F-Theory and Singular Elliptic Fibrations

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Universitaet des Saarlandes
May 16, 2023

Outline

Presentation based on work with . . .

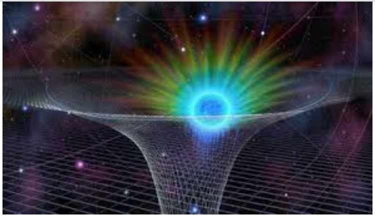
- OSCAR computer algebra: (https://www.oscar-system.org/)
  - L. Kastner: Toric geometry in OSCAR (2303.08110)
  - A. P. Turner, M. Zach, A. Frühbis-Krüger: FTheoryTools (work in progress)
- M. Cvetič, R. Donagi, M. Ong: Vector-like spectra

  (2007.00009, 2102.10115, 2104.08297, 2205.00008, 2303.08144, and work in progress)

Outline

- Brief introduction to String and F-Theory.
- The physics of resolved singularities and FTheoryTools.
- Vector-like spectra in F-Theory.

String theory = General relativity + Standard Model?



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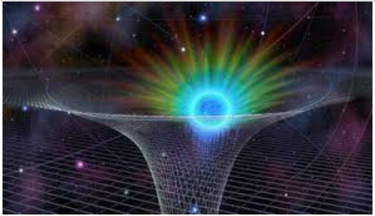

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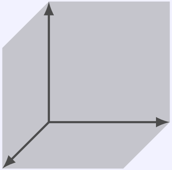

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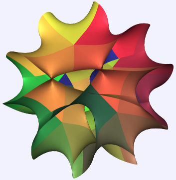
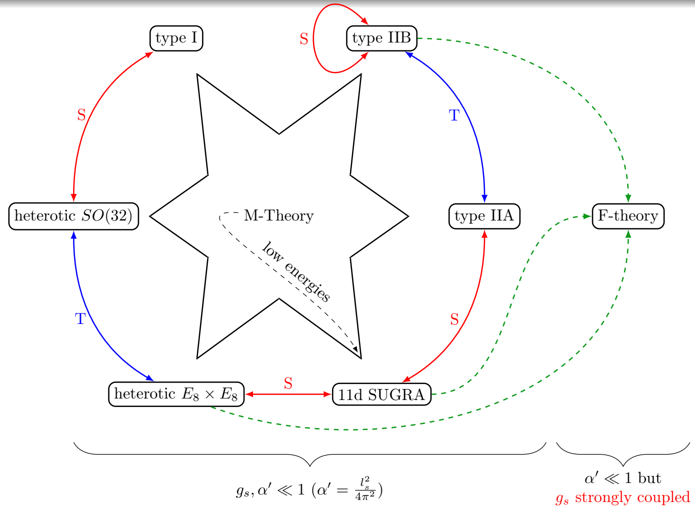


4-dim. world $\mathcal{W}$ ‘small‘ 6 real-dim. manifold $\mathcal{B}_3$

Challenge: Find $\mathcal{B}_3$ s.t. ST reproduces 4d physics.

Different types of String theory



Type IIB string theory with D7-branes [Vafa '96], [Morrison Vafa '96]

- Assumption for spacetime: $\mathbb{R}^{1,3} \times (B_2 \times \mathbb{C})$.
- Axio-dilaton: $\tau \colon B_2 \times \mathbb{C} \to \mathbb{C}$, $p \mapsto C_0(p) + \frac{i}{g_s(p)}$.
- Perturbation theory for $g_s \ll 1$: $q = \sum_{n=0}^{\infty} q_n \cdot g_s^n \sim q_0$.
- Put in D7-brane $D_7 = \mathbb{R}^{1,3} \times B_2$:
  - Located at $z = z_0$ in complex plane $\mathbb{C}$ orthogonal to D_7.
  - Sources τ via $\Delta\tau = \delta^{(2)}(z - z_0)$.

  $\Rightarrow$ $\tau(z) = \frac{1}{2\pi i} \ln(z - z_0)$: **Strong** coupling at $z = z_0$.

$\Rightarrow$ No perturbative description!

- Physics invariant under $SL(2, \mathbb{Z})$ transformation of τ:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1\,. \tag{1}$$

$\Rightarrow$ Axio-dilation $\leftrightarrow$ complex structure modulus of elliptic curve!

F-theory vacua

Recent review: [Weigand '18]

Defining data

- Elliptic fibration $\pi \colon Y_4 \twoheadrightarrow B_3$:
  Origin: Axio dilaton τ as complex structure of elliptic curve.
- Gauge background $G_4 \in H^{2,2}(Y_4)$:
  Origin: M-theory 3-form C_3 with $G_4 = dC_3$.
- More data (e.g. $A \in H^4_D(Y_4, \mathbb{Z}(2))$, ...)

Fun facts

- **M**-theory: **M**embrane, **M**other, ...
- **F**-theory: **F**iber, **F**ather, ...
- Non-trivial F-theory vacuum $\leftrightarrow$ **singular** Y_4.

Singularities meet F-theory

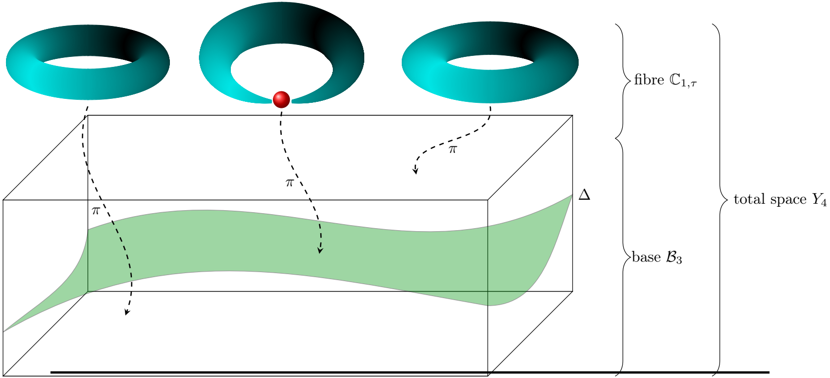
- Strategy 1: Do not resolve the singularities
  - Hard to extract the physics.
  - Some attempts exist in the literature.

    [Anderson Heckman Katz '13], [Collinucci Savelli '14], [Collinucci Giacomelli Savelli Valandro '16]
- Strategy 2: Resolve the singularities ($\leftrightarrow$ Coulomb branch of dual 3d M-theory)
  - For (simple) physics interpretation, must resolve **crepantly**.
  - Employ (weighted) blowup sequence. ... [Arena Jefferson Obinna '23]

  $\Rightarrow$ Challenges to find a crepant resolution:
    - Q-factorial terminal singularities cannot be resolved crepantly.
    - Hard to identify Q-factorial terminal singularities.
    - No algorithm for crepant (weighted blowup) resolution.
  - Sometimes find non-flat fibrations: Physics not clear.

    [Lawrie Schafer-Nameki '12], [Apruzzi Heckman Morrison Tizzano '18], ...
- Goal of `FTheoryTools`: Automate strategy 2.

Singular elliptic fibration



IIB-SUGRA	Geometry
union of loci of D7-branes in IIB-compactification	Singular locus Δ of elliptic fibration $\mathbb{C}_{1,\tau} \hookrightarrow Y_4 \overset{\pi}{\twoheadrightarrow} \mathcal{B}_3$

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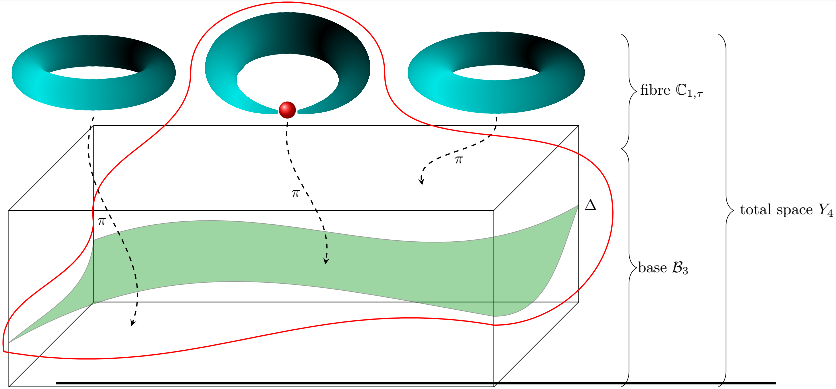

