F-Theory: Exemplifying OSCAR's Pursuit for Multidisciplinary Excellence

Martin Bies

RPTU Kaiserslautern-Landau

Third Annual Meeting 2023 of SFB-TRR 195
Saarbruecken, Germany
September 25, 2023

Toric Geometry in OSCAR with L. Kastner, (2303.08110)
`FTheoryTools` (WIP) with A. Frühbis-Krüger, A. P. Turner, M. Zach,
Studies in F-theory with M. Cvetič, R. Donagi, M. Ong. (2303.08144, 2307.02535).
Special thanks to the entire OSCAR team!

SFB-TRR 195: Summary of the research programme

Computing examples has always been a key component of mathematical research. Modern computers paired with sophisticated mathematical software tools have taken the possibilities of such calculations to a new level. In the realm of algebra and its applications, where exact calculations are inevitable, the necessary software tools are provided by computer algebra systems. Current challenges in this area arise from the increasing complexity of examples, higher levels of abstraction and the need for interdisciplinary methods. The TRR 195 aims at taking a leading role in meeting these challenges.

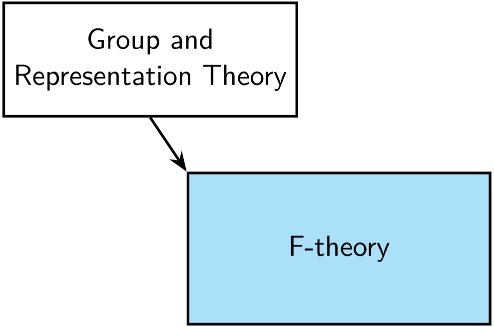
SFB-TRR 195: Summary of the research programme

Computing examples has always been a key component of mathematical research. Modern computers paired with sophisticated mathematical software tools have taken the possibilities of such calculations to a new level. In the realm of algebra and its applications, where exact calculations are inevitable, the necessary software tools are provided by computer algebra systems. Current challenges in this area arise from the increasing complexity of examples, higher levels of abstraction and the ***need for interdisciplinary methods.*** *The TRR 195 aims at taking a leading role in meeting these challenges.*

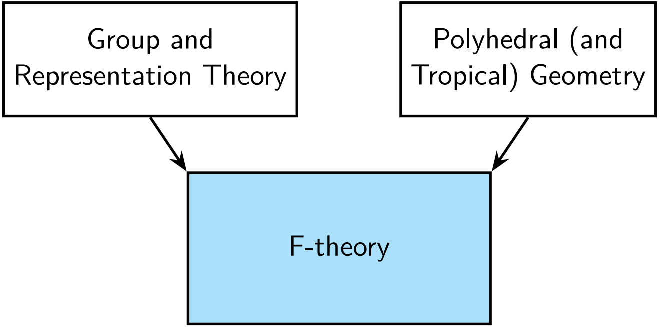
F-theory: An interdisciplinary application for the SFB-TRR 195

F-theory

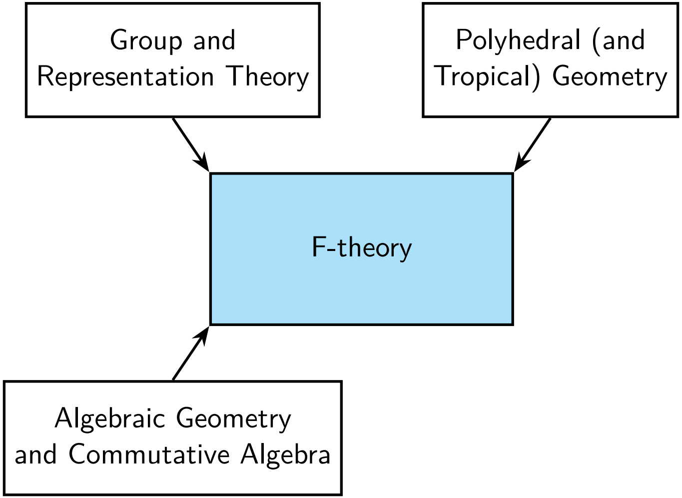
F-theory: An interdisciplinary application for the SFB-TRR 195



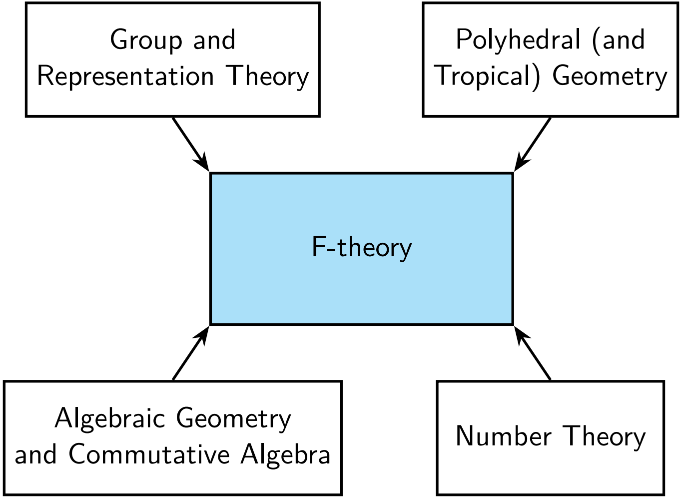
F-theory: An interdisciplinary application for the SFB-TRR 195



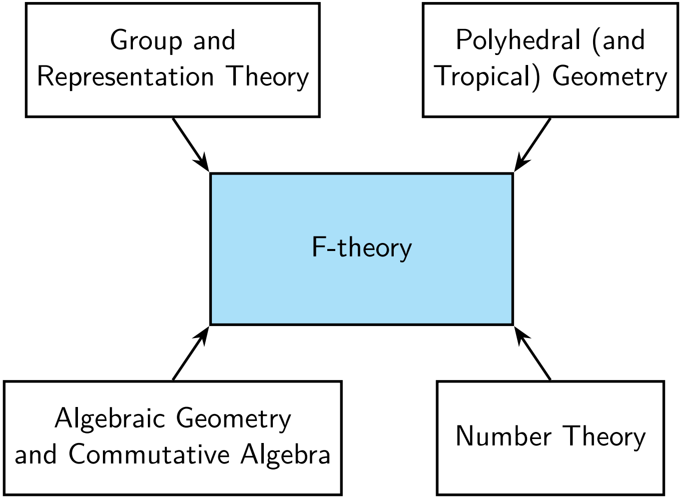
F-theory: An interdisciplinary application for the SFB-TRR 195



F-theory: An interdisciplinary application for the SFB-TRR 195



F-theory: An interdisciplinary application for the SFB-TRR 195



(Non-commutative algebra and free probability theory.)

F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.

F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.

F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.
- **Research Goal:** Discern solutions aligned with observed **particle physics**.

(Cosmological implications are largely ignored in my work.)

F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.
- **Research Goal:** Discern solutions aligned with observed **particle physics**.

(Cosmological implications are largely ignored in my work.)

- **Focus:** F-theory – solutions to string theory with a strong geometric backbone.

F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.
- **Research Goal:** Discern solutions aligned with observed **particle physics**.

(Cosmological implications are largely ignored in my work.)

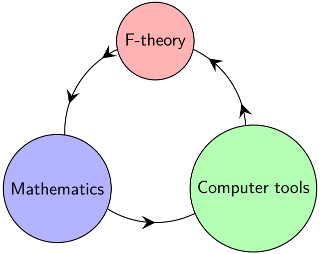
- **Focus:** F-theory – solutions to string theory with a strong geometric backbone.
- **Progress:** Still uncovering the depths of F-theory.

F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.
- **Research Goal:** Discern solutions aligned with observed **particle physics**.

(Cosmological implications are largely ignored in my work.)

- **Focus:** F-theory – solutions to string theory with a strong geometric backbone.
- **Progress:** Still uncovering the depths of F-theory.
- **Strategy:**

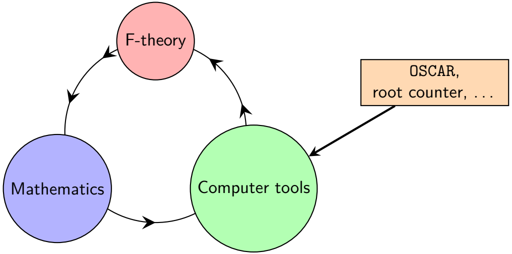


F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.
- **Research Goal:** Discern solutions aligned with observed **particle physics**.

