brevity.

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Finding G_4-Flux Generators I – Focus on **Vertical** Fluxes

$$H^{2,2}(\widehat{Y}_4, \mathbb{C}) = H^{2,2}_{\text{hor}} \oplus H^{2,2}_{\text{vert}} \oplus H^{2,2}_{\text{rem}}$$

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- **Horizontal fluxes** $H^{2,2}_{\text{hor}}$: from variations of the holomorphic $(4,0)$-form Ω.
- **Vertical fluxes** $H^{2,2}_{\text{vert}}$: spanned by wedge products of $(1,1)$-forms:

$$H^{2,2}_{\text{vert}} = \text{Span}\left(H^{1,1} \wedge H^{1,1}\right)$$

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- **Remainder fluxes** $H^{2,2}_{\text{rem}}$: everything else.

Goal for the remainder of this talk:
Study vertical G_4-fluxes of "The F-theory geometry with most flux vacua".

Finding G_4-Flux Generators II – Focus on **Ambient** Vertical Fluxes

- Computing a full basis of $H^{2,2}_{\text{vert}}(\widehat{Y}_4)$ is often computationally prohibitive.

Finding G_4-Flux Generators II – Focus on **Ambient** Vertical Fluxes

- Computing a full basis of $H^{2,2}_{\text{vert}}(\widehat{Y}_4)$ is often computationally prohibitive.
- If $\widehat{Y}_4 \subset X_\Sigma$ is a hypersurface in a toric space, can use:

$$H^{2,2}(X_\Sigma, \mathbb{Q})\Big|_{\widehat{Y}_4} \subseteq H^{2,2}_{\text{vert}}(\widehat{Y}_4, \mathbb{Q}) \,.$$

Finding G_4-Flux Generators II – Focus on **Ambient** Vertical Fluxes

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- We call these **ambient vertical G_4-fluxes**.

Finding G_4-Flux Generators II – Focus on **Ambient** Vertical Fluxes

- Computing a full basis of $H^{2,2}_{\text{vert}}(\widehat{Y}_4)$ is often computationally prohibitive.
- If $\widehat{Y}_4 \subset X_\Sigma$ is a hypersurface in a toric space, can use:

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(More refined approaches do exists, for instance Braun, Watari 2014 – 1408.6167.)

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Goal for the remainder of this talk:
Study ambient vertical G_4-fluxes of “The F-theory geometry with most flux vacua”.

Finding G_4-Flux Generators III – Enumerating Generators

Toric ambient space with 313 rays. Naïvely, $\binom{313}{2} = 48,828$ generators.

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julia> t = literature_model(arxiv_id = "1511.03209" )
Global Tate model over a concrete base -- The F-theory geometry with most flux vacua based on arXiv paper 1511
.03209 Eq. (2.11)

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julia> betti_number(ambient_space(t_res), 2)
308

julia> betti_number(ambient_space(t_res), 4)
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julia> length(chosen_g4_flux_gens(t_res))
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Refined: **629 generators** for ambient vertical G_4-fluxes.

Off to the Quantization Condition

- Singular elliptically fibered Calabi–Yau 4-fold: $\pi \colon Y_4 \to B_3$.
- Crepant resolution: $\widehat{\pi} \colon \widehat{Y}_4 \to B_3$.
- A G_4-flux is an element

$$G_4 \in H^{2,2}(\widehat{Y}_4, \mathbb{R}) := H^{2,2}(\widehat{Y}_4, \mathbb{C}) \cap H^4(\widehat{Y}_4, \mathbb{R}) \, , \checkmark$$

satisfying the quantization condition

$$G_4 + \frac{1}{2} c_2(\widehat{Y}_4) \in H^4(\widehat{Y}_4, \mathbb{Z})$$

and a set of additional physical consistency conditions (transversality, flux breaking, etc.), not discussed in this talk for brevity.

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Solving the Quantization Condition – Strategy

- **Task:** Find all ambient vertical G_4-fluxes with $G_4 + \frac{1}{2} c_2(\widehat{Y}_4) \in H^4(\widehat{Y}_4, \mathbb{Z})$.

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- **Challenge 1:** Identify $c_2(\widehat{Y}_4)$
  1. Compute $c_i(X_\Sigma)$ (smooth toric ambient space).
  2. Apply adjunction formula: $c_2(\widehat{Y}_4) = \hat{c}_2|_{\widehat{Y}_4}$ for suitable $\hat{c}_2 \in H^{(2,2)}(X_\Sigma)$.

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- PD: Poincaré dual
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$$= \sum_{k=1}^{629} \mu_k \int_{X_\Sigma} g_k \wedge \text{PD}(H) \wedge \text{PD}(D_i) \wedge \text{PD}(D_j) + \int_{X_\Sigma} \frac{\hat{c}_2}{2} \wedge \text{PD}(H) \wedge \text{PD}(D_i) \wedge \text{PD}(D_j) \,.$$

- PD: Poincaré dual
- D_i: toric divisor basis of X_Σ
- H: toric divisor corresponding to $\widehat{Y}_4$
- g_k: generators of ambient vertical G_4-fluxes, $\mu_k \in \mathbb{Q}$

Solving the Quantization Condition II – Default Algorithm for Integrals

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Limitation for "The F-theory geometry with most flux vacua"

- Cohomology ring R: quotient of polynomial ring in **313 vars** by ideal with **46, 547 generators**.
- Checking triviality of q is computationally prohibitive.