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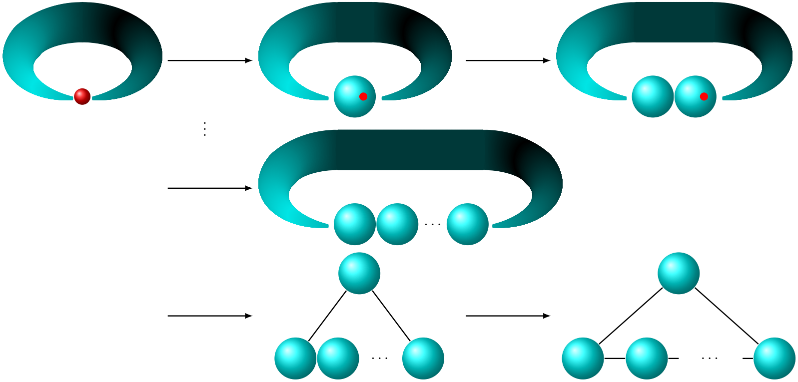
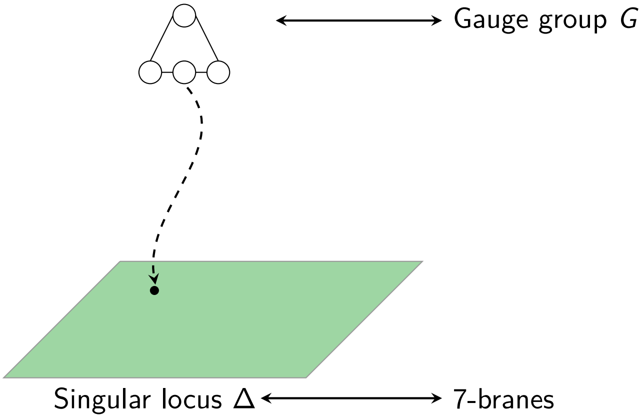
Cartoon of blow-up resolution



In general obtain ...

... **affine Dynkin diagrams** of A-, B-, C-, D-, E-, F_4 and G_2-type

Massless matter [Katz Vafa '96], [Witten '96], [Grassi,Morrison '00 & '11], [Morrison,Taylor '11], [Grassi,Halverson,Shaneson '13], [Cvetič,Klevers,Piragua,Taylor '15], [Anderson,Gray,Raghuram,Taylor '15], [Klevers,Taylor '16], [Klevers,Morrison,Raghuram,Taylor '17], . . .



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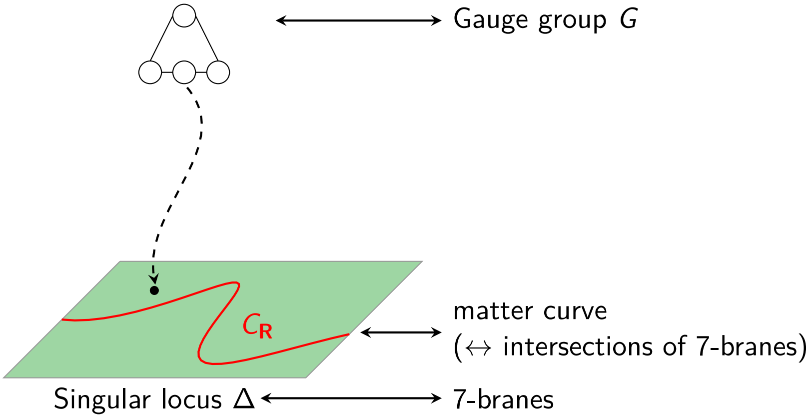

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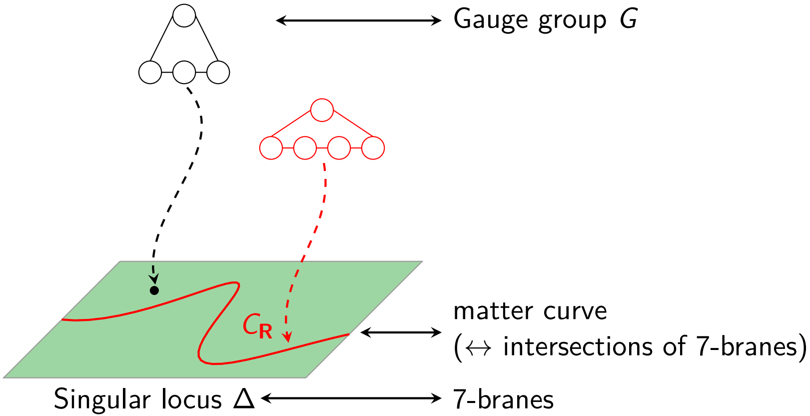

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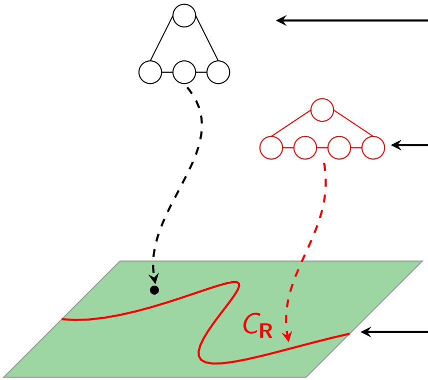

$\longleftrightarrow$ Gauge group G

$\longleftrightarrow$ Massless matter:
Formal linear sum of $\mathbb{P}^1_i$ fibrations
($\leftrightarrow$ weight $\beta^a(\mathbf{R})$ of irrep. $\mathbf{R}$ of G)

$\longleftrightarrow$ matter curve
($\leftrightarrow$ intersections of 7-branes)

Singular locus Δ $\longleftrightarrow$ 7-branes

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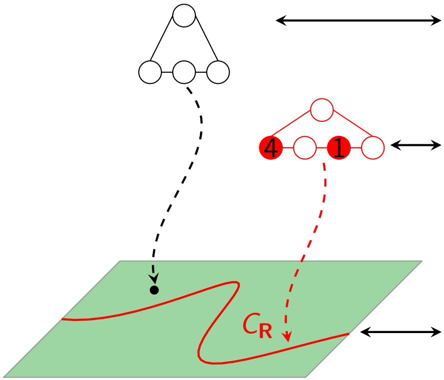

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Questions so far?

Goals for FTheoryTools

1. Many models studied in large detail in F-theory literature:
   - Resolutions, topological data, . . . known.
   - Study same model with different techniques.

   $\Rightarrow$ **LiteratureModels**

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   ⇒ `LiteratureModels`
2. Generalize/implement techniques:
   - A lot of toric functionality in OSCAR [Bies Kastner '23]
   - Many interesting techniques known [Jefferson Taylor Turner '21], [Jefferson Turner '22], ...
   - Sometimes, we need/wish to go beyond the toric regime (e.g. non-toric (crepant) blowup).

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    - Sometimes, we need/wish to go beyond the toric regime (e.g. non-toric (crepant) blowup).
3. Find algorithm for **crepant** resolution:
    - Many details known in F-theory literature.
    - Crepant is "exotic" condition in mathematics.
    - ⇒ No algorithm known yet, but we can try ...

An F-theory global Tate model More details: [Weigand '18]

- Consider $\mathbb{P}^{2,3,1}$ with coordinates $[x : y : z]$.
- Let B_3 be a complete, Kaehler 3-fold s.t. there exist

$$0 \neq a_i \in H^0\left(B_3, \overline{K}_{B_3}^{\otimes i}\right), \qquad i \in \{1, 2, 3, 4, 6\}.$$

- Define the Tate polynomial ("long Weierstrass equation"):

$$P_T = y^2 + a_1 xyz + a_3 yz^3 - x^3 - a_2 x^2 z^2 - a_4 xz^4 - a_6 z^6 .$$

- Fix $p \in B_3$. Then $V(P_T(p)) \subset \mathbb{P}^{2,3,1}$ is a torus surface.

$\Rightarrow$ Elliptic fibration $\pi \colon Y_4 \twoheadrightarrow B_3$ (section $[x : y : z] = [1 : 1 : 0]$)

("Global": P_T defines the model for every $p \in B_3$.)

Global Tate model to Weierstrass model More details: [Weigand '18]

- Consider global Tate model defined by $a_i \in H^0\left(B_3, \overline{K}_{B_3}^{\otimes i}\right)$ and

$$P_T = y^2 + a_1 xyz + a_3 yz^3 - x^3 - a_2 x^2 z^2 - a_4 xz^4 - a_6 z^6 \,.$$

- We define a few quantities:

$$b_2 = 4a_2 + a_1^2 \,, \quad b_4 = 2a_4 + a_1 a_3 \,, \quad b_6 = 4a_6 + a_3^2 \,,$$

$$f = -\frac{1}{48}\left(b_2^2 - 24 b_4\right) \,, \quad g = \frac{1}{864}\left(b_2^3 - 36 b_2 b_4 + 216 b_6\right) \,.$$

$\Rightarrow$ (Short) Weierstrass model defined by f, g and

$$P_W = y^2 - x^3 - fxz^4 - gz^6 \,.$$

The singular loci of the Tate/Weierstrass model are

$$V(\Delta) = V(4f^3 + 27g^2) \subset B_3 \,.$$

An $SU(5) \times U(1)$ F-theory global Tate model

Fine tune F-theory global Tate model

Wish to have particular singularity over hypersurface $V(w) \subset B_3$.

