From F-theory Standard Models to Freyd Categories and back

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Overview

Presentation based on work with ...

- T. Weigand, C. Mayrhofer, C. Pehle

  1402.5144, 1706.04616, 1706.08528, 1802.08860

- S. Posur 1909.00172
- M. Barakat, S. Gutsche, S. Posur, K. M. Saleh

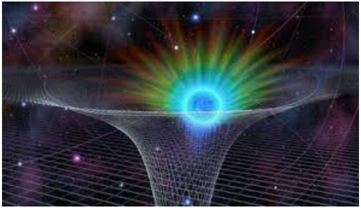
  Various gap and CAP-packages on *https://github.com/homalg-project*

- M. Cvetič, L. Lin, M. Liu Work in progress

Outline

- Physics: Counting exact massless spectra in F-theory
- Mathematics: Monoidal structures on Freyd categories
- Physics: Applications to F-theory model building

String theory = General relativity + Standard Model?



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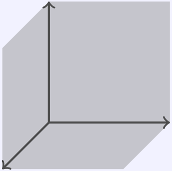

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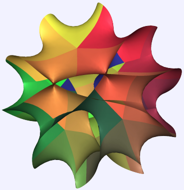
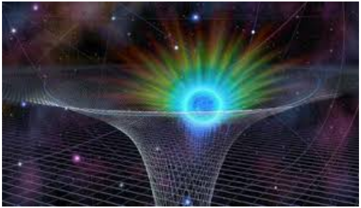


our 4-dim. world $\mathcal{W}$ ‘small’ 6-dim. manifold $\mathcal{B}_3$

Challenge: Find $\mathcal{B}_3$ s.t. ST reproduces 4d physics

String theory = General relativity + Standard Model?



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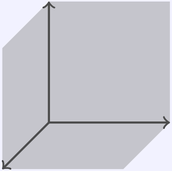

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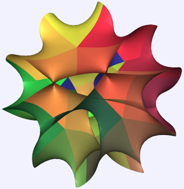
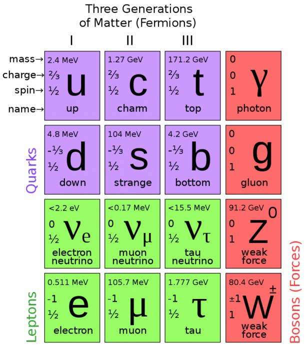


our 4-dim. world $\mathcal{W}$ 'small' 6-dim. manifold $\mathcal{B}_3$

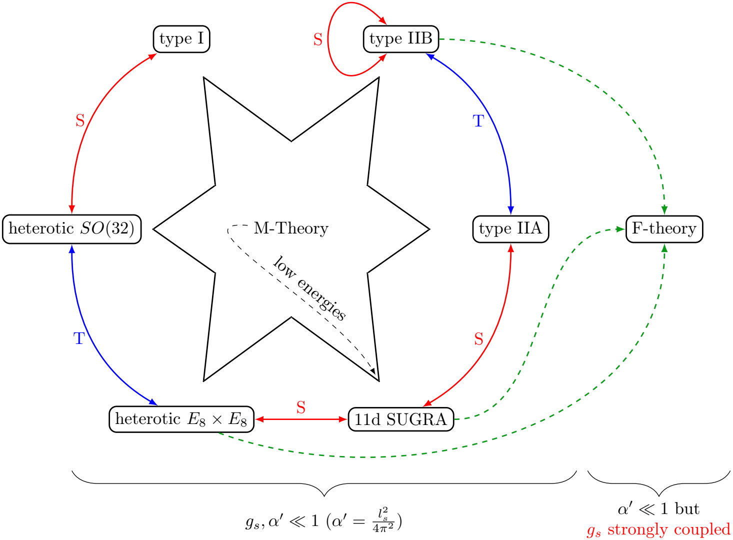
Challenge: Find $\mathcal{B}_3$ s.t. ST reproduces Standard model

Exact massless spectra - what and why?



- For phenomenology:
  - number of Higgs doublets
  - amount of vector-like exotics
- Conceptually:
  - affects RG flow e.g. of couplings
  - enters Higgsing and transitions between vacua
  - depends on complex structure moduli
  - goes beyond rigid data
  - leads to rich mathematics (coherent sheaves, Freyd categories, monoidal structures, . . .)

Which type of string theory is best for constructing the SM?



SM constructions in perturbative string theory

- $E_8 \times E_8$: [Candelas Horowitz Strominger Witten '85], [Greene Kirklin Miron Ross '86], [Braun He Ovrut Pantev '05], [Bouchard Donagi '05], [Anderson Gray He Lukas '10], [Anderson Gray Lukas Palti '11 & '12], ...
- type II: [Berkooz Douglas Leigh '96], [Aldazabal Franco Ibanez Rabadan Uranga '00], [Ibanez Marchesano Rabadan '00], [Blumenhagen Kors Lust Ott '01], [Cvetič Shiu Uranga '01], ...
- Exact vector-like spectra without exotics [Braun He Ovrut Pantev '05], [Bouchard Donagi '05]
- Difficulties:
    - global consistency
    - Yukawa couplings

SM constructions in F-theory

- Geometrization: [Vafa '96], [Morrison Vafa '96]
  - Global consistency ↔ consistent elliptic fibration
  - Yukawa couplings ↔ intersections of matter curves