(Cosmological implications are largely ignored in my work.)

- **Focus:** F-theory – solutions to string theory with a strong geometric backbone.
- **Progress:** Still uncovering the depths of F-theory.
- **Strategy:**

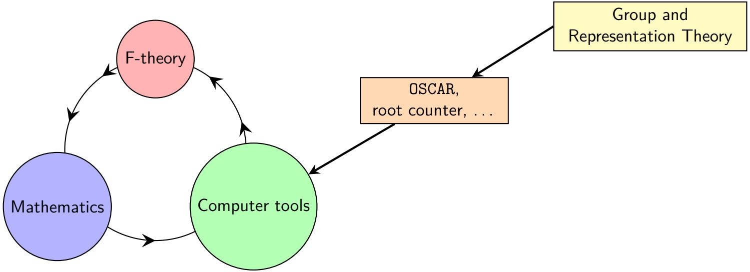


F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.
- **Research Goal:** Discern solutions aligned with observed **particle physics**.

(Cosmological implications are largely ignored in my work.)

- **Focus:** F-theory – solutions to string theory with a strong geometric backbone.
- **Progress:** Still uncovering the depths of F-theory.
- **Strategy:**

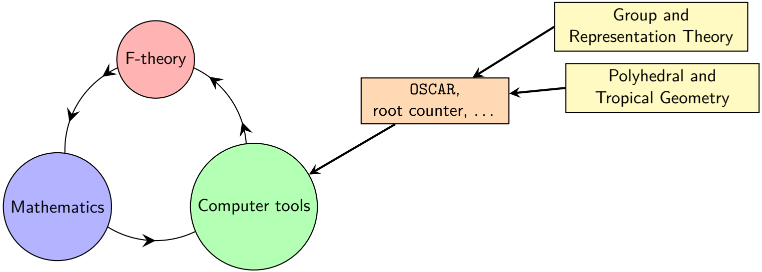


F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.
- **Research Goal:** Discern solutions aligned with observed **particle physics**.

(Cosmological implications are largely ignored in my work.)

- **Focus:** F-theory – solutions to string theory with a strong geometric backbone.
- **Progress:** Still uncovering the depths of F-theory.
- **Strategy:**

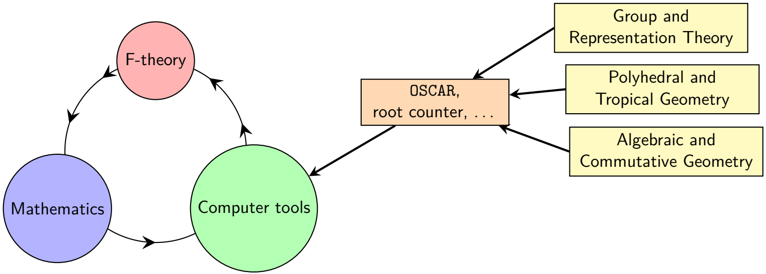


F-theory: Background and Motivation (More details: [Weigand '18])

- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.
- **Research Goal:** Discern solutions aligned with observed **particle physics**.

(Cosmological implications are largely ignored in my work.)

- **Focus:** F-theory – solutions to string theory with a strong geometric backbone.
- **Progress:** Still uncovering the depths of F-theory.
- **Strategy:**

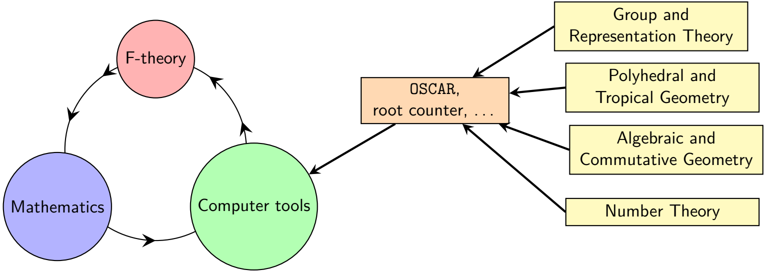


F-theory: Background and Motivation (More details: [Weigand '18])

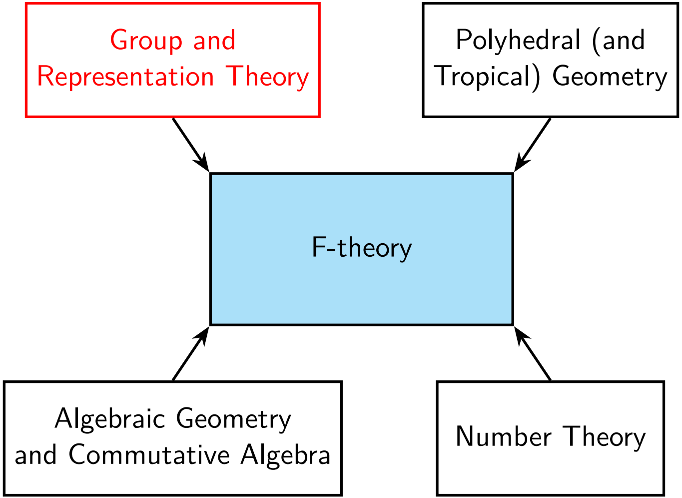
- **Passion from Undergrad:** String theory as a unifying theory of **all** natural phenomena.
- **String Landscape:** A vast solution set to string theory.
- **Research Goal:** Discern solutions aligned with observed **particle physics**.

(Cosmological implications are largely ignored in my work.)

- **Focus:** F-theory – solutions to string theory with a strong geometric backbone.
- **Progress:** Still uncovering the depths of F-theory.
- **Strategy:**



Groups meets F-theory



Groups meet F-theory: Gauge Groups

- Groups reveal symmetries and simplify problems in physics.
- **Gauge Groups**:
  - **Lagrangian:** Functional, which determines system dynamics when minimized.
  - **Gauge:** Regulates redundant degrees of freedom in the Lagrangian.
  - **Gauge group:** Mappings between gauges.
  - Gauge group of electromagnetism, weak, and strong force: $SU(3)_C \times SU(2)_W \times U(1)_Y$.
  - Quantum field theories and string theory based on group principles.
    $\rightarrow$ Particles classified by gauge group representation theory.

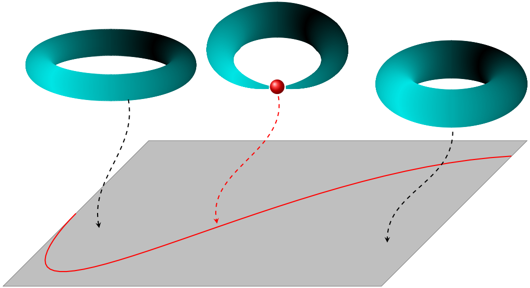
$\Rightarrow$ Gauge groups: Key to F-theory.

Groups meet F-theory: Gauge Groups from Elliptic Fibrations (Details: [Weigand '18])

- **Axio-dilaton** τ: Key to F-theory, a section of a holomorphic $SL(2,\mathbb{Z})$ bundle.
- Value of τ at spacetime point sets complex structure of an elliptic curve.

$\Rightarrow$ Value of τ/choice of elliptic curve varies with spacetime point.

$\Rightarrow$ Elliptic fibration: Book-keeping device of axio-dilaton τ



- Crucial: **Gauge group** in F-theory **from singularities** of elliptic fibration. (Cf. Kodaira classification)

Groups meet F-theory: Weierstrass Models

- Consider the weighted projective space $\mathbb{P}^{2,3,1}$ with coordinates $[x : y : z]$.
- An elliptic curve in Weierstrass form ($f, g \in \mathbb{C}$):

$$C = \{[x : y : z] \in \mathbb{P}^{2,3,1} \mid y^2 - x^3 - fxz^4 - gz^6 = 0\}$$

- C becomes singular when $4f^3 + 27g^2 = 0$.
- To construct an elliptic fibration over base B, we use sections:

$$f \in H^0(B, \overline{K}_B^{\otimes 4}), \quad g \in H^0(B, \overline{K}_B^{\otimes 6}) \,.$$

(This implies $x \in H^0(B, \overline{K}_B^{\otimes 2})$, $y \in H^0(B, \overline{K}_B^{\otimes 3})$, $z \in H^0(B, \mathcal{O}_B)$.)