One particular model [Krause Mayrhofer Weigand '11]

- Assume that B_3 allows us to factor the sections a_i:

$$a_1 = a_1\,,\ a_2 = a_{2,1} w\,,\ a_3 = a_{3,2} w^2\,,\ a_4 = a_{4,3} w^3\,,\ a_6 \equiv 0\,.$$

$\Rightarrow$ $\Delta = 4f^3 + 27g^2 = w^5 \cdot P$, with complicated polynomial P.

- Singularities:
  - $\mathrm{ord}_{V(w)}(f, g, \Delta) = (0, 0, 5)$: I_5-singularity $\leftrightarrow$ $SU(5)$
  - $\mathrm{ord}_{V(P)}(f, g, \Delta) = (0, 0, 1)$: I_1-singularity $\leftrightarrow$ "Not relevant"

$U(1)$ from Mordell-Weil group of elliptic fibration ...

(More information: Kodaira classification, Tate table, Weierstrass table)

Resolution for $SU(5) \times U(1)$ F-theory global Tate model

- Blowup sequence worked out in literature [Krause Mayrhofer Weigand '11]:

$$\begin{aligned}
(x, y, w) &\to (xe_1, ye_1, we_1)\,, \\
(y, e_1) &\to (ye_4, e_1 e_4)\,, \\
(x, e_4) &\to (xe_2, e_4 e_2)\,, \\
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- Demonstrate with **experimental stage of** `FTheoryTools`:
https://docs.oscar-system.org/dev/Experimental/FTheoryTools/tate/

Status of FTheoryTools

- History:
  - Oct. '22: Development began with A. Turner (Original repository),
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  - Desire refined data of the resolved space.
- Great opportunity:
  - Testing ground for new techniques (e.g. weighted blowups).
  - User feedback very much appreciated.

Questions so far?

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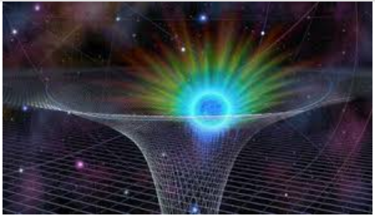

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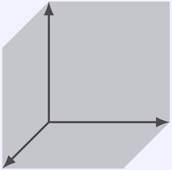

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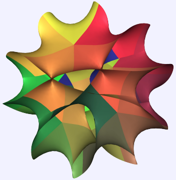

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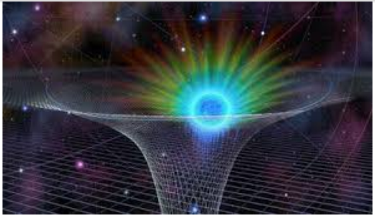

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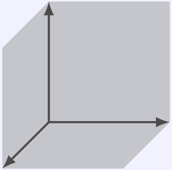

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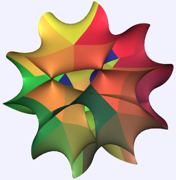

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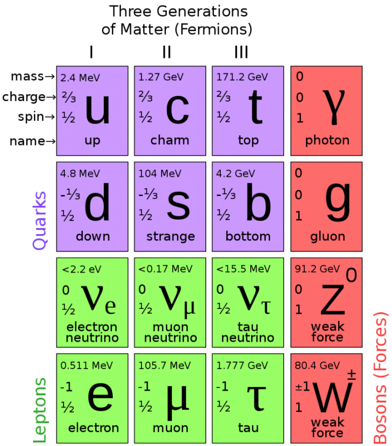

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  - Same look and feel, but different mass.
  - Experiment: At least 3 families.
  - → Chiral spectra in ST.

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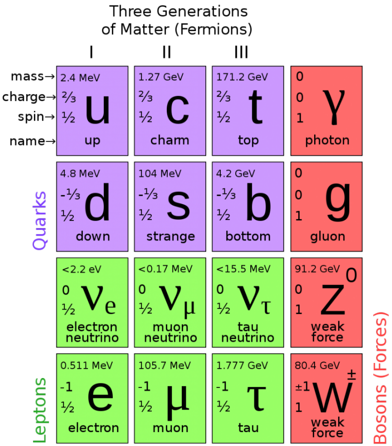

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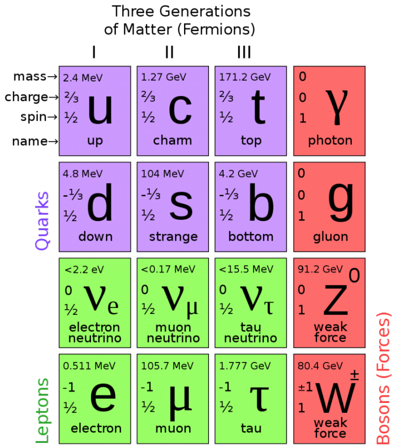

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  - Values determined in experiment.
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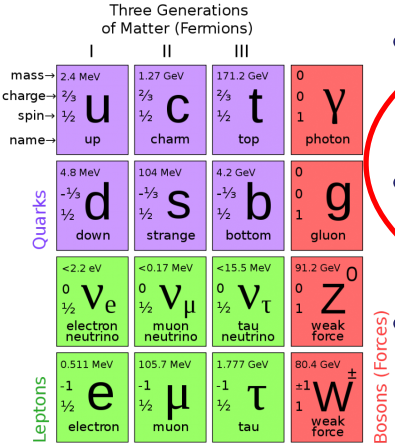

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SM constructions in perturbative string theory

Chiral spectrum (i.e. 3 particle families)

- $E_8 \times E_8$: [Candelas Horowitz Strominger Witten '85], [Greene Kirklin Miron Ross '86], [Braun He Ovrut Pantev '05], [Anderson Gray He Lukas '10], [Anderson Gray Lukas Palti '11 & '12], ...
- type II: [Berkooz Douglas Leigh '96], [Aldazabal Franco Ibanez Rabadan Uranga '00], [Ibanez Marchesano Rabadan '00], [Blumenhagen Kors Lust Ott '01], [Cvetič Shiu Uranga '01], ...

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Chiral **and vector-like** spectrum (i.e. 3 particle families, 1 Higgs)

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Challenges

- Global consistency,
- Yukawa couplings.

SM constructions in F-theory

Geometry “solves“ perturbative challenges

- Global consistency ↔ elliptic fibration [Vafa ‘96], [Morrison Vafa ‘96]
- Yukawa couplings ↔ intersections of matter curves

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- *Quadrillion SMs from F-theory (QSMs)* [Cvetič Halverson Lin Liu Tian ‘19]

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  [Donagi, Wijnholt ‘12], [Cvetic Lin Liu Zhang Zoccarato ‘19]

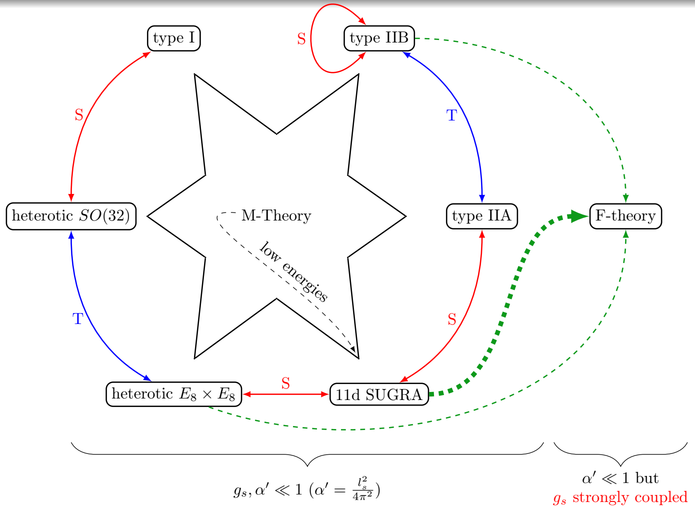
Chiral spectra

- [Krause Mayrhofer Weigand ‘12], [Cvetič Klevers Pena Oehlmann Reuter ‘15], [Lin Weigand ‘16], [Cvetič Lin Liu Oehlmann ‘18], . . .
- *Quadrillion SMs from F-theory (QSMs)* [Cvetič Halverson Lin Liu Tian ‘19]

Vector-like spectra

- [M.B. Mayrhofer Pehle Weigand ‘14], [M.B. Mayrhofer Weigand ‘17], [M.B. ‘18], [M.B. Cvetič Donagi Lin Liu Ruehle ‘20
- VL-spectra of *QSMs* [M.B. Cvetič Donagi Liu Ong ‘21], [M.B. Cvetič Liu ‘21], [M.B. Cvetič Donagi Ong ‘22], WIP.

G_4-fluxes and M-theory 3-form C_3



Origin of G_4-flux: M-theory 3-form C_3

G_4-flux

- 11d SUGRA action ($G_4 = dC_3$):

$$S_{11D} \sim \int_{M_{11}} d^{11}x \left(\sqrt{-\det G} R - \frac{G_4 \wedge * G_4}{2} - \frac{C_3 \wedge G_4 \wedge G_4}{6} \right)$$

- $G_4 = dC_3 \in H^{2,2}(\hat{Y}_4)$ is field strength.