    [Donagi, Wijnholt '12], [Cvetic Lin Liu Zhang Zoccarato '19]
- SM constructions [Krause Mayrhofer Weigand '12], [Cvetič Klevers Pena Oehlmann Reuter '15], [Lin Weigand '16], [Cvetič Lin Liu Oehlmann '18]
- Most recently: *A Quadrillion Standard Models from F-theory* [Cvetič Halverson Lin Liu Tian '19]
- Vector-like spectra computed only in **toy** models [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]

⇒ Analyse spectra of *Quadrillion SMs* and find model without vector-like exotics

F-theory – Generalities

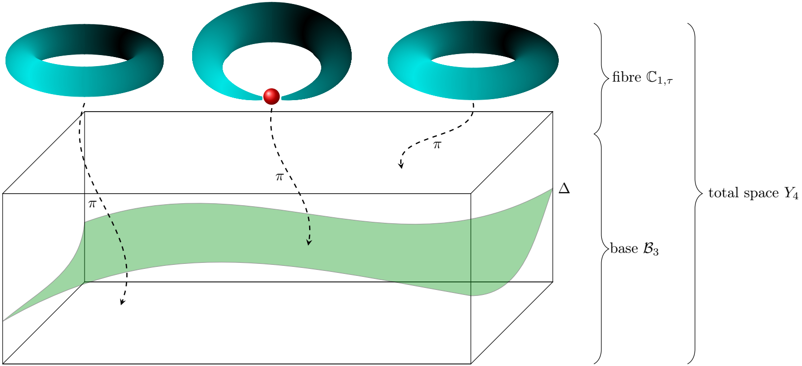
Defining data recent review: [Weigand '18]

- **Singular** elliptic fibration $\pi \colon Y_4 \twoheadrightarrow B_3$
  Origin: Interpret axio dilaton τ as complex structure of torus and fibre this torus over B_3
- Gauge background $G_4 \in H^{2,2}(Y_4)$
  Origin: M-theory 3-form C_3 with $G_4 = dC_3$
- Additional non-geometric data (e.g. T-branes)

How to deal with singularities?

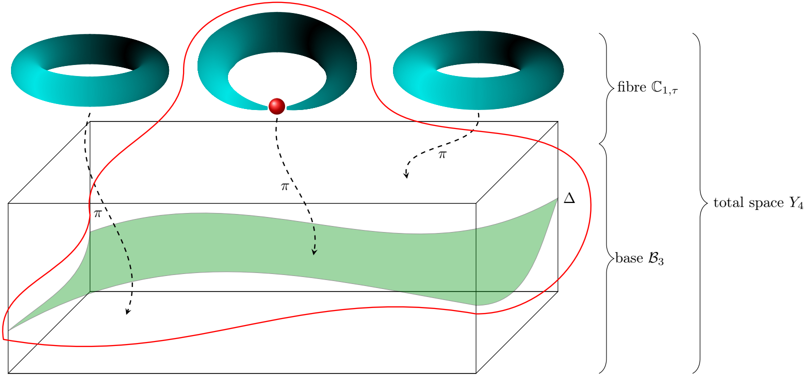
- Non-minimal [Lawrie Schafer-Nameki '12], [Apruzzi Heckman Morrison Tizzano '18], ...
- Minimal
  - Do not resolve them [Anderson Heckman Katz '13], [Collinucci Savelli '14], [Collinucci Giacomelli Savelli Valandro '16]
  - Resolve them ($\leftrightarrow$ Coulomb branch of dual 3d M-theory)

Singular elliptic fibration



IIB-SUGRA	Geometry
union of loci of D7-branes in IIB-compactification	Singular locus Δ of elliptic fibration $\mathbb{C}_{1,\tau} \hookrightarrow Y_4 \overset{\pi}{\twoheadrightarrow} \mathcal{B}_6$

Singular elliptic fibration



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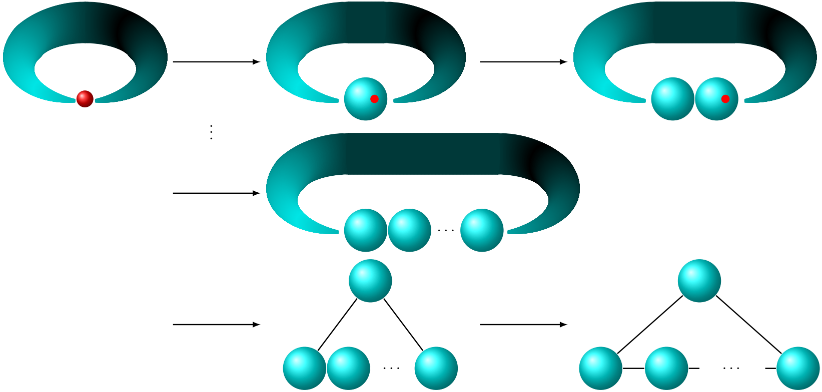
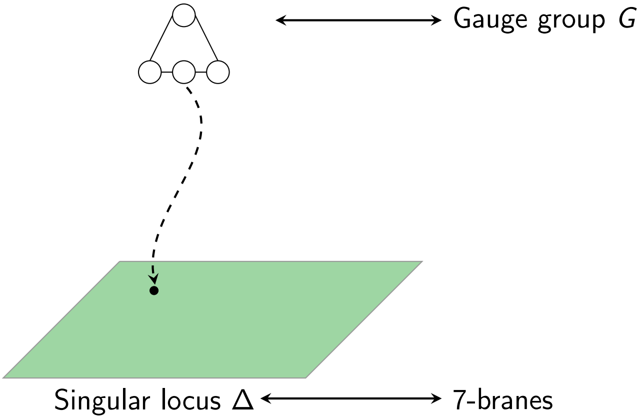
Cartoon of blow-up resolution



In general obtain . . .