  - Singularities appear at $\Delta = \{p \in B \mid 4f(p)^3 + 27g(p)^2 = 0\}$.
  - Gauge group set by vanishing orders of $\left(f, g, 4f^3 + 27g^2\right)$ at Δ:

Weierstrass table in OSCAR documentation.

Groups meet F-theory: Tate Models

Often, we seek elliptic fibration with singularities corresponding to a chosen gauge group G. Typically, this is easier with Tate models:

- Define Tate model similar to Weierstrass model, but use $a_i \in H^0(B, \overline{K}_B^{\otimes i})$ and

$$P_T = y^2 + a_1 xyz + a_3 yz^3 - x^3 - a_2 x^2 z^2 - a_4 xz^4 - a_6 z^6 \,.$$

- Recover Weierstrass model:

$$b_2 = 4a_2 + a_1^2 \,, \quad b_4 = 2a_4 + a_1 a_3 \,, \quad b_6 = 4a_6 + a_3^2 \,,$$
$$f = -\frac{b_2^2 - 24 b_4}{48} \,, \quad g = \frac{b_2^3 - 36 b_2 b_4 + 216 b_6}{864} \,, \quad P_W = y^2 - x^3 - fxz^4 - gz^6 \,.$$

(Conversely, expressing a Weierstrass model as a Tate model is generally possible only locally.)

  - Singularities appear at $\Delta = \{p \in B \mid 4f(p)^3 + 27g(p)^2 = 0\}$.
  - Gauge group set by vanishing orders of $(a_1, a_2, a_3, a_4, a_6)$ at Δ:

Tate table in `OSCAR` documentation.

Groups meet F-theory: An example of a Tate model [Krause Mayrhofer Weigand '11]

- Goal: Engineer an $SU(5)$ Tate model with singularity over $\{w = 0\} \subset B$.
  1. Look up vanishing orders from Tate table: $(0, 1, 2, 3, 5)$.
  2. Factor $a_i \in H^0(B, \overline{K}_B^{\otimes i})$ accordingly (assuming this is possible):

$$a_1 = a_1 \,, \quad a_2 = a_{2,1} w \,, \quad a_3 = a_{3,2} w^2 \,, \quad a_4 = a_{4,3} w^3 \,, \quad a_6 = a_{6,5} w^5 \,.$$

  $\Rightarrow$ Voila! Global Tate model with $SU(5)$ singularity over $\{w = 0\}$.

- Make a yet more special choice [Krause Mayrhofer Weigand '11]

$$a_1 = a_1 \,, \; a_2 = a_{2,1} w \,, \; a_3 = a_{3,2} w^2 \,, \; a_4 = a_{4,3} w^3 \,, \; a_6 \equiv 0 \,.$$

$\rightarrow$ Enhances gauge group to $SU(5) \times U(1)$ ($\leftrightarrow$ Mordell-Weil group of elliptic fibration).
$\rightarrow$ This will be our working example for most of this talk.

Groups meet F-theory: An OSCAR example

With OSCAR we create this $SU(5) \times U(1)$ global Tate model as follows:

```
base_ring, (a10, a21, a32, a43, a65, w) = QQ["a10", "a21", "a32", "a43", "a65", "w"]
base_grading = [1 2 3 4 6 0; 0 -1 -2 -3 -5 1]
a1 = a10
a2 = a21 * w
a3 = a32 * w^2
a4 = a43 * w^3
a6 = a65 * w^5
ais = [a1, a2, a3, a4, a6]
t = global_tate_model(base_ring, base_grading, 3, ais)
t2 = global_tate_model(base_ring, base_grading, 3, ais; toric_sample = false)
```

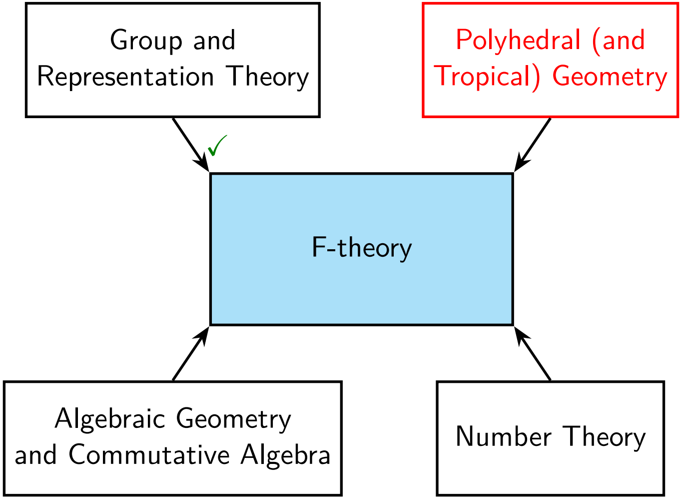
Groups meet F-theory: An OSCAR example

With OSCAR we create this $SU(5) \times U(1)$ global Tate model as follows:

```
base_ring, (a10, a21, a32, a43, a65, w) = QQ["a10", "a21", "a32", "a43", "a65", "w"]
base_grading = [1 2 3 4 6 0; 0 -1 -2 -3 -5 1]
a1 = a10
a2 = a21 * w
a3 = a32 * w^2
a4 = a43 * w^3
a6 = a65 * w^5
ais = [a1, a2, a3, a4, a6]
t = global_tate_model(base_ring, base_grading, 3, ais)
t2 = global_tate_model(base_ring, base_grading, 3, ais; toric_sample = false)
```

Note: For t we create a particular geometry as base space – a special **toric** space.

Toric geometry meets F-theory



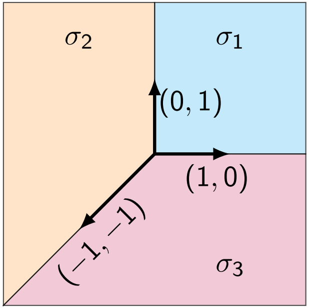
Toric geometry meets F-theory: An Overview [Cox, Little, Schenk '11]

A **toric** variety is an algebraic variety containing an **algebraic torus** $(\mathbb{C}^*)^n$ as an open dense subset, such that the action of the torus on itself extends to the whole variety.

Toric geometry meets F-theory: An Overview [Cox, Little, Schenk '11]

A **toric** variety is an algebraic variety containing an **algebraic torus** $(\mathbb{C}^*)^n$ as an open dense subset, such that the action of the torus on itself extends to the whole variety.

- **Key Insight**:
  - Defined by combinatorial data from convex polyhedral cones.
  - Analyzed with polyhedral geometry and combinatorics.

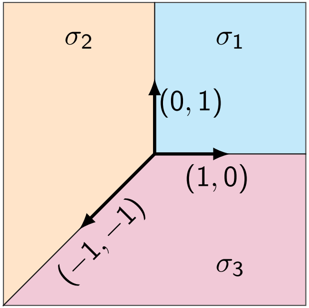


Fan of the 2-dimensional projective space $\mathbb{P}^2$ with three maximal cones σ_1, σ_2, and σ_3.

Toric geometry meets F-theory: An Overview [Cox, Little, Schenk '11]

A **toric** variety is an algebraic variety containing an **algebraic torus** $(\mathbb{C}^*)^n$ as an open dense subset, such that the action of the torus on itself extends to the whole variety.

- **Key Insight**:
  - Defined by combinatorial data from convex polyhedral cones.
  - Analyzed with polyhedral geometry and combinatorics.
- **Applications**:



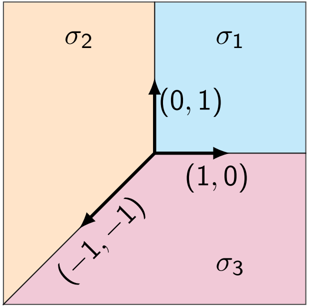
Fan of the 2-dimensional projective space $\mathbb{P}^2$

with three maximal cones σ_1, σ_2, and σ_3.

Toric geometry meets F-theory: An Overview [Cox, Little, Schenk '11]

A **toric** variety is an algebraic variety containing an **algebraic torus** $(\mathbb{C}^*)^n$ as an open dense subset, such that the action of the torus on itself extends to the whole variety.