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- $G_4 = dC_3 \in H^{2,2}(\hat{Y}_4)$ is field strength.

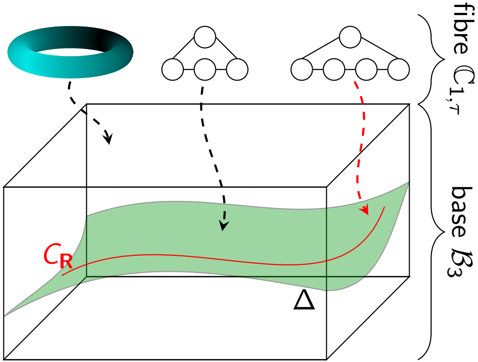
Full gauge data from Deligne cohomology [Curio,Donagi '98], [Donagi,Wijnholt '12/13], [Intriligator,Jockers,Mayr,Morrison,Plesser '12], [Anderson,Heckman,Katz '13]

- Definition of Deligne cohomology:

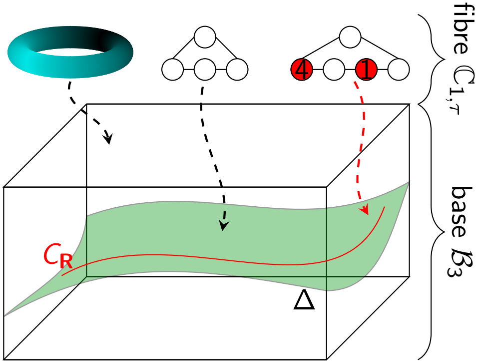
$$0 \to J^2(\hat{Y}_4) \hookrightarrow H^4_D(\hat{Y}_4, \mathbb{Z}) \twoheadrightarrow H^{2,2}(\hat{Y}_4) \to 0$$

- Easy-to-work-with parametrisation: $A \in \mathrm{CH}^2(\hat{Y}_4)$ [Green Murre Voisin '94]

Recipe [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]

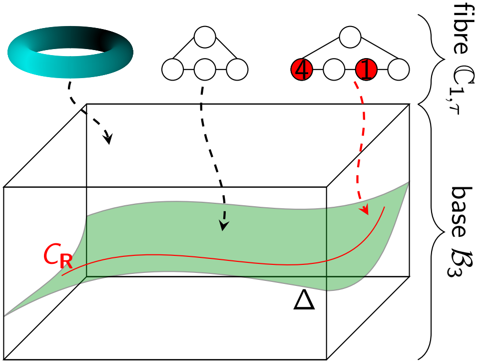


Recipe [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]



1. Massless matter:
   $\leftrightarrow S_{\mathbf{R}} = 4\mathbb{P}^1_A + \mathbb{P}^1_C \in \mathrm{CH}^2(\hat{Y}_4)$

Recipe [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]



1. Massless matter:
   $\leftrightarrow S_{\mathbf{R}} = 4\mathbb{P}^1_A + \mathbb{P}^1_C \in \mathrm{CH}^2(\hat{Y}_4)$
2. **Full** G_4-gauge data:
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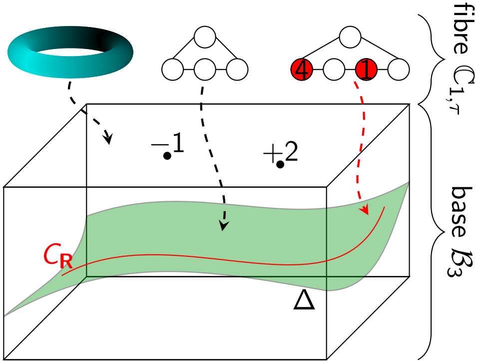

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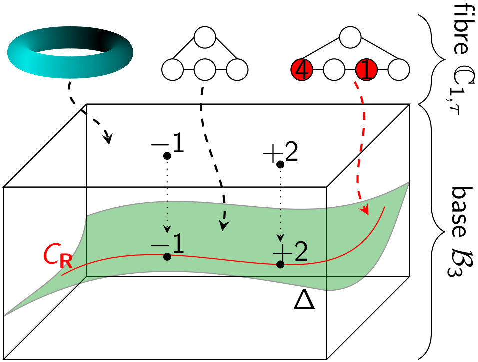

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   $\leftrightarrow S_\mathbf{R} = 4\mathbb{P}^1_A + \mathbb{P}^1_C \in \mathrm{CH}^2(\hat{Y}_4)$
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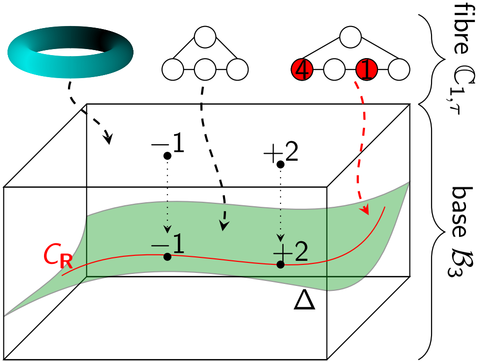
3. $S_\mathbf{R}$ and A intersect in points of $\hat{Y}_4$
4. $\pi_* (S_\mathbf{R} \cdot A) \hat{=}$ points in $C_\mathbf{R}$

Recipe [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]



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4. $\pi_* (S_{\mathbf{R}} \cdot A) \hat{=}$ points in $C_{\mathbf{R}}$
5. line bundle $\mathcal{L}_{\mathbf{R}} (S_{\mathbf{R}}, A)$ on $C_{\mathbf{R}}$
   $\mathcal{O}_{C_{\mathbf{R}}} (\pi_* (S_{\mathbf{R}} \cdot A)) \otimes \sqrt{K_{C_{\mathbf{R}}}}$

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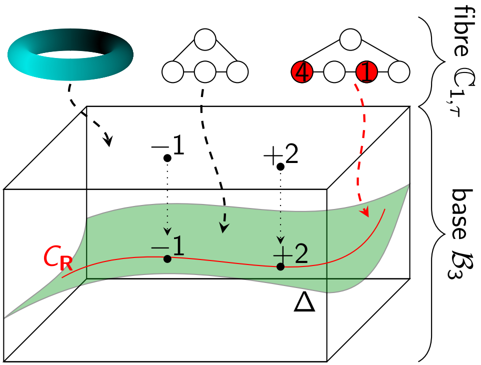

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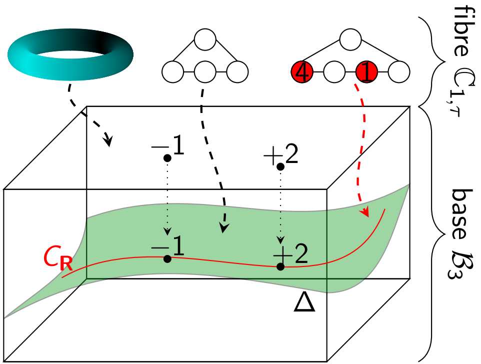
   Choice of spin bundle: Delicate!

Recipe [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]



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Choice of spin bundle: Delicate!

Consequence [Katz,Sharpe '02] [Beasley,Heckman,Vafa '08] [Donagi,Wijnholt '08]

$$\mathcal{N} = 1 \text{ chiral multiplets} \quad \leftrightarrow \quad H^0 \left(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}} \left(S_{\mathbf{R}}, A \right) \right)$$
$$\mathcal{N} = 1 \text{ anti-chiral multiplets} \quad \leftrightarrow \quad H^1 \left(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}} \left(S_{\mathbf{R}}, A \right) \right)$$

Desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}}$ = # chiral fields in rep $\mathbf{R}$	# $n_{\overline{\mathbf{R}}}$ = chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$			
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$			
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$			
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$			
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$			
How to compute?			

Desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \; n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$			3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$			3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$			3
How to compute?			

Desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \; n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$			3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$			3
$C_{(\mathbf{1},\mathbf{1})_{1}} = V(s_1, s_5)$			3
How to compute?			$\chi = \int_{S_{\mathbf{R}}} G_4$

Desired vector-like spectra in the QSMs

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$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$			3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$			3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$			3
How to compute?			$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

Desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \; n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	3	0	3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4	1	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	3	0	3
How to compute?			$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

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Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\#\ n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
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$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4 $(4,1) = (3,0) \oplus (1,1) =$ leptons + Higgs	1	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	3	0	3
How to compute?			$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

Desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	3	0	3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4	1	3
	$(4,1) = (3,0) \oplus (1,1) =$ leptons + Higgs		
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	3	0	3
How to compute?	$h^0(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$	$h^1(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$	$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

Desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}}$ = # chiral fields in rep $\mathbf{R}$	# $n_{\overline{\mathbf{R}}}$ = chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	3	0	3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4	1	3
	$(4,1) = (3,0) \oplus (1,1)$ = leptons + Higgs		
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	3	0	3
How to compute?	$h^0(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$ [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17] [M.B. '18] and references therein	$h^1(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$	$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

Desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \, n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	3	0	3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4	1	3
	$(4,1) = (3,0) \oplus (1,1) =$ leptons + Higgs		
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	3	0	3
How to compute?	$h^0(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$	$h^1(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$	$\chi = \deg(\mathcal{L}_{\mathbf{R}}) - g(C_{\mathbf{R}}) + 1$ $\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]
	[M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17] [M.B. '18] and references therein		

Necessary condition for $\mathcal{L}_{\mathbf{R}}$ [M.B. Cvetič Donagi Liu Ong '21]

Matter curve $C_{\mathbf{R}}$	Necessary root bundle condition for $\mathcal{L}_{\mathbf{R}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	$\mathcal{L}^{\otimes 36}_{(\mathbf{3},\mathbf{2})_{1/6}} = K^{\otimes 24}_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}$
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	$\mathcal{L}^{\otimes 36}_{(\mathbf{1},\mathbf{2})_{-1/2}} = K^{\otimes 22}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30 \cdot Y_1)$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	$\mathcal{L}^{\otimes 36}_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = K^{\otimes 24}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}}$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	$\mathcal{L}^{\otimes 36}_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = K^{\otimes 22}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}} \otimes \mathcal{O}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}}(-30 \cdot Y_3)$
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	$\mathcal{L}^{\otimes 36}_{(\mathbf{1},\mathbf{1})_1} = K^{\otimes 24}_{C_{(\mathbf{1},\mathbf{1})_1}}$

Exponents of root bundle constraints for base 3-folds B_3 with $K_{B_3}^3 = 18$. See [M.B. Cvetič Donagi Liu Ong '21] for exponents of B_3 with other $K_{B_3}^3$.

Necessary condition for $\mathcal{L}_{\mathbf{R}}$ [M.B. Cvetič Donagi Liu Ong '21]

Matter curve $C_{\mathbf{R}}$	Necessary root bundle condition for $\mathcal{L}_{\mathbf{R}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	$\mathcal{L}^{\otimes 36}_{(\mathbf{3},\mathbf{2})_{1/6}} = K^{\otimes 24}_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}$
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	$\mathcal{L}^{\otimes 36}_{(\mathbf{1},\mathbf{2})_{-1/2}} = K^{\otimes 22}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30 \cdot Y_1)$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	$\mathcal{L}^{\otimes 36}_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = K^{\otimes 24}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}}$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	$\mathcal{L}^{\otimes 36}_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = K^{\otimes 22}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}} \otimes \mathcal{O}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}}(-30 \cdot Y_3)$
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	$\mathcal{L}^{\otimes 36}_{(\mathbf{1},\mathbf{1})_1} = K^{\otimes 24}_{C_{(\mathbf{1},\mathbf{1})_1}}$

Exponents of root bundle constraints for base 3-folds B_3 with $K^3_{B_3} = 18$. See [M.B. Cvetič Donagi Liu Ong '21] for exponents of B_3 with other $K^3_{B_3}$.

- Constraints highly non-trivial:
  Infinitely many line bundles with $\chi = 3$ but only finitely many root bundles.

Necessary condition for $\mathcal{L}_{\mathbf{R}}$ [M.B. Cvetič Donagi Liu Ong '21]

Matter curve $C_{\mathbf{R}}$	Necessary root bundle condition for $\mathcal{L}_{\mathbf{R}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	$\mathcal{L}^{\otimes 36}_{(\mathbf{3},\mathbf{2})_{1/6}} = K^{\otimes 24}_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}$
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	$\mathcal{L}^{\otimes 36}_{(\mathbf{1},\mathbf{2})_{-1/2}} = K^{\otimes 22}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30 \cdot Y_1)$
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$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	$\mathcal{L}^{\otimes 36}_{(\mathbf{1},\mathbf{1})_1} = K^{\otimes 24}_{C_{(\mathbf{1},\mathbf{1})_1}}$

Exponents of root bundle constraints for base 3-folds B_3 with $K^3_{B_3} = 18$. See [M.B. Cvetič Donagi Liu Ong '21] for exponents of B_3 with other $K^3_{B_3}$.

- Constraints highly non-trivial:
  Infinitely many line bundles with $\chi = 3$ but only finitely many root bundles.
- Must not drop common exponents ($x^2 = 2^2 \not\Rightarrow x = 2$).

Necessary condition for $\mathcal{L}_{\mathbf{R}}$ [M.B. Cvetič Donagi Liu Ong '21]

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$\Rightarrow$ Vector-like spectra of QSMs from studying root bundles.

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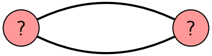
Refined idea

Vector-like spectra of the QSMs from **root bundles** on **nodal curves**.

Example: Spin bundles on nodal curve



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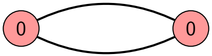

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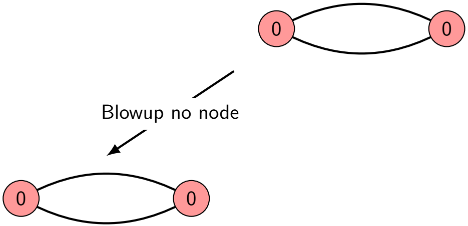

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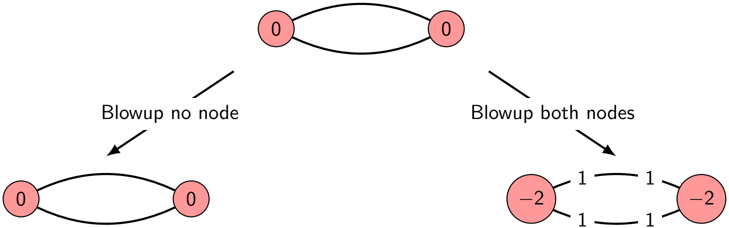

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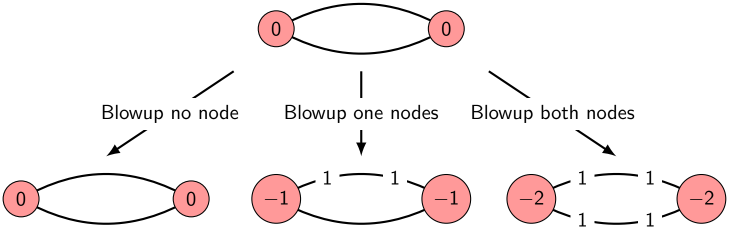

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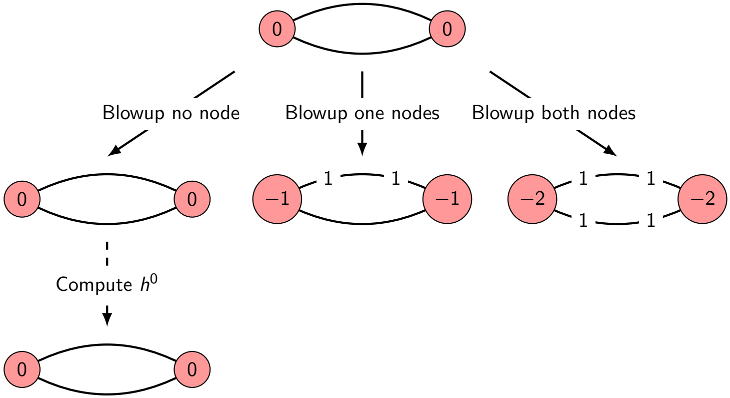

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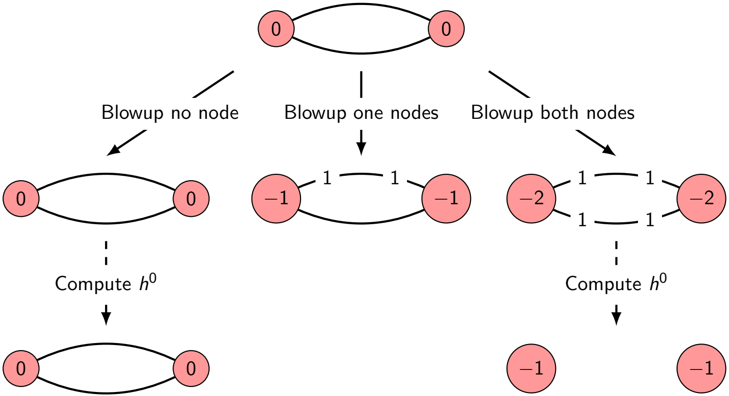