. . . **affine Dynkin diagrams** of A-, B-, C-, D-, E-, F_4 and G_2-type

Massless matter [Katz Vafa '96], [Witten '96], [Grassi,Morrison '00 & '11], [Morrison,Taylor '11], [Grassi,Halverson,Shaneson '13], [Cvetič,Klevers,Piragua,Taylor '15], [Anderson,Gray,Raghuram,Taylor '15], [Klevers,Taylor '16], [Klevers,Morrison,Raghuram,Taylor '17], ...



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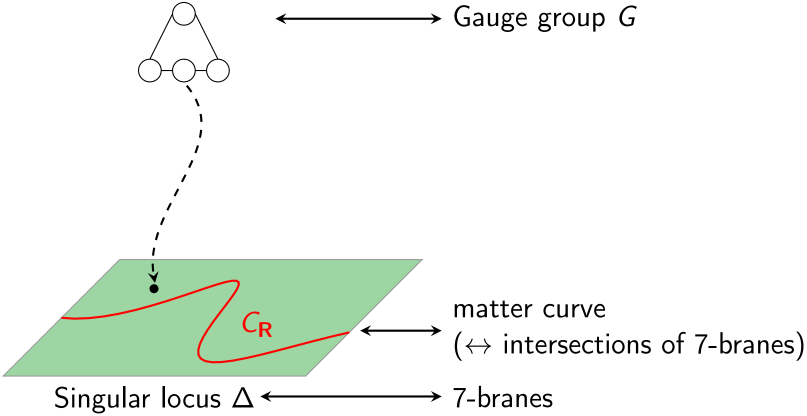

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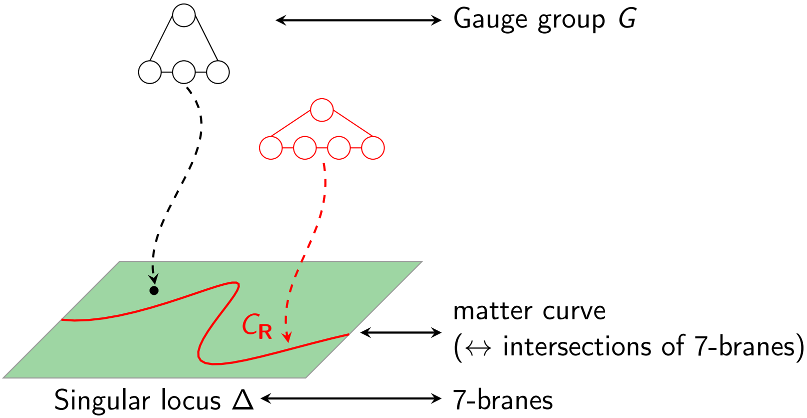

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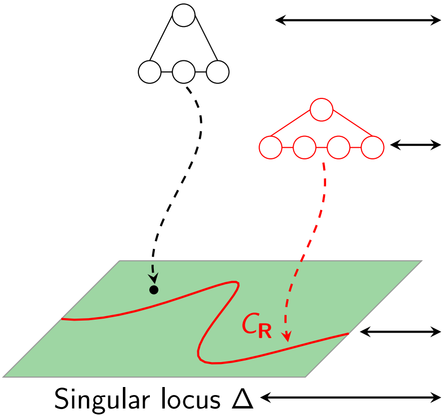


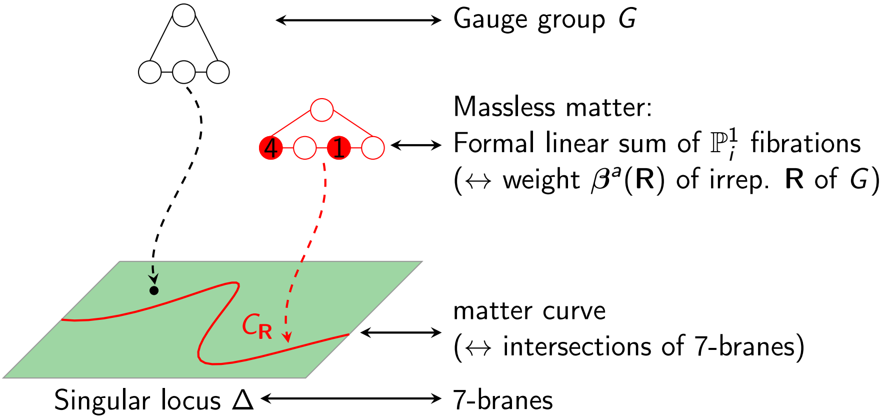
$\longleftrightarrow$ Gauge group G

$\longleftrightarrow$ Massless matter:
Formal linear sum of $\mathbb{P}^1_i$ fibrations
($\leftrightarrow$ weight $\beta^a(\mathbf{R})$ of irrep. $\mathbf{R}$ of G)