- **Key Insight**:
  - Defined by combinatorial data from convex polyhedral cones.
  - Analyzed with polyhedral geometry and combinatorics.
- **Applications**:
  - **Algebraic Geometry**:
    - Intersection theory, Chow ring, sheaf cohomology and more.

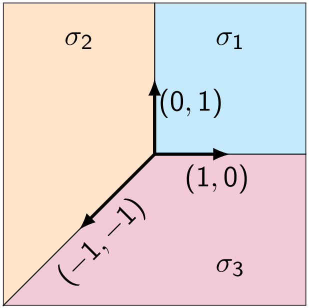


Fan of the 2-dimensional projective space $\mathbb{P}^2$ with three maximal cones σ_1, σ_2, and σ_3.

Toric geometry meets F-theory: An Overview [Cox, Little, Schenk '11]

A **toric** variety is an algebraic variety containing an **algebraic torus** $(\mathbb{C}^*)^n$ as an open dense subset, such that the action of the torus on itself extends to the whole variety.

- **Key Insight**:
  - Defined by combinatorial data from convex polyhedral cones.
  - Analyzed with polyhedral geometry and combinatorics.
- **Applications**:
  - **Algebraic Geometry**:
    - Intersection theory, Chow ring, sheaf cohomology and more.
  - **Mirror Symmetry**: [Cox, Katz '99]
    - Deep connection in string theory and algebraic geometry.
    - Relies of reflexive lattice polytope and its polar dual.

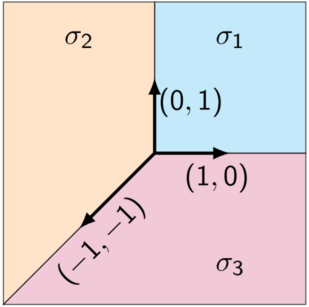


Fan of the 2-dimensional projective space $\mathbb{P}^2$ with three maximal cones σ_1, σ_2, and σ_3.

Toric geometry meets F-theory: An Overview [Cox, Little, Schenk '11]

A **toric** variety is an algebraic variety containing an **algebraic torus** $(\mathbb{C}^*)^n$ as an open dense subset, such that the action of the torus on itself extends to the whole variety.

- **Key Insight**:
  - Defined by combinatorial data from convex polyhedral cones.
  - Analyzed with polyhedral geometry and combinatorics.
- **Applications**:
  - **Algebraic Geometry**:
    - ★ Intersection theory, Chow ring, sheaf cohomology and more.
  - **Mirror Symmetry**: [Cox, Katz '99]
    - ★ Deep connection in string theory and algebraic geometry.
    - ★ Relies of reflexive lattice polytope and its polar dual.
  - **Geometric Modeling**:
    - ★ Useful in computer-aided geometric design.
    - ★ Foundation for many `OSCAR` constructions of F-theory models.

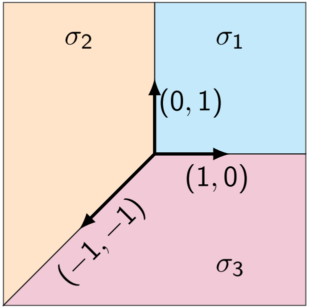


Fan of the 2-dimensional projective space $\mathbb{P}^2$ with three maximal cones σ_1, σ_2, and σ_3.

Toric geometry meets F-theory: An Overview [Cox, Little, Schenk '11]

A **toric** variety is an algebraic variety containing an **algebraic torus** $(\mathbb{C}^*)^n$ as an open dense subset, such that the action of the torus on itself extends to the whole variety.

- **Key Insight**:
  - Defined by combinatorial data from convex polyhedral cones.
  - Analyzed with polyhedral geometry and combinatorics.
- **Applications**:
  - **Algebraic Geometry**:
    - ★ Intersection theory, Chow ring, sheaf cohomology and more.
  - **Mirror Symmetry**: [Cox, Katz '99]
    - ★ Deep connection in string theory and algebraic geometry.
    - ★ Relies of reflexive lattice polytope and its polar dual.
  - **Geometric Modeling**:
    - ★ Useful in computer-aided geometric design.
    - ★ Foundation for many OSCAR constructions of F-theory models.



Fan of the 2-dimensional projective space $\mathbb{P}^2$ with three maximal cones σ_1, σ_2, and σ_3.

See also M.B. & L. Kastner, Toric Geometry in OSCAR, *ComputerAlgebraRundbrief #72*, 2023.

Toric geometry meets F-theory: Mirror Symmetry and F-theory QSMs

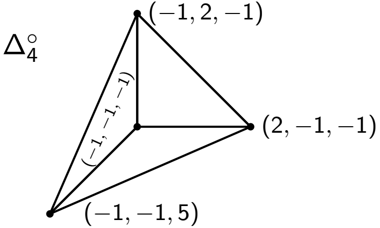
- **Mirror Symmetry**: Relates physics/geometry of pairs of Calabi-Yau manifolds.

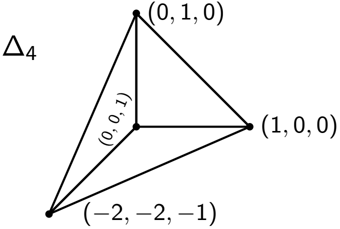
Toric geometry meets F-theory: Mirror Symmetry and F-theory QSMs

- **Mirror Symmetry**: Relates physics/geometry of pairs of Calabi-Yau manifolds.
- **Polar dual**: Many constructions use toric geometry. Via triangulation, a toric variety is defined by a reflexive lattice polytope $P \in \mathbb{R}^d$. The mirror is the polar dual polytope P° with $\langle P, P^\circ \rangle \geq -1$.

Toric geometry meets F-theory: Mirror Symmetry and F-theory QSMs

- **Mirror Symmetry**: Relates physics/geometry of pairs of Calabi-Yau manifolds.
- **Polar dual**: Many constructions use toric geometry. Via triangulation, a toric variety is defined by a reflexive lattice polytope $P \in \mathbb{R}^d$. The mirror is the polar dual polytope P° with $\langle P, P^\circ \rangle \geq -1$.

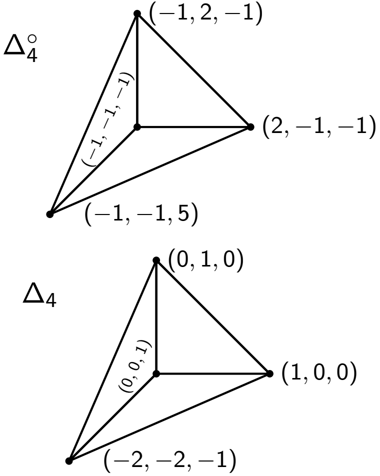




Toric geometry meets F-theory: Mirror Symmetry and F-theory QSMs

- **Mirror Symmetry**: Relates physics/geometry of pairs of Calabi-Yau manifolds.
- **Polar dual**: Many constructions use toric geometry. Via triangulation, a toric variety is defined by a reflexive lattice polytope $P \in \mathbb{R}^d$. The mirror is the polar dual polytope P° with $\langle P, P^\circ \rangle \geq -1$.
- **Kreuzer-Skarke list**: Complete list of all 3- and 4-dim. reflexive lattice polytopes.

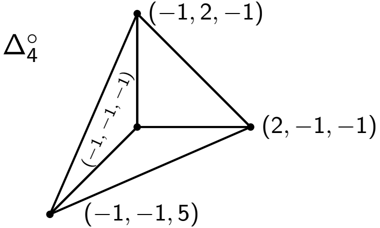
[Kreuzer, Skarke '98], [Kreuzer, Skarke '00]

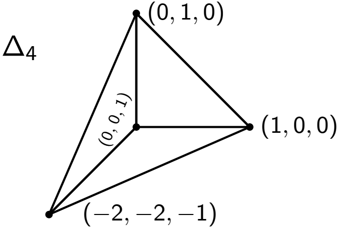


Toric geometry meets F-theory: Mirror Symmetry and F-theory QSMs

- **Mirror Symmetry**: Relates physics/geometry of pairs of Calabi-Yau manifolds.
- **Polar dual**: Many constructions use toric geometry. Via triangulation, a toric variety is defined by a reflexive lattice polytope $P \in \mathbb{R}^d$. The mirror is the polar dual polytope P° with $\langle P, P^\circ \rangle \geq -1$.
- **Kreuzer-Skarke list**: Complete list of all 3- and 4-dim. reflexive lattice polytopes.