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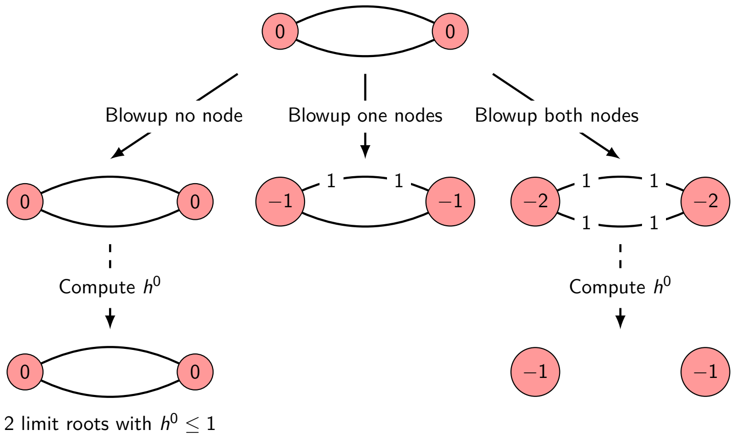

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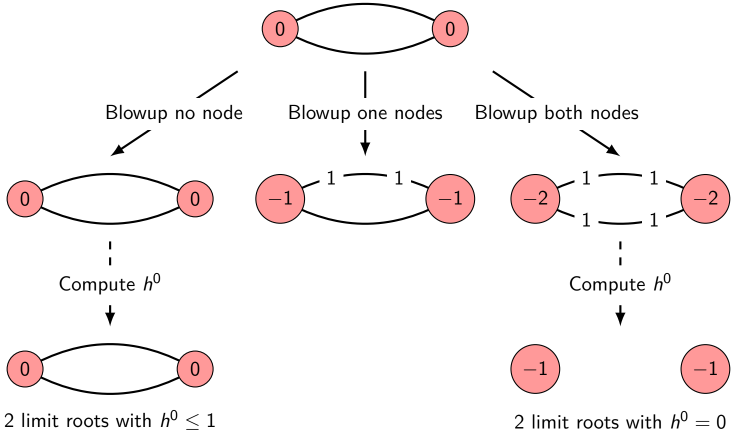

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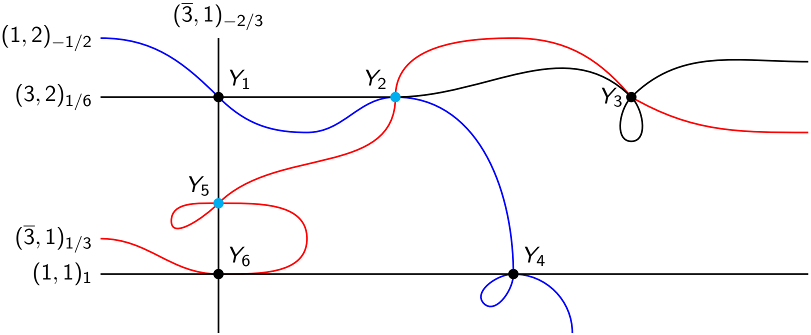

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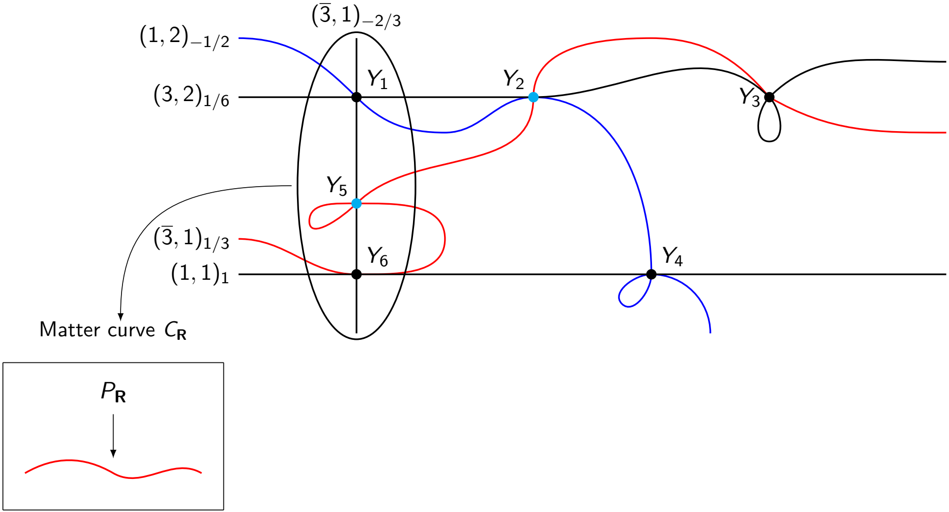

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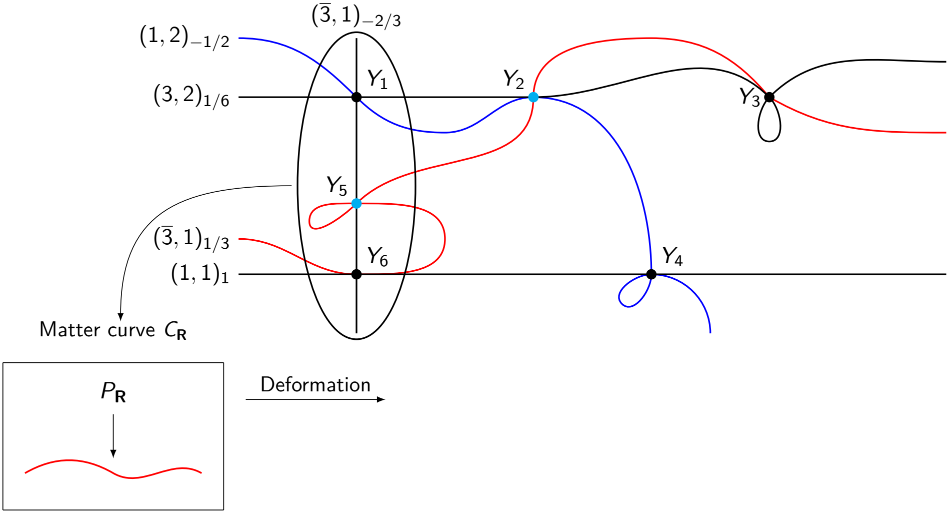

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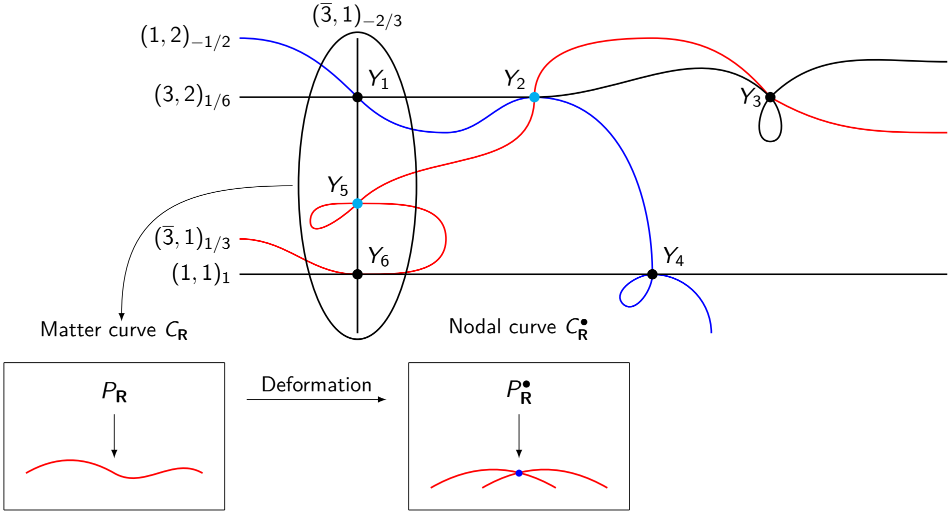

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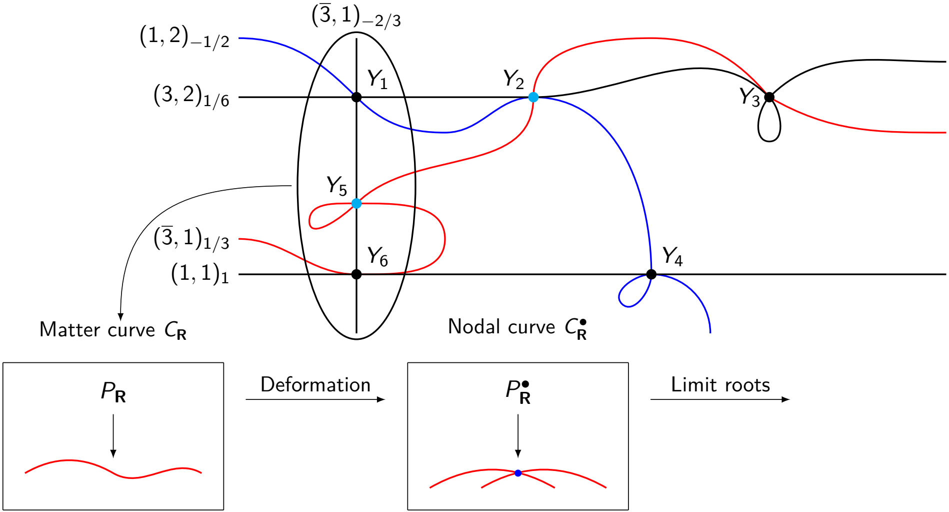