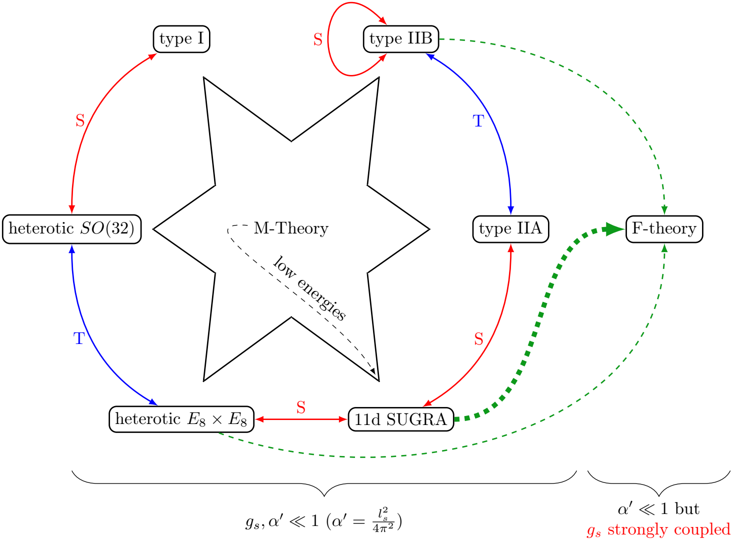
$\longleftrightarrow$ matter curve
($\leftrightarrow$ intersections of 7-branes)

Singular locus Δ $\longleftrightarrow$ 7-branes

Massless matter [Katz Vafa '96], [Witten '96], [Grassi,Morrison '00 & '11], [Morrison,Taylor '11], [Grassi,Halverson,Shaneson '13], [Cvetič,Klevers,Piragua,Taylor '15], [Anderson,Gray,Raghuram,Taylor '15], [Klevers,Taylor '16], [Klevers,Morrison,Raghuram,Taylor '17], ...



G_4-fluxes and M-theory 3-form C_3



Origin of G_4-flux: M-theory 3-form C_3

11d SUGRA action ($G_4 = dC_3$)

$$S_{11D} = \frac{M_{11D}^9}{2} \int_{M_{11}} d^{11}x \left(\sqrt{-\det G} R - \frac{G_4 \wedge * G_4}{2} - \frac{C_3 \wedge G_4 \wedge G_4}{6} \right)$$

Consequence

- M2-branes couple electrically to 3-form gauge potential C_3
- $G_4 = dC_3 \in H^{2,2}(\hat{Y}_4)$ is field strength

Questions

- What specifies gauge data beyond field strength G_4?

$\Rightarrow$ Look for structure which combines information on
  - field strength $G_4 \in H^{2,2}(\hat{Y}_4)$
  - Wilson line d.o.f. $\oint C_3$

Full gauge data from Deligne cohomology

Natural candidate in mathematics [Curio,Donagi '98], [Donagi,Wijnholt '12/13], [Anderson,Heckman,Katz '13], [Intriligator,Jockers,Mayr,Morrison,Plesser '12]

$$0 \to J^2(\hat{Y}_4) \hookrightarrow H^4_D(\hat{Y}_4, \mathbb{Z}(2)) \twoheadrightarrow H^{2,2}(\hat{Y}_4) \to 0$$

$J^2(\hat{Y}_4) \simeq \frac{H^3(\hat{Y}_4,\mathbb{C})}{H^{2,1}(\hat{Y}_4)+H^3(\hat{Y}_4,\mathbb{Z})}$	$\leftrightarrow$	Wilson lines $\oint C_3$
$H^4_D(\hat{Y}_4, \mathbb{Z}(2))$	$\leftrightarrow$	full gauge data
$H^{2,2}(\hat{Y}_4)$	$\leftrightarrow$	field strength G_4

Drawback

- $H^4_D(\hat{Y}_4, \mathbb{Z}(2))$ is hard to handle (practically)

$\Rightarrow$ Easy-to-work-with parametrisation: $\mathrm{CH}^2(\hat{Y}_4)$ [Green Murre Voisin '94]

Describe full G_4-gauge data by $A \in \mathrm{CH}^2(\hat{Y}_4)$

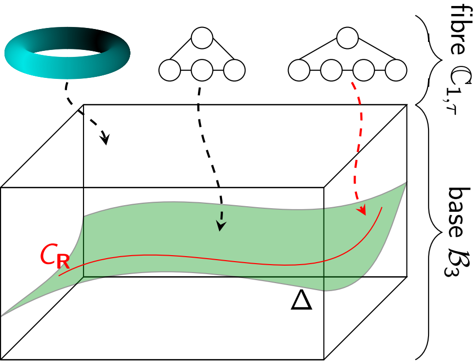
How does this parametrization work? [H. Esnault, E. Viehweg '88] – see also [Braun,Collinucci,Valandro '11]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}^2_{\mathrm{hom}}(\hat{Y}_4) & \longrightarrow & \mathrm{CH}^2(\hat{Y}_4) & \xrightarrow{\gamma_2} & H^{2,2}_{\mathrm{alg}}(\hat{Y}_4) & \longrightarrow & 0 \\ & & \downarrow \mathrm{AJ} & & \downarrow \hat{\gamma}_2 & & \downarrow & & \\ 0 & \longrightarrow & J^2(\hat{Y}_4) & \longrightarrow & H^4_D(\hat{Y}_4, \mathbb{Z}(2)) & \xrightarrow{\hat{c}_2} & H^{2,2}(\hat{Y}_4) & \longrightarrow & 0 \end{array}$$

Definition of Chow group $\mathrm{CH}^2(\hat{Y}_4)$

- Rational equivalence:
  $C_1 \sim C_2 \in Z_2(\hat{Y}_4)$ iff $C_1 - C_2$ is zero/pole of a **rational** function defined on 3-dim. irreducible subspace of $\hat{Y}_4$
- $\mathrm{CH}^2(\hat{Y}_4) = \{\text{rational equivalence classes of 2-cycles}\}$