  [Kreuzer, Skarke '98], [Kreuzer, Skarke '00]
- **F-theory QSMs** [Cvetič Halverson Ling Liu Tian '19]: $\mathcal{O}(10^{15})$ F-theory solutions with attractive pyhsics features.
  - Based on 708 3-dim. reflexive polytopes.
  - Triangulation yields $\mathcal{O}(10^{15})$ different toric spaces.





Toric geometry meets F-theory: Finding Bases for F-theory Models

- F-theory models often explore fibrations over **families of bases**.
- Identifying a specific base geometry aids refined studies.

Toric geometry meets F-theory: Finding Bases for F-theory Models

- F-theory models often explore fibrations over **families of bases**.
- Identifying a specific base geometry aids refined studies.
- We approximate a “generic” member of the family with toric geometry:

```
base_ring, (a10, a21, a32, a43, a65, w) = QQ["a10", "a21", "a32", "a43", "a65", "w"]
base_grading = [1 2 3 4 6 0; 0 -1 -2 -3 -5 1]
a1 = a10
a2 = a21 * w
a3 = a32 * w^2
a4 = a43 * w^3
a6 = a65 * w^5
ais = [a1, a2, a3, a4, a6]
t = global_tate_model(base_ring, base_grading, 3, ais)
```

Toric geometry meets F-theory: Finding Bases for F-theory Models

- F-theory models often explore fibrations over **families of bases**.
- Identifying a specific base geometry aids refined studies.
- We approximate a "generic" member of the family with toric geometry:

```
base_ring, (a10, a21, a32, a43, a65, w) = QQ["a10", "a21", "a32", "a43", "a65", "w"]
base_grading = [1 2 3 4 6 0; 0 -1 -2 -3 -5 1]
a1 = a10
a2 = a21 * w
a3 = a32 * w^2
a4 = a43 * w^3
a6 = a65 * w^5
ais = [a1, a2, a3, a4, a6]
t = global_tate_model(base_ring, base_grading, 3, ais)
```

⇒ **Task:** Find a 3-dim. toric variety without torus factor and such that its Cox ring $S = \mathbb{Q}[a_{10}, a_{21}, a_{32}, a_{43}, a_{65}, w,]$ is graded by `base_grading`.

Toric geometry meets F-theory: Bases for F-theory tools II

```
julia> cox_ring(base_space(t))
Multivariate polynomial ring in 6 variables over QQ graded by
  a10 -> [1 0]
  a21 -> [2 -1]
  a32 -> [3 -2]
  a43 -> [4 -3]
  a65 -> [6 -5]
  w -> [0 1]

julia> stanley_reisner_ideal(base_space(t))
ideal(a32*a43*a65, a10*a21*w, a21*a43*a65*w, a21*a32*a65*w, a21*a32*a43*w, a10*a32*a43*w,
          a10*a21*a43*a65, a10*a21*a32*a65, a10*a21*a32*a43, a10*a43*a65*w, a10*a32*a65*w)
```

Toric geometry meets F-theory: Bases for F-theory tools II

```
julia> cox_ring(base_space(t))
Multivariate polynomial ring in 6 variables over QQ graded by
  a10 -> [1 0]
  a21 -> [2 -1]
  a32 -> [3 -2]
  a43 -> [4 -3]
  a65 -> [6 -5]
  w -> [0 1]

julia> stanley_reisner_ideal(base_space(t))
ideal(a32*a43*a65, a10*a21*w, a21*a43*a65*w, a21*a32*a65*w, a21*a32*a43*w, a10*a32*a43*w,
          a10*a21*a43*a65, a10*a21*a32*a65, a10*a21*a32*a43, a10*a43*a65*w, a10*a32*a65*w)
```

- **Stricter rule:** To replicate literature results, base must "behave" like 3-dim. affine space.

Toric geometry meets F-theory: Bases for F-theory tools II

```
julia> cox_ring(base_space(t))
Multivariate polynomial ring in 6 variables over QQ graded by
  a10 -> [1 0]
  a21 -> [2 -1]
  a32 -> [3 -2]
  a43 -> [4 -3]
  a65 -> [6 -5]
  w -> [0 1]

julia> stanley_reisner_ideal(base_space(t))
ideal(a32*a43*a65, a10*a21*w, a21*a43*a65*w, a21*a32*a65*w, a21*a32*a43*w, a10*a32*a43*w,
          a10*a21*a43*a65, a10*a21*a32*a65, a10*a21*a32*a43, a10*a43*a65*w, a10*a32*a65*w)
```

- **Stricter rule:** To replicate literature results, base must "behave" like 3-dim. affine space.
- **Issue:** The generators $a_{32}a_{43}a_{65}$ and $a_{10}a_{21}w$ of the Stanley-Reisner ideal conflict.

Toric geometry meets F-theory: Bases for F-theory tools II

```
julia> cox_ring(base_space(t))
Multivariate polynomial ring in 6 variables over QQ graded by
  a10 -> [1 0]
  a21 -> [2 -1]
  a32 -> [3 -2]
  a43 -> [4 -3]
  a65 -> [6 -5]
  w -> [0 1]

julia> stanley_reisner_ideal(base_space(t))
ideal(a32*a43*a65, a10*a21*w, a21*a43*a65*w, a21*a32*a65*w, a21*a32*a43*w, a10*a32*a43*w,
          a10*a21*a43*a65, a10*a21*a32*a65, a10*a21*a32*a43, a10*a43*a65*w, a10*a32*a65*w)
```

- **Stricter rule:** To replicate literature results, base must “behave” like 3-dim. affine space.
- **Issue:** The generators $a_{32} a_{43} a_{65}$ and $a_{10} a_{21} w$ of the Stanley-Reisner ideal conflict.

$\Rightarrow$ Need for geometries beyond the toric regime?

Toric geometry meets F-theory: (Toric) Resolutions (More details: [Weigand '18])

- Recall: Starting point in F-theory is a **singular** elliptic fibration.
  - It is hard off the physics from the singular geometry.
  - **Common approach:** Resolve the singularities & study the smoothed-out geometry.

Toric geometry meets F-theory: (Toric) Resolutions (More details: [Weigand '18])

- Recall: Starting point in F-theory is a **singular** elliptic fibration.
  - It is hard off the physics from the singular geometry.
  - **Common approach:** Resolve the singularities & study the smoothed-out geometry.
- Key demand: We seek a **crepant** resolution.
  - As long as elliptic fibration remains Calabi-Yau ($\leftrightarrow$ crepant), physics remains "similar".
  - Crepancy largely ignored in resolution algorithms.
  - Crepancy prevents resolution of $\mathbb{Q}$-factorial terminal singularities.

Toric geometry meets F-theory: (Toric) Resolutions (More details: [Weigand '18])

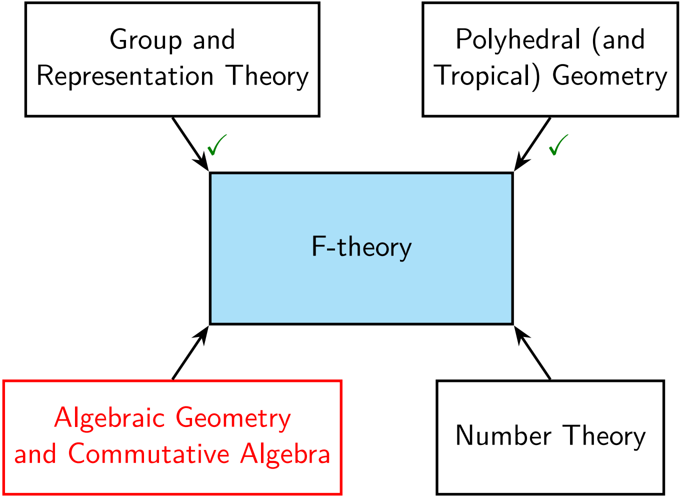
- Recall: Starting point in F-theory is a **singular** elliptic fibration.
  - It is hard off the physics from the singular geometry.
  - **Common approach:** Resolve the singularities & study the smoothed-out geometry.
- Key demand: We seek a **crepant** resolution.
  - As long as elliptic fibration remains Calabi-Yau ($\leftrightarrow$ crepant), physics remains "similar".
  - Crepancy largely ignored in resolution algorithms.
  - Crepancy prevents resolution of $\mathbb{Q}$-factorial terminal singularities.
- Vanilla sceanrio: Sequence of toric blowups is sufficient.
  - The $SU(5) \times U(1)$ global Tate model discussed before has this feature. [Krause Mayrhofer Weigand '11]
  - Idea: Set up a database for such findings? Already in place in `OSCAR`!