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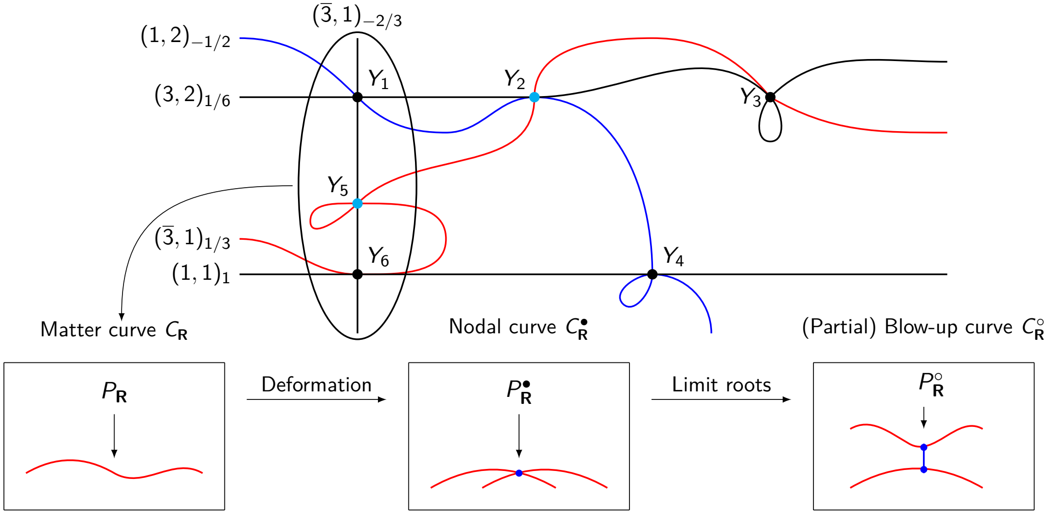

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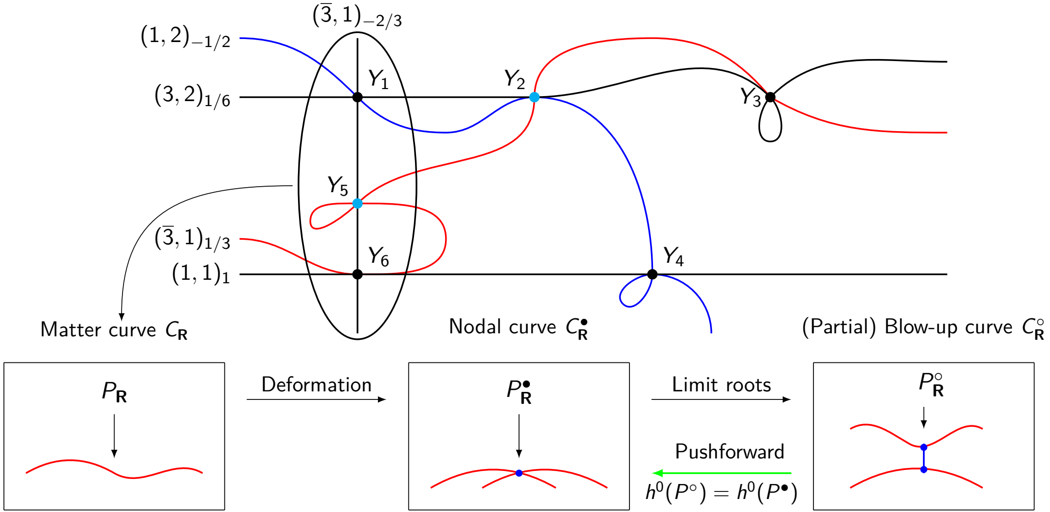

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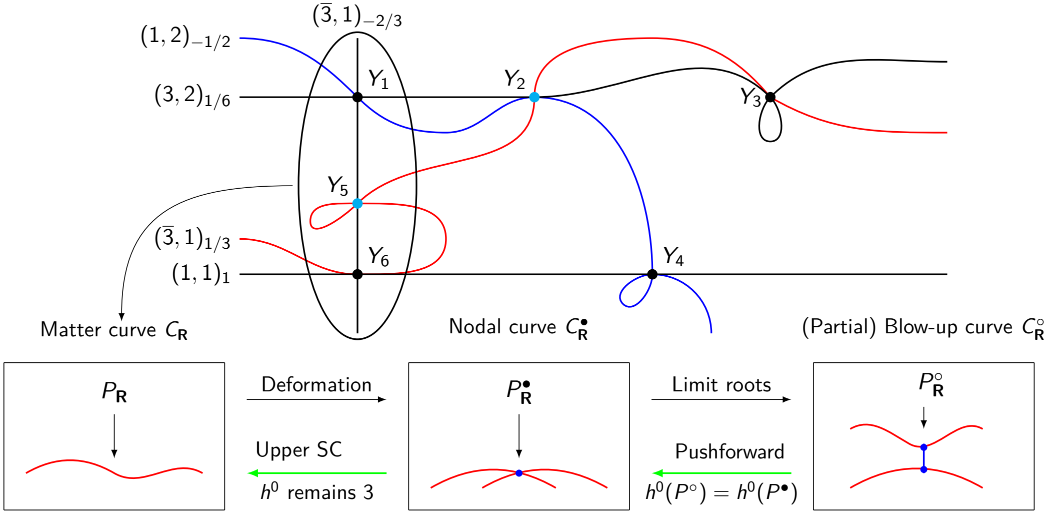

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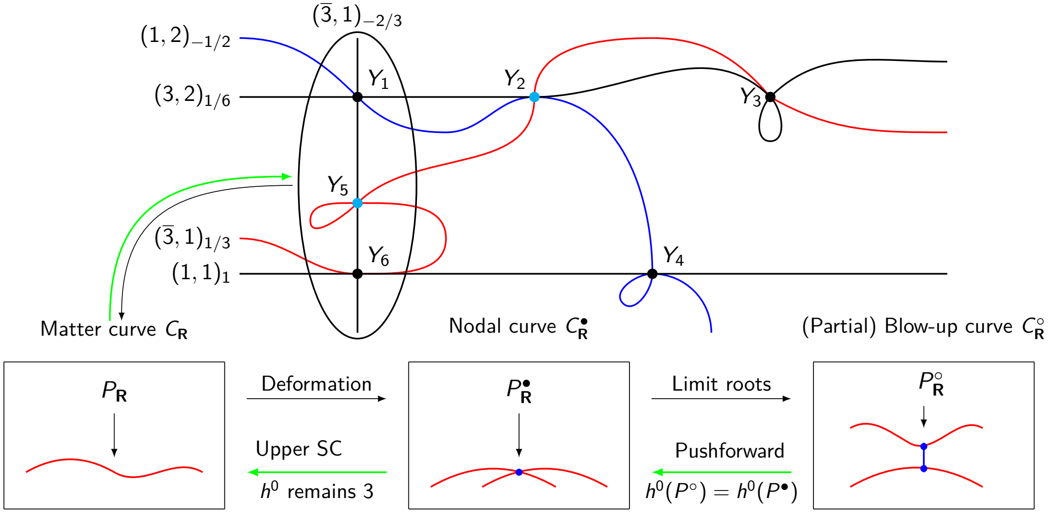
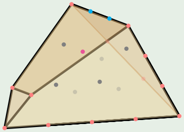


Philosophy: Local, bottom-up and FRST invariant

[M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22]

Advantage: Triangulation invariant estimate of VL spectra for huge families of QSMs



Δ° → fine regular star triangulations → Family $B_3(\Delta^\circ)$ of toric F-theory base 3-folds - - -▸ **Same** nodal matter curve $C^\bullet_{\mathbf{R}}$ $\forall X_\Sigma \in B_3(\Delta^\circ)$

[Klevers Peña Oehlmann Piragua Reuter '14], [Cvetič Klevers Peña Oehlmann Reuter '15], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], ...

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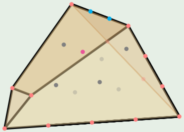

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Application of OSCAR

Compute triangulations in [M.B. Cvetič Donagi Ong '22].

Towards “good“ physical roots

(Naive) Brill-Noether theory for **root bundles**

Discriminate the r^{2g} line bundles $\mathcal{L} \in \mathrm{Pic}(C)$ with $\mathcal{L}^r = T$:

$$r^{2g} = N_0 + N_1 + N_2 + \ldots \, ,$$

N_i is the number of those root bundles $\mathcal{L}$ with $h^0(C, \mathcal{L}) = i$.