Solving the Quantization Condition III – Special Algorithm for Integrals

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2. By choosing suitable rationally-equivalent cycles:

$$\int_{X_\Sigma} g_k \wedge \mathrm{PD}(H) \wedge \mathrm{PD}(D_i) \wedge \mathrm{PD}(D_j) = \sum_{a,b,c,d} \lambda_{abcd} \left| \mathbb{V}(x_a, x_b, x_c, x_d, P) \right|,$$

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3. Simplify $\mathbb{V}(x_a, x_b, x_c, x_d, P)$ and compute its cardinality from hard-coded cases.

Solving the Quantization Condition IV – Details on Special Algorithm

- **Monte-Carlo approach:** In toric geometry, algebraic cycles can be moved into general position using linear relations and the Stanley–Reisner ideal. Randomly select a rationally-equivalent representative until integrals become computable. *Tested on hundreds of geometries, thousands of integrals.*

Solving the Quantization Condition IV – Details on Special Algorithm

- **Monte-Carlo approach:** In toric geometry, algebraic cycles can be moved into general position using linear relations and the Stanley–Reisner ideal. Randomly select a rationally-equivalent representative until integrals become computable. *Tested on hundreds of geometries, thousands of integrals.*
- **Hard-coded edge cases:** Some intersections reduce to few variables, e.g.

$$\mathbb{V}(x_a, x_b, x_c, x_d, P) = \mathbb{V}\left(q_1 z_1 + q_2 z_2\right) \qquad q_i \neq 0$$

z_1, z_2 must not vanish simultaneously (Stanley–Reisner ideal) and scaling relation $[z_1 : z_2] \sim [\lambda z_1 : \lambda z_2]$, $z_1, z_2 \neq 0$. Leads to **unique** solution:

$$(z_1, z_2) = (1, -q_1/q_2) \sim (-q_2/q_1, 1)\,.$$

(Three more, similar edge cases: see arXiv:2506.13849.)

Quantization Condition V – Computed Intersection Numbers

```
julia> t_res.__attrs[:inter_dict]
Dict{NTuple{4, Int64}, ZZRingElem} with 14154797 entries:
  (78, 102, 103, 127)  => 0
  (87, 254, 289, 289)  => 0
  (74, 259, 260, 278)  => 0
  (46, 147, 257, 260)  => 0
  (9, 26, 103, 211)    => 0
  (66, 104, 206, 311)  => 0
  (50, 51, 102, 139)   => 0
  (31, 103, 103, 233)  => 0
  (42, 103, 148, 181)  => 0
  (61, 80, 103, 304)   => 0
  (53, 53, 91, 311)    => 0
  (11, 236, 236, 254)  => 0
  (47, 103, 154, 159)  => 0
  (20, 73, 103, 252)   => 0
```

Computed roughly **14 million intersection numbers** (about 10.000 are non-zero).

Ambient vertical G_4-fluxes for Taylor, Wang 2015 – 1511.03209

Can now solve the quantization condition, i.e. find those $\mu_k \in \mathbb{Q}$ with

$$\sum_{k=1}^{629} \mu_k \int_{X_\Sigma} g_k \wedge \mathrm{PD}(H) \wedge \mathrm{PD}(D_i) \wedge \mathrm{PD}(D_j) + \int_{X_\Sigma} \frac{\hat{c}_2}{2} \wedge \mathrm{PD}(H) \wedge \mathrm{PD}(D_i) \wedge \mathrm{PD}(D_j) \in \mathbb{Z}\,.$$

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julia> family_of_g4_fluxes = special_flux_family(t_res)
Family of G4 fluxes:
  - Elementary quantization checks: satisfied
  - Transversality checks: satisfied
  - Non-abelian gauge group: breaking pattern not analyzed

julia> size(matrix_integral(family_of_g4_fluxes))
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Ambient vertical G_4-fluxes parametrized by $\mathbb{Z}^{224} \times \mathbb{Q}^{127}$.

Gauge Group Breaking by G_4-Flux for Taylor, Wang 2015 – 1511.03209

- Singularity structure/Gauge group:

$$G = E_8^9 \times F_4^8 \times G_2^{16} \times \mathrm{SU}(2)^{16}.$$

- G_4-flux can break G.
- Those G_4s that leave G unbroken (easy to find) form a family $\mathbb{Z}^1 \times \mathbb{Q}^{127}$:

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- Outlook:
  - For a random flux, which subgroup $H \subseteq G$ survives?
  - What is the probability to break G to a given H?
  - $D3$-tadpole adds an essential (Diophantine) constraint on allowed G_4-fluxes.

Outlook

`FTheoryTools`

- Analyze singular elliptic fibrations, their resolutions, and geometry.
- Includes (and expands) a database of established F-theory models.

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Thank you for your attention!