Toric geometry meets F-theory: Blowups II

```
julia> m = literature_model(arxiv_id = "1109.3454", equation = "3.1");

julia> cox_ring(resolve(m, 1))
Multivariate polynomial ring in 13 variables over QQ graded by
  a1 -> [1 0 0 0 0 0 0 0]
  a21 -> [0 1 0 0 0 0 0 0]
  a32 -> [-1 2 0 0 0 0 0 0]
  a43 -> [-2 3 0 0 0 0 0 0]
  w -> [0 0 1 0 0 0 0 0]
  x -> [0 0 0 1 0 0 0 0]
  y -> [0 0 0 0 1 0 0 0]
  z -> [0 0 0 0 0 1 0 0]
  e1 -> [0 0 0 0 0 0 1 0]
  e4 -> [0 0 0 0 0 0 0 1]
  e2 -> [1 -1 -1 -1 1 -1 -1 0]
  e3 -> [1 0 0 1 -1 1 0 -1]
  s -> [-2 2 2 -1 0 2 1 1]
```

Algebraic geometry meets F-theory



Algebraic Geometry meets F-theory: Desires and Attempts

- Desires:
  - Represent and analyze non-toric generic members of base families.
  - Cover significant fraction of non-toric solutions to F-theory.
  - Incorporate crepant resolution techniques, even if they exceed the toric scope.

(E.g. [Arena Jefferson Obinna '23])

$\Rightarrow$ WIP with Anne Frühbis-Krüger, Andrew Turner, Matthias Zach.

Algebraic Geometry meets F-theory: Desires and Attempts

- Desires:
  - Represent and analyze non-toric generic members of base families.
  - Cover significant fraction of non-toric solutions to F-theory.
  - Incorporate crepant resolution techniques, even if they exceed the toric scope.

(E.g. [Arena Jefferson Obinna '23])

- Current work/ideas:
  - Toric varieties should benefit from the functionality schemes offer.
  - F-theory tools should accepts schemes as base.
  - Study elliptic fibrations over family of bases $\rightarrow$ "Computational base moduli space"
  - Models over "arbitrary" base must be evaluable at concrete base, be it toric or a scheme.

$\Rightarrow$ WIP with Anne Frühbis-Krüger, Andrew Turner, Matthias Zach.

Algebraic Geometry meets F-theory: Desires and Attempts

- Desires:
  - Represent and analyze non-toric generic members of base families.
  - Cover significant fraction of non-toric solutions to F-theory.
  - Incorporate crepant resolution techniques, even if they exceed the toric scope.

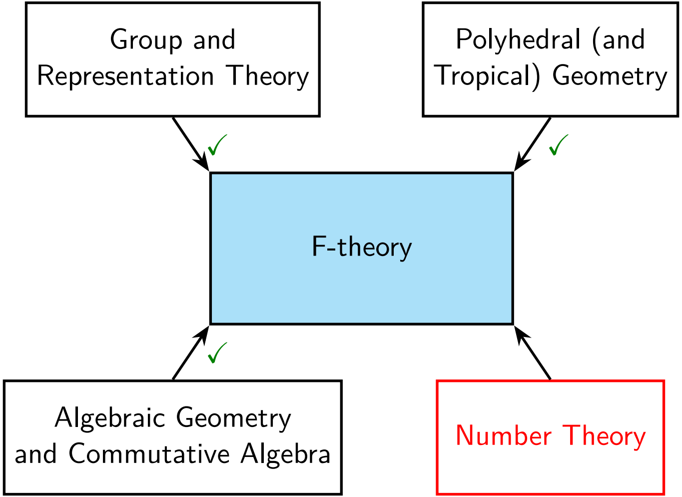
(E.g. [Arena Jefferson Obinna '23])

- Current work/ideas:
  - Toric varieties should benefit from the functionality schemes offer.
  - F-theory tools should accepts schemes as base.
  - Study elliptic fibrations over family of bases $\rightarrow$ "Computational base moduli space"
  - Models over "arbitrary" base must be evaluable at concrete base, be it toric or a scheme.

$\Rightarrow$ WIP with Anne Frühbis-Krüger, Andrew Turner, Matthias Zach.

$\Rightarrow$ Eventually, back to decipher the physics encoded in those geometries.

Number theory touches F-theory



Number Theory touches F-theory: Why root bundles on nodal curves?

- **F-theory QSMs:** 10^{15} solutions with promising relations to particle physics.

Number Theory touches F-theory: Why root bundles on nodal curves?

- **F-theory QSMs:** 10^{15} solutions with promising relations to particle physics.
- Missing information: Count of Higgs pairs ($\leftrightarrow$ Physics Nobel Prize).

Number Theory touches F-theory: Why root bundles on nodal curves?

- **F-theory QSMs:** 10^{15} solutions with promising relations to particle physics.
- Missing information: Count of Higgs pairs ($\leftrightarrow$ Physics Nobel Prize).
- Detailed study in recent years.

[M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22], [M.B. '23], [M.B. Cvetič Donagi Liu Ong '23]

Number Theory touches F-theory: Why root bundles on nodal curves?

- **F-theory QSMs:** 10^{15} solutions with promising relations to particle physics.
- Missing information: Count of Higgs pairs ($\leftrightarrow$ Physics Nobel Prize).
- Detailed study in recent years.

[M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22], [M.B. '23], [M.B. Cvetič Donagi Liu Ong '23]

- Leads to **root bundles on nodal curves**:
  - Each QSM has canonical, nodal curve $C^\bullet$. (Origin: Toric K3 desingularizations. Locally, Nodal singularity: $\{x \cdot y = 0\}$.)
  - Physics **should** pick $P^\bullet \in \mathrm{Pic}(C^\bullet)$ s.t. $h^0(C^\bullet, P^\bullet)$ is Higgs pair count.
  - Current understanding: $r \in \mathbb{Z}_{\geq 2}$, $E^\bullet \in \mathrm{Pic}(C^\bullet)$ set by F-theory geometry, s.t.

$$r \cdot P^\bullet = E\,.$$

Gives r^{2g} candidates for $P^\bullet$ (g = arithmetic genus of $C^\bullet$).

Number Theory touches F-theory: Why root bundles on nodal curves?

- **F-theory QSMs:** 10^{15} solutions with promising relations to particle physics.
- Missing information: Count of Higgs pairs ($\leftrightarrow$ Physics Nobel Prize).
- Detailed study in recent years.

[M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22], [M.B. '23], [M.B. Cvetič Donagi Liu Ong '23]

- Leads to **root bundles on nodal curves**:
  - Each QSM has canonical, nodal curve $C^\bullet$. (Origin: Toric K3 desingularizations. Locally, Nodal singularity: $\{x \cdot y = 0\}$.)
  - Physics **should** pick $P^\bullet \in \mathrm{Pic}(C^\bullet)$ s.t. $h^0(C^\bullet, P^\bullet)$ is Higgs pair count.
  - Current understanding: $r \in \mathbb{Z}_{\geq 2}$, $E^\bullet \in \mathrm{Pic}(C^\bullet)$ set by F-theory geometry, s.t.

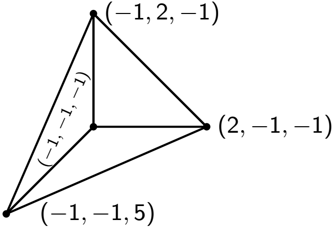
$$r \cdot P^\bullet = E\,.$$

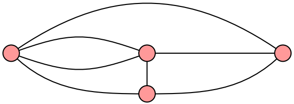
Gives r^{2g} candidates for $P^\bullet$ ($g =$ arithmetic genus of $C^\bullet$).