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Current standing

- Systematic answer unknown (to my knowledge).
- Sometimes, we only know to compute a lower bound to h^0.
- h^0 can jump (nodes specially aligned, special descent data) [M.B. Cvetič Donagi Ong ‘22]

$$r^{2g} = \left(\widetilde{N}_0 + \widetilde{N}_{\geq 0}\right) + \left(\widetilde{N}_1 + \widetilde{N}_{\geq 1}\right) + \ldots .$$

Brill-Noether numbers of $(\overline{\mathbf{3}}, \mathbf{2})_{1/6}$ in QSMs

- First estimates computed in [M.B. Cvetič Liu '21]:
  - count **"simple"** root bundles with minimal h^0,
  - no estimate for $\widetilde{N}_{\geq i}$.
- Refinements/extensions in [M.B. Cvetič Donagi Ong '22]:
  - enumerate **all** root bundles,
  - discriminate via line bundle cohomology on rational tree-like nodal curves,

QSM-family	# FRSTs	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$
Δ_8°	$\sim 10^{15}$	57.3%	?	?	?
Δ_4°	$\sim 10^{11}$	53.6%	?	?	?
Δ_{134}°	$\sim 10^{10}$	48.7%	?	?	?
Δ_{128}°, Δ_{130}°, Δ_{136}°, Δ_{236}°	$\sim 10^{11}$	42.0%	?	?	?

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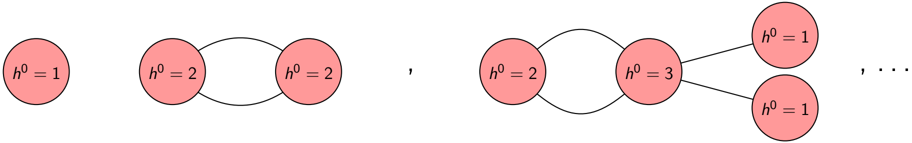
QSM-family (KS polytope)	# FRSTs	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$
Δ_8°	$\sim 10^{15}$	76.4%	23.6%		
Δ_4°	$\sim 10^{11}$	99.0%	1.0%		
Δ_{134}°	$\sim 10^{10}$	99.8%	0.2%		
$\Delta_{128}^\circ, \Delta_{130}^\circ, \Delta_{136}^\circ, \Delta_{236}^\circ$	$\sim 10^{11}$	99.9%	0.1%		

Can we do better for $B_3(\Delta_4^\circ)$? The 1% contains ...

- Stationary circuits with $h^0 = 3$:



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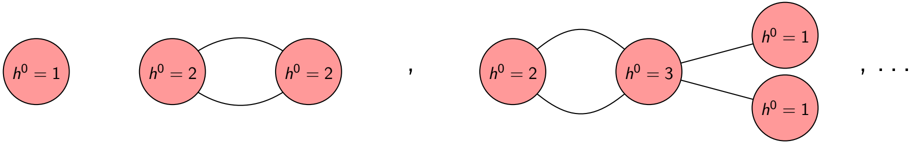
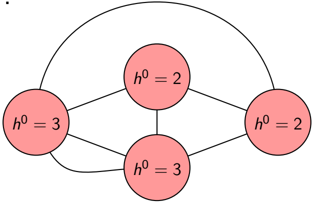


- Jumping circuit with $h^0 = 4$:



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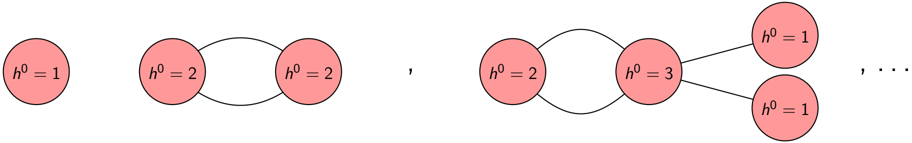

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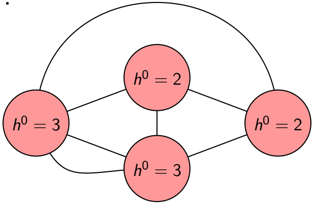

Consequence

$B_3(\Delta_4^\circ)$: 99.995% of solutions to **necessary** root bundle constraint have $h^0 = 3$.

Brill-Noether numbers of $(\overline{3}, 2)_{1/6}$ in QSMs [M.B. Cvetič Donagi Ong '22]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ°_{88}	74.9	22.1	2.5	0.5	0.0	0.0		
Δ°_{110}	82.4	14.1	3.1	0.4	0.0			
Δ°_{272}, Δ°_{274}	78.1	18.0	3.4	0.5	0.0	0.0		
Δ°_{387}	73.8	21.9	3.5	0.7	0.0	0.0		
Δ°_{798}, Δ°_{808}, Δ°_{810}, Δ°_{812}	77.0	17.9	4.4	0.7	0.0	0.0		
Δ°_{254}	95.9	0.5	3.5	0.0	0.0	0.0		
Δ°_{52}	95.3	0.7	3.9	0.0	0.0	0.0		
Δ°_{302}	95.9	0.5	3.5	0.0	0.0			
Δ°_{786}	94.8	0.3	4.8	0.0	0.0	0.0		
Δ°_{762}	94.8	0.3	4.9	0.0	0.0	0.0		
Δ°_{417}	94.8	0.3	4.8	0.0	0.0	0.0	0.0	
Δ°_{838}	94.7	0.3	5.0	0.0	0.0	0.0		
Δ°_{782}	94.6	0.3	5.0	0.0	0.0	0.0		
Δ°_{377}, Δ°_{499}, Δ°_{503}	93.4	0.2	6.2	0.0	0.1	0.0		
Δ°_{1348}	93.7	0.0	6.2	0.0	0.1		0.0	
Δ°_{882}, Δ°_{856}	93.4	0.3	6.2	0.0	0.1	0.0	0.0	
Δ°_{1340}	92.3	0.0	7.6	0.0	0.1		0.0	
Δ°_{1879}	92.3	0.0	7.5	0.0	0.1		0.0	
Δ°_{1384}	90.9	0.0	8.9	0.0	0.2		0.0	

WIP: Develop yet more refined techniques [M.B. Cvetič Donagi Ong – to appear soon]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ°_{88}	96.6700	0.3361	2.9850		0.0089			
Δ°_{110}	95.6268	0.8372	3.5179	0.0050	0.0131			
Δ°_{272}, Δ°_{274}	95.5097	0.5155	3.9552	0.0016	0.0180			
Δ°_{387}	95.1923	0.4981	4.2773		0.0323			
Δ°_{798}, Δ°_{808}, Δ°_{810}, Δ°_{812}	93.8268	0.8795	5.2390	0.0029	0.0518			
Δ°_{254}	96.3942	0.0687	3.5193	0.0003	0.0175			
Δ°_{52}	96.0587	0.0171	3.9066	0.0000	0.0176			
Δ°_{302}	96.3960	0.0636	3.5222	0.0001	0.0181			
Δ°_{786}	95.0714	0.0393	4.8466	0.0002	0.0425			
Δ°_{762}	95.0167	0.0369	4.9052	0.0005	0.0407			
Δ°_{417}	95.0745	0.0433	4.8389	0.0003	0.0429		0.0001	
Δ°_{838}	94.9092	0.0215	5.0216	0.0000	0.0477			
Δ°_{782}	94.9019	0.0161	5.0359	0.0000	0.0461			
Δ°_{377}, Δ°_{499}, Δ°_{503}	93.6500	0.0347	6.2312	0.0005	0.0836			
Δ°_{1348}	93.7075	0.0112	6.1978	0.0001	0.0833		0.0001	
Δ°_{882}, Δ°_{856}	93.6546	0.0425	6.2190	0.0009	0.0825		0.0005	
Δ°_{1340}	92.2989	0.0064	7.5515	0.0001	0.1427		0.0004	
Δ°_{1879}	92.3015	0.0108	7.5447	0.0002	0.1421		0.0007	
Δ°_{1384}	90.8524	0.0031	8.9219	0.0001	0.2213		0.0012	

Root bundles: Summary and outlook

- **Statistical observation**
  In QSMs, absence of vector-like exotics in $(\overline{\mathbf{3}}, \mathbf{2})_{1/6}$, $(\overline{\mathbf{3}}, \mathbf{1})_{-2/3}$, $(\mathbf{1}, \mathbf{1})_1$ likely,
  but ...
  - **Sufficient** condition for quantization of G_4-flux? [Jefferson Taylor Turner '21].
  - F-theory gauge potential
    - may select (proper) subset of these root bundles,
    - lead to correlated choices on distinct matter curves.
- Vector-like spectra on $C_{\mathbf{R}}^{\bullet}$ "upper bound" to those on $C_{\mathbf{R}}$.
  $\leftrightarrow$ Understand "drops" from **Yukawa interactions**? [Cvetič Lin Liu Zhang Zoccarato '19]
  $\rightarrow$ Towards the Higgs ...
- Computationally, Higgs curve currently too challenging.
  - Need **Brill-Noether theory for root bundles on nodal curves**.
    Map from (dual) graphs (and a couple of integers) to Brill-Noether numbers.
    $\leftrightarrow$ Arena for **machine learning**?

$\rightarrow$ **Probability/statistics** for F-theory setups to arise **without vector-like exotics**.

Thank you for your attention!