Recipe [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]



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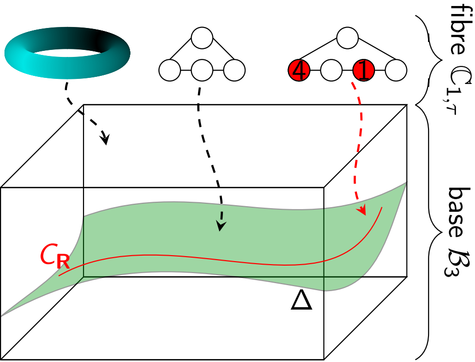

1. Massless matter:
   $\leftrightarrow S_{\mathbf{R}} = 4\mathbb{P}^1_A + \mathbb{P}^1_C \in \mathrm{CH}^2(\hat{Y}_4)$

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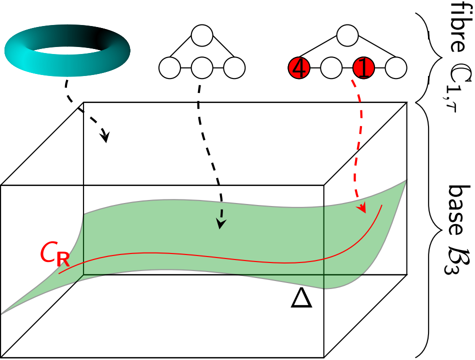

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2. **Full** G_4-gauge data:
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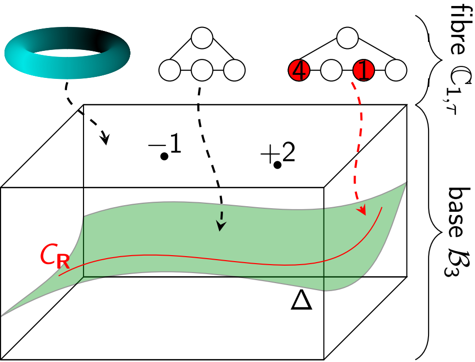

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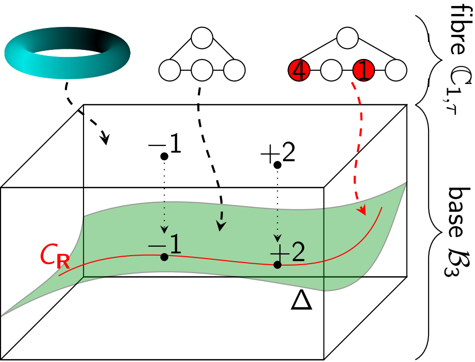

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4. $\pi_* \left(S_{\mathbf{R}} \cdot A \right) \mathrel{\hat{=}}$ points in $C_{\mathbf{R}}$

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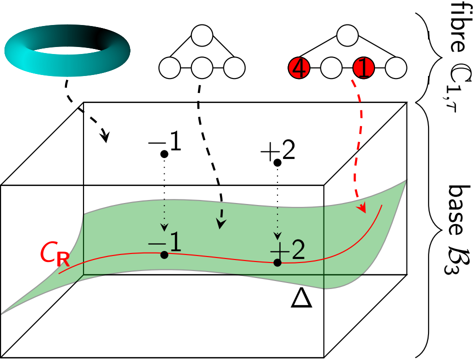

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3. $S_{\mathbf{R}}$ and A intersect in points of $\hat{Y}_4$
4. $\pi_* (S_{\mathbf{R}} \cdot A) \mathrel{\hat{=}}$ points in $C_{\mathbf{R}}$
5. line bundle $L(S_{\mathbf{R}}, A)$ on $C_{\mathbf{R}}$
   $\mathcal{O}_{C_{\mathbf{R}}}(\pi_*(S_{\mathbf{R}} \cdot A)) \otimes \sqrt{K_{C_{\mathbf{R}}}}$

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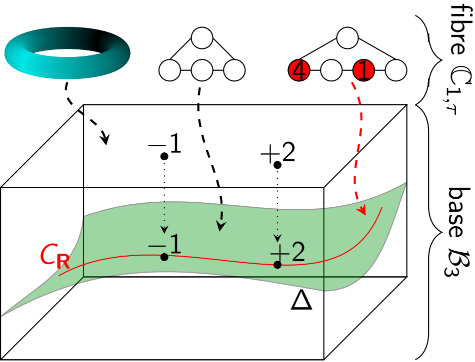

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Consequence [Katz,Sharpe'02] [Beasley,Heckman,Vafa'08] [Donagi,Wijnholt'08]

$\mathcal{N} = 1$ chiral multiplets $\leftrightarrow H^0\left(C_\mathbf{R}, L\left(S_\mathbf{R}, A \right) \right)$

$\mathcal{N} = 1$ anti-chiral multiplets $\leftrightarrow H^1\left(C_\mathbf{R}, L\left(S_\mathbf{R}, A \right) \right)$

Towards coherent sheaves

Challenge: Sheaf cohomologies of $L(S^a_{\mathbf{R}}, A)$ hard to determine

- $L(S^a_{\mathbf{R}}, A)$ in general not pullback [M.B. Mayrhofer Weigand '17]
- Hypercharge flux must not be a pullback [Braun, Collinucci, Valandro '14]

Simplification: assume embedding $\iota \colon \hat{Y}_4 \hookrightarrow X$ in 'simple' space X