$\Rightarrow$ Determine $h^0(C^\bullet, P^\bullet)$ for all r^{2g} candidates $P^\bullet$ to gain insights into F-theory QSM.

Number Theory touches F-theory: An Example of Brill-Noether Numbers

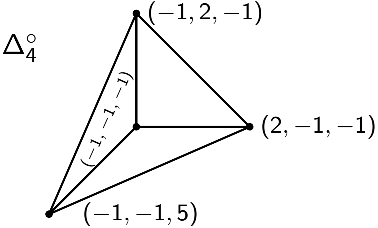
Δ_4°

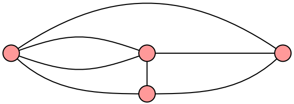




- Goal: Enumerate all 12^8 solutions $P^\bullet$ to $12P^\bullet = 12K_{C^\bullet}$ and find their $h^0(C^\bullet, P^\bullet)$.

Number Theory touches F-theory: An Example of Brill-Noether Numbers





- Goal: Enumerate all 12^8 solutions $P^\bullet$ to $12P^\bullet = 12K_{C^\bullet}$ and find their $h^0(C^\bullet, P^\bullet)$.
- Results: (based on [Caporaso Casagrande Cornalba '07], but significantly extended)
  - '21 Update: 53.6% of 12^8 roots had 3 global sections. (All other roots untouched.)
  - '23 Update: $h^0(C^\bullet, P^\bullet) = 4$ for 12^4 roots, and the rest has 3 sections.

Roots Count	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$
12^8	$12^4 \cdot \left(12^4 - 1\right)$	0	12^4	0

  - $h^0(C^\bullet, P^\bullet)$ may depend on finer than combinatorial data (descent data). For such cases we compute an optimal lower bound an list them in every 2nd column.

Number Theory touches F-theory: Brill-Noether numbers [M.B. Cvetič Liu '21]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ_8°	57.3	?	?	?	?	?	?	?

Number Theory touches F-theory: Brill-Noether numbers [M.B. Cvetič Liu '21]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ°_{8}	57.3	?	?	?	?	?	?	?
Δ°_{4}	53.6	?	?	?	?	?	?	?
Δ°_{134}	48.7	?	?	?	?	?	?	?
Δ°_{128}, Δ°_{130}, Δ°_{136}, Δ°_{236}	42.0	?	?	?	?	?	?	?
Δ°_{88}	61.1	?	?	?	?	?	?	?
Δ°_{110}	57.8	?	?	?	?	?	?	?
Δ°_{272}, Δ°_{274}	57.5	?	?	?	?	?	?	?
Δ°_{387}	57.3	?	?	?	?	?	?	?
Δ°_{798}, Δ°_{808}, Δ°_{810}, Δ°_{812}	54.0	?	?	?	?	?	?	?
Δ°_{254}	54.7	?	?	?	?	?	?	?
Δ°_{52}	54.7	?	?	?	?	?	?	?
Δ°_{302}	54.7	?	?	?	?	?	?	?
Δ°_{786}	51.3	?	?	?	?	?	?	?
Δ°_{762}	51.3	?	?	?	?	?	?	?
Δ°_{417}	51.3	?	?	?	?	?	?	?
Δ°_{838}	51.3	?	?	?	?	?	?	?
Δ°_{782}	51.3	?	?	?	?	?	?	?
Δ°_{377}, Δ°_{499}, Δ°_{503}	48.2	?	?	?	?	?	?	?
Δ°_{1348}	48.2	?	?	?	?	?	?	?
Δ°_{882}, Δ°_{856}	48.2	?	?	?	?	?	?	?
Δ°_{1340}	45.2	?	?	?	?	?	?	?
Δ°_{1879}	45.2	?	?	?	?	?	?	?
Δ°_{1384}	42.5	?	?	?	?	?	?	?

Number Theory touches F-theory: Brill-Noether numbers [M.B. Cvetič Donagi Ong '22]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ°_{8}	76.4	23.6						
Δ°_{4}	99.0	1.0						
Δ°_{134}	99.8	0.2						
Δ°_{128}, Δ°_{130}, Δ°_{136}, Δ°_{236}	99.9	0.1						
Δ°_{88}	74.9	22.1	2.5	0.5	0.0	0.0		
Δ°_{110}	82.4	14.1	3.1	0.4	0.0			
Δ°_{272}, Δ°_{274}	78.1	18.0	3.4	0.5	0.0	0.0		
Δ°_{387}	73.8	21.9	3.5	0.7	0.0	0.0		
Δ°_{798}, Δ°_{808}, Δ°_{810}, Δ°_{812}	77.0	17.9	4.4	0.7	0.0	0.0		
Δ°_{254}	95.9	0.5	3.5	0.0	0.0	0.0		
Δ°_{52}	95.3	0.7	3.9	0.0	0.0	0.0		
Δ°_{302}	95.9	0.5	3.5	0.0	0.0			
Δ°_{786}	94.8	0.3	4.8	0.0	0.0	0.0		
Δ°_{762}	94.8	0.3	4.9	0.0	0.0	0.0		
Δ°_{417}	94.8	0.3	4.8	0.0	0.0	0.0	0.0	
Δ°_{838}	94.7	0.3	5.0	0.0	0.0	0.0		
Δ°_{782}	94.6	0.3	5.0	0.0	0.0	0.0		
Δ°_{377}, Δ°_{499}, Δ°_{503}	93.4	0.2	6.2	0.0	0.1	0.0		
Δ°_{1348}	93.7	0.0	6.2	0.0	0.1		0.0	
Δ°_{882}, Δ°_{856}	93.4	0.3	6.2	0.0	0.1	0.0	0.0	
Δ°_{1340}	92.3	0.0	7.6	0.0	0.1		0.0	
Δ°_{1879}	92.3	0.0	7.5	0.0	0.1		0.0	
Δ°_{1384}	90.9	0.0	8.9	0.0	0.2		0.0	

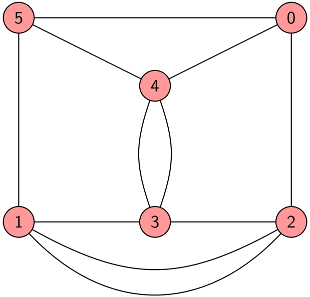
Number Theory touches F-theory: Brill-Noether numbers [M.B. Cvetič Donagi Ong '23]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ_8°	99.9421		0.0579					
Δ_4°	99.9952		0.0048					
Δ_{134}°	99.9952		0.0048					
Δ_{128}°, Δ_{130}°, Δ_{136}°, Δ_{236}°	99.9952		0.0048					
Δ_{88}°	96.6700	0.3361	2.9850		0.0089			
Δ_{110}°	95.6268	0.8372	3.5179	0.0050	0.0131			
Δ_{272}°, Δ_{274}°	95.5097	0.5155	3.9552	0.0016	0.0180			
Δ_{387}°	95.1923	0.4981	4.2773		0.0323			
Δ_{798}°, Δ_{808}°, Δ_{810}°, Δ_{812}°	93.8268	0.8795	5.2390	0.0029	0.0518			
Δ_{254}°	96.3942	0.0687	3.5193	0.0003	0.0175			
Δ_{52}°	96.0587	0.0171	3.9066	0.0000	0.0176			
Δ_{302}°	96.3960	0.0636	3.5222	0.0001	0.0181			
Δ_{786}°	95.0714	0.0393	4.8466	0.0002	0.0425			
Δ_{762}°	95.0167	0.0369	4.9052	0.0005	0.0407			
Δ_{417}°	95.0745	0.0433	4.8389	0.0003	0.0429		0.0001	
Δ_{838}°	94.9092	0.0215	5.0216	0.0000	0.0477			
Δ_{782}°	94.9019	0.0161	5.0359	0.0000	0.0461			
Δ_{377}°, Δ_{499}°, Δ_{503}°	93.6500	0.0347	6.2312	0.0005	0.0836			
Δ_{1348}°	93.7075	0.0112	6.1978	0.0001	0.0833		0.0001	
Δ_{882}°, Δ_{856}°	93.6546	0.0425	6.2190	0.0009	0.0825		0.0005	
Δ_{1340}°	92.2989	0.0064	7.5515	0.0001	0.1427		0.0004	
Δ_{1879}°	92.3015	0.0108	7.5447	0.0002	0.1421		0.0007	
Δ_{1384}°	90.8524	0.0031	8.9219	0.0001	0.2213		0.0012	