- Extend $L(S^a_{\mathbf{R}}, A)$ 'by zero' outside of matter curve $C_{\mathbf{R}}$

$\Rightarrow$ Obtain coherent sheaf $\mathcal{F} \in \mathfrak{Coh}(X)$, i.e. locally

$$\mathcal{F}|_U \cong \operatorname{cok}\left(\mathcal{O}_X^{\oplus I}\Big|_U \xleftarrow{M} \mathcal{O}_X^J\Big|_U \right) ,$$

M is s.t. $\mathcal{F}$ matches $L(S^a_{\mathbf{R}}, A)$ on $C_{\mathbf{R}}$ and is otherwise trivial

- Example: Structure sheaf of $V(P) = \{P = 0\}$ is given by

$$\mathcal{O}_{V(P)} \cong \operatorname{cok}\left(\mathcal{O}_X \xleftarrow{P} \mathcal{O}_X \right)$$

$\Rightarrow$ Q: Can we handle these sheaves for 'simple' spaces X?

Toric varieties as ambient spaces

Why?

- Toric varieties form a very large class of geometries
- Many aspects of toric varieties are computationally under control, e.g. intersection theory

What? Example – projective space

- $\mathbb{P}^{n-1} = (\mathbb{C}^n - \{\mathbf{0}\})/\mathbb{C}^*$ with

$$\mathbb{C}^* \colon \lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n) \,.$$

- Coordinate ring (Cox ring): $S = \mathbb{C}[x_1, x_2, \ldots, x_n]$, $\deg(x_i) = 1$
- Stanley-Reisner ideal ('forbidden locus'): $I_{SR} = \langle x_1 x_2 \cdot \cdots \cdot x_n \rangle$.

Coherent sheaves on toric varieties

Sheafification functor [Cox Little Schenck '11] – see also [Barakat Lange-Hegermann '12]

- $S\text{-fpgrmod}$: **category** of finitely presented graded S-modules
- Any $A \in S\text{-fpgrmod}$ is of the form

$$A \cong \operatorname{cok}\left(\bigoplus_{i=1}^{n} S(d_i) \xleftarrow{M} \bigoplus_{j=1}^{m} S(e_j) \right) .$$

- Q: Does $A \in S\text{-fpgrmod}$ correspond to coherent sheaf on X_Σ?
  A: Yes, there exists the sheafification functor

$$\widetilde{\ } : S\text{-fpgrmod} \to \mathfrak{Coh} X_\Sigma \, , \; M \mapsto \widetilde{M} \qquad (S \mapsto \mathcal{O}_{X_\Sigma})$$

Consequence

$S\text{-fpgrmod}$ models coherent sheaves on X_Σ

Towards Freyd categories and monoidal structures

Counting global sections on toric varieties [Cox Little Schenck '11], [Smith '98], [Blumenhagen Jurke Rahn Roschy '10], [M.B. '18]

- $H^0(X_\Sigma, \mathcal{F}) = \Gamma\left(\mathscr{H}om_{\mathcal{O}_{X_\Sigma}}(\mathcal{O}_{X_\Sigma}, \mathcal{F})\right)$
- Algebraic counterpart in S-fpgrmod:
  For **suitable** $F, I \in S$-fpgrmod with $\widetilde{F} \cong \mathcal{F}$ and $\widetilde{I} \cong \mathcal{O}_{X_\Sigma}$ have

$$H^0(X_\Sigma, \mathcal{F}) \cong \underline{\mathrm{Hom}}_S(I, F)_{=0} \, .$$

Towards efficient computer models ...

- S-fpgrmod is a Freyd category
- Internal hom $\underline{\mathrm{Hom}}_S$ is part of monoidal structure

$\Rightarrow$ What can we learn about monoidal structures on Freyd categories?

Questions so far?

1. Massless matter in **resolved** elliptic fibration $\hat{Y}_4$
   $\leftrightarrow$ $\mathbb{P}^1$-fibration over matter curve
2. Parametrize G_4-flux beyond field strength
   $\leftrightarrow$ Chow group $\mathrm{CH}^2(\hat{Y}_4)$
   ($2_{\mathbb{C}}$-cycles modulo rational equivalence)
3. Count massless matter
   $\leftrightarrow$ cohomologies of coherent sheaves
4. Explicit computations in toric spaces

Coherent sheaf	$\leftrightarrow$	Object in Freyd category $\mathcal{A}(\mathbf{A})$
Sheaf cohomologies	$\leftrightarrow$	Monoidal structure on $\mathcal{A}(\mathbf{A})$

Freyd categories – generalities [P. Freyd '65], [A. Beligiannis '00]

Why are Freyd categories interesting?

- Completely constructive [Posur '17]
  - `CAP`-package *Freyd categories*
  - Computer models for coherent (toric) sheaves in *SheafCohomologyOnToricVarieties*
- Unified framework for f.p. (graded) modules and f.p. functors
- Iteration yields approach to free Abelian category

Any **additive** category $\mathbf{A}$ admits a Freyd category $\mathcal{A}(\mathbf{A})$ s.t.

$\mathbf{A} \subseteq \mathcal{A}(\mathbf{A})$ and $\mathcal{A}(\mathbf{A})$ has cokernels

Work by [M.B. Posur '19] – what and why?

What did we find?

Promonoidal structures on $\mathbf{A}$ $\leftrightarrow$ monoidal structure on $\mathcal{A}(\mathbf{A})$

This is important because ...

- it provides tensor products of finitely presented functors
- it allows studies of monoidal structures on free Abelian categories [M. Prest '09]
- it offers simple approach to Day convolution [B. Day '70 & '72] in f.p. context
- it provides efficient structure for computer implementations of Freyd categories (in particular $\underline{\mathrm{Hom}}_S$ for S-fpgrmod)

[M.B. Posur '19] – How?

How does the corresondance of (pro)monoidal structures arise?

Follows from **multilinear 2-categorical universal property of Freyd categories**: There exists an equivalence of categories

$$\mathrm{Hom}((\mathbf{A}_i)_{i \in \underline{n}}, \mathbf{B}) \simeq \mathscr{H}om^{\mathrm{r}}\left((\mathcal{A}(\mathbf{A}_i))_{i \in \underline{n}}, \mathbf{B}\right)$$

Program

1. Constructive approach to Freyd categories
2. **Bi**linear 2-categorical universal property
3. Application to (pro)monoidal structures

Freyd categories – objects, morphisms and cokernels

Notation

- $a, b, c, \ldots$ are objects of $\mathbf{A}$
- $A, B, C, \ldots$ are objects of $\mathcal{A}(\mathbf{A})$

Objects

Be $a \xleftarrow{\rho_a} r_a \in \mathrm{Mor}(\mathbf{A})$, then $A \equiv (a \xleftarrow{\rho_a} r_a) \in \mathrm{Obj}(\mathcal{A}(\mathbf{A}))$

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Be $a \xleftarrow{\rho_a} r_a \in \mathrm{Mor}(\mathbf{A})$, then $A \equiv (a \xleftarrow{\rho_a} r_a) \in \mathrm{Obj}(\mathcal{A}(\mathbf{A}))$

Morphism $\{\alpha, \omega_\alpha\}\colon (a \xleftarrow{\rho_a} r_a) \to (b \xleftarrow{\rho_a} r_b)$

$$\begin{array}{ccc} a & \xleftarrow[\rho_a]{} & r_a \\ & & \\ b & \xleftarrow{\rho_b} & r_b \end{array} \qquad \longleftrightarrow \qquad \begin{array}{c} A \\ \big\downarrow \{\alpha, \omega_\alpha\} \\ B \end{array}$$

Freyd categories – objects, morphisms and cokernels

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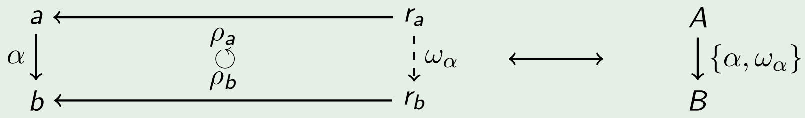

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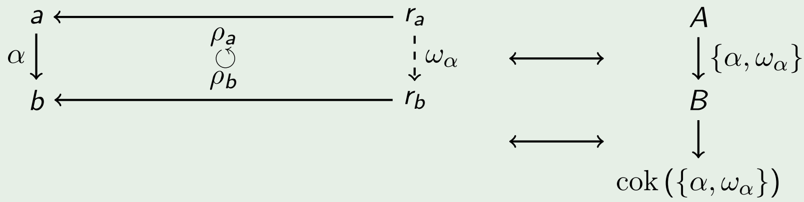

Freyd categories – objects, morphisms and cokernels

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Morphism $\{\alpha, \omega_\alpha\}$: $(a \xleftarrow{\rho_a} r_a) \to (b \xleftarrow{\rho_b} r_b)$

$$
\begin{array}{ccccc}
a & \xleftarrow{\qquad\qquad} & r_a & & A \\
\alpha \downarrow & \rho_a \circlearrowleft \rho_b & \downarrow \omega_\alpha & \longleftrightarrow & \downarrow \{\alpha, \omega_\alpha\} \\
b & \xleftarrow{\qquad\qquad} & r_b & & B \\
\mathrm{id}_b \downarrow & & \mathrm{id}_{r_b} \oplus 0 \downarrow & \longleftrightarrow & \downarrow \\
b & \xleftarrow{\rho_b \oplus \alpha} & r_b \oplus a & & \mathrm{cok}\left(\{\alpha, \omega_\alpha\}\right)
\end{array}
$$

More constructions for $\mathcal{A}(\mathbf{A})$

Systematic analysis and implementation in CAP

- Systematic analysis [Posur '17]

$\Rightarrow$ Constructive approach to direct sums, pullbacks, . . .

- Implementation in CAP-package ▸ Freyd categories

Central philosophy of CAP

- Derive complicated construction from simpler constructions (*https://homalg-project.github.io/capdays-2018/ program/*)
- Example: Pullback $\leftrightarrow$ product + difference + kernel ▸ Details

$\Rightarrow$ Goal: Algorithms for monoidal structures of Freyd categories

Definition of two categories

Category $\mathrm{Hom}((\mathbf{A}_i)_{i \in 1,2}, \mathbf{B})$

- Objects: Bilinear functors $\mathbf{A}_1 \times \mathbf{A}_2 \xrightarrow{F} \mathbf{B}$
- Morphisms: Natural transformations

Category $\mathscr{H}om^{\mathrm{r}}((\mathcal{A}(\mathbf{A}_i))_{i \in 1,2}, \mathbf{B})$