Number Theory touches F-theory: Brill-Noether numbers [M.B. Cvetič Donagi Ong '23]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ°_8	99.9421		0.0579					
Δ°_4	99.9952		0.0048					
Δ°_{134}	99.9952		0.0048					
Δ°_{128}, Δ°_{130}, Δ°_{136}, Δ°_{236}	99.9952		0.0048					
Δ°_{88}	96.6700	0.3361	2.9850		0.0089			
Δ°_{110}	95.6268	0.8372	3.5179	0.0050	0.0131			
Δ°_{272}, Δ°_{274}	95.5097	0.5155	3.9552	0.0016	0.0180			
Δ°_{387}	95.1923	0.4981	4.2773		0.0323			
Δ°_{798}, Δ°_{808}, Δ°_{810}, Δ°_{812}	93.8268	0.8795	5.2390	0.0029	0.0518			
Δ°_{254}	96.3942	0.0687	3.5193	0.0003	0.0175			
Δ°_{52}	96.0587	0.0171	3.9066	0.0000	0.0176			
Δ°_{302}	96.3960	0.0636	3.5222	0.0001	0.0181			
Δ°_{786}	95.0714	0.0393	4.8466	0.0002	0.0425			
Δ°_{762}	95.0167	0.0369	4.9052	0.0005	0.0407			
Δ°_{417}	95.0745	0.0433	4.8389	0.0003	0.0429		0.0001	
Δ°_{838}	94.9092	0.0215	5.0216	0.0000	0.0477			
Δ°_{782}	94.9019	0.0161	5.0359	0.0000	0.0461			
Δ°_{377}, Δ°_{499}, Δ°_{503}	93.6500	0.0347	6.2312	0.0005	0.0836			
Δ°_{1348}	93.7075	0.0112	6.1978	0.0001	0.0833		0.0001	
Δ°_{882}, Δ°_{856}	93.6546	0.0425	6.2190	0.0009	0.0825		0.0005	
Δ°_{1340}	92.2989	0.0064	7.5515	0.0001	0.1427		0.0004	
Δ°_{1879}	92.3015	0.0108	7.5447	0.0002	0.1421		0.0007	
Δ°_{1384}	90.8524	0.0031	8.9219	0.0001	0.2213		0.0012	

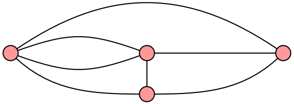
Number Theory touches F-theory: Dual Graph Δ°_{254}



$$20P^{\bullet} = 16K_{C^{\bullet}}$$

Number Theory touches F-theory: Summary and outlook

- An F-theory inspired cryptosystem?
  - Task: For a given integer partition, find graph and root constraint, s.t. the ensuing Brill-Noether numbers match the given partition.
  - Example: $12^8 = 12^4 \cdot \left(12^4 - 1\right) + 0 + 12^4 + 0$ (public key) leads to (private key)

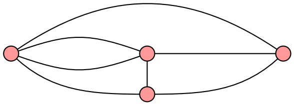


$$12P^\bullet = 12K_{C^\bullet}$$

  - Daunting task! Inverse possible with tailor made software `RootCounter`.

Number Theory touches F-theory: Summary and outlook

- An F-theory inspired cryptosystem?
  - Task: For a given integer partition, find graph and root constraint, s.t. the ensuing Brill-Noether numbers match the given partition.
  - Example: $12^8 = 12^4 \cdot \left(12^4 - 1\right) + 0 + 12^4 + 0$ (public key) leads to (private key)



$$12 P^\bullet = 12 K_{C^\bullet}$$

  - Daunting task! Inverse possible with tailor made software `RootCounter`.
- Connection between graphs and Brill-Noether numbers begs to be investigated.
  - Could benefit from machine learning tools and analytic/algebraic insights.
  - Once systematics clear, we can apply this to complex curves, extending previous work.

  $\Rightarrow$ Better approximation of F-theory QSMs' Higgs counts (& vector-like spectra).

Summary: F-TheoryTools – Summary and outlook

- Initial challenge: **Crepant** resolution of singular elliptic fibration.
  Crepant prevents resolving $\mathbb{Q}$-factorial terminal singularities. Currently, apply entire F-theory toolkit and assume non-resolvability should those standard methods fail.

Summary: `F-TheoryTools` – Summary and outlook

- Initial challenge: **Crepant** resolution of singular elliptic fibration.
  Crepant prevents resolving $\mathbb{Q}$-factorial terminal singularities. Currently, apply entire F-theory toolkit and assume non-resolvability should those standard methods fail.
- Anticipated key features:
  1. State-of-the-art resolution techniques, in particular beyond toric blowups.
  2. Database for utilizing established literature constructions.

Summary: F-TheoryTools – Summary and outlook

- Initial challenge: **Crepant** resolution of singular elliptic fibration.
  Crepant prevents resolving $\mathbb{Q}$-factorial terminal singularities. Currently, apply entire F-theory toolkit and assume non-resolvability should those standard methods fail.
- Anticipated key features:
  1. State-of-the-art resolution techniques, in particular beyond toric blowups.
  2. Database for utilizing established literature constructions.
- Post-2024 objective: Automate extracting physics from $\widehat{Y}_4$.
  - Topological intersection numbers ($\leftrightarrow$ chiral spectrum).
  - Intersection theory in the Chow ring ($\leftrightarrow$ vector-like spectra).
  - Nodal curves and Brill-Noether numbers ($\leftrightarrow$ approx. Higgs counts for F-theory QSMs).
  - Learn (some features) of the Mordell-Weil group ($\leftrightarrow$ Abelian gauge factors).
  - Include powerful established techniques from the F-theory community.

(... [Ling Weigand '16], [Jefferson Taylor Turner '21], [Jefferson Turner '22], ...)

Summary: `F-TheoryTools` – Summary and outlook

- Initial challenge: **Crepant** resolution of singular elliptic fibration.
  Crepant prevents resolving $\mathbb{Q}$-factorial terminal singularities. Currently, apply entire F-theory toolkit and assume non-resolvability should those standard methods fail.
- Anticipated key features:
  1. State-of-the-art resolution techniques, in particular beyond toric blowups.
  2. Database for utilizing established literature constructions.
- Post-2024 objective: Automate extracting physics from $\widehat{Y}_4$.
  - Topological intersection numbers ($\leftrightarrow$ chiral spectrum).
  - Intersection theory in the Chow ring ($\leftrightarrow$ vector-like spectra).
  - Nodal curves and Brill-Noether numbers ($\leftrightarrow$ approx. Higgs counts for F-theory QSMs).
  - Learn (some features) of the Mordell-Weil group ($\leftrightarrow$ Abelian gauge factors).
  - Include powerful established techniques from the F-theory community.

(... [Ling Weigand '16], [Jefferson Taylor Turner '21], [Jefferson Turner '22], ...)

$\Rightarrow$ Numerous exciting applications by extending and streamlining standard processes!

Summary: `F-TheoryTools` – Summary and outlook

- Initial challenge: **Crepant** resolution of singular elliptic fibration.
  Crepant prevents resolving $\mathbb{Q}$-factorial terminal singularities. Currently, apply entire F-theory toolkit and assume non-resolvability should those standard methods fail.
- Anticipated key features:
  1. State-of-the-art resolution techniques, in particular beyond toric blowups.
  2. Database for utilizing established literature constructions.
- Post-2024 objective: Automate extracting physics from $\widehat{Y}_4$.
  - Topological intersection numbers ($\leftrightarrow$ chiral spectrum).
  - Intersection theory in the Chow ring ($\leftrightarrow$ vector-like spectra).
  - Nodal curves and Brill-Noether numbers ($\leftrightarrow$ approx. Higgs counts for F-theory QSMs).
  - Learn (some features) of the Mordell-Weil group ($\leftrightarrow$ Abelian gauge factors).
  - Include powerful established techniques from the F-theory community.

(... [Ling Weigand '16], [Jefferson Taylor Turner '21], [Jefferson Turner '22], ...)

$\Rightarrow$ Numerous exciting applications by extending and streamlining standard processes!

Thank you for your attention!