- Objects: Bilinear functors $\mathcal{A}(\mathbf{A}_1) \times \mathcal{A}(\mathbf{A}_2) \xrightarrow{F} \mathbf{B}$ such that

$$0 \leftarrow F(\mathrm{cok}(\alpha_1), \mathrm{cok}(\alpha_2)) \leftarrow F(a_1, a_2) \xleftarrow{\begin{pmatrix} F(\mathrm{id}_{a_1}, \alpha_2) \\ F(\alpha_1, \mathrm{id}_{a_2}) \end{pmatrix}} \begin{matrix} F(a_1, b_2) \\ \oplus \\ F(b_1, a_2) \end{matrix}$$

  is exact for any two morphisms $a_1 \xleftarrow{\alpha_1} b_1$, $a_2 \xleftarrow{\alpha_2} b_2$
- Morphisms: Natural transformations

Universal property and strategy of proof

Bilinear 2-categorical universal property of Freyd categories

There exists an equivalence of categories

$$\mathrm{Hom}((\mathbf{A}_i)_{i \in 1,2}, \mathbf{B}) \simeq \mathscr{H}\!om^{\mathrm{r}}\left((\mathcal{A}(\mathbf{A}_i))_{i \in 1,2}, \mathbf{B}\right)$$

Revision: Equivalence of categories $C \simeq D$ consists of ...

- functor $F\colon C \to D$
- functor $G\colon D \to C$
- natural isomorphism $\epsilon\colon FG \to \mathrm{id}_D$ (among others $FG(X) \cong X$ for all objects X of D)
- natural isomorphism $\eta\colon GF \to \mathrm{id}_C$ (among others $GF(Y) \cong Y$ for all objects Y of C)

Strategy of proof ▸ Details

1. $\mathrm{Hom}((\mathbf{A}_i)_{i\in 1,2}, \mathbf{B}) \to \mathscr{H}\!om^{\mathrm{r}}((\mathcal{A}(\mathbf{A}_i))_{i\in 1,2}, \mathbf{B}) : F \mapsto \widehat{F}$
   Demand that for $A_1 \in \mathcal{A}(\mathbf{A}_i)$ the following row is exact
   $$0 \longleftarrow \widehat{F}(A_1, A_2) \longleftarrow F(a_1, a_2) \xleftarrow{\begin{pmatrix} F(\mathrm{id}_{a_1}, \rho_{a_2}) \\ F(\rho_{a_1}, \mathrm{id}_{a_2}) \end{pmatrix}} \begin{matrix} F(a_1, r_{a_2}) \\ \oplus F(r_{a_1}, a_2) \end{matrix}$$
2. $\mathscr{H}\!om^{\mathrm{r}}((\mathcal{A}(\mathbf{A}_i))_{i\in 1,2}, \mathbf{B}) \to \mathrm{Hom}((\mathbf{A}_i)_{i\in 1,2}, \mathbf{B}) : G \mapsto G|_{\mathbf{A}_1\times\mathbf{A}_2}$
   Restrict the given functor G to $\mathbf{A}_1 \times \mathbf{A}_2$.
3. Show that for $F \in \mathrm{Hom}((\mathbf{A}_i)_{i\in 1,2}, \mathbf{B})$: $F \cong \widehat{F}\Big|_{\mathbf{A}_1\times\mathbf{A}_2}$
4. Show that for $G \in \mathscr{H}\!om^{\mathrm{r}}((\mathcal{A}(\mathbf{A}_i))_{i\in 1,2}, \mathbf{B})$: $G \cong \widehat{\left(G|_{\mathbf{A}_1\times\mathbf{A}_2}\right)}$

Algorithmic lift of $T\colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A})$

Step 1: Fix notation:

- For $a_1, a_2 \in \mathrm{Obj}(\mathbf{A})$, denote $T(a_1, a_2) \in \mathrm{Obj}(\mathcal{A}(\mathbf{A}))$ by

$$\left(g_T(a_1, a_2) \xleftarrow{\rho_T(a_1,a_2)} r_T(a_1, a_2) \right) .$$

- For $a_1 \xleftarrow{\alpha_1} b_1$, $a_2 \xleftarrow{\alpha_2} b_2$, denote $T(\alpha, \beta) \in \mathrm{Mor}(\mathcal{A}(\mathbf{A}))$ by

$$\begin{array}{ccc}
g_T(b_1, b_2) & \xleftarrow[\rho_T(b_1, b_2)]{\qquad\qquad} & r_T(b_1, b_2) \\
\Big\downarrow \delta_T(\alpha_1, \alpha_2) & \circlearrowleft & \omega_T(\alpha_1, \alpha_2) \Big\downarrow \\
g_T(a_1, a_2) & \xleftarrow{\rho_T(a_1, a_2)} & r_T(a_1, a_2)
\end{array}$$

Algorithmic lift of $T \colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A})$

Recall definition of $\hat{T}(A_1, A_2)$ from exact sequence

$$0 \longleftarrow \widehat{T}(A_1, A_2) \longleftarrow T(a_1, a_2) \xleftarrow{\begin{pmatrix} T(\mathrm{id}_{a_1}, \rho_{a_2}) \\ T(\rho_{a_1}, \mathrm{id}_{a_2}) \end{pmatrix}} \begin{matrix} T(a_1, r_{a_2}) \\ \oplus T(r_{a_1}, a_2) \end{matrix}$$

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Step 2: Express morphism by objects/morphisms in A

$$\hat{T}(A_1, A_2) = \mathrm{cok} \left[\begin{array}{ccc} \begin{matrix} g_T(a_1, r_{a_2}) \\ \oplus g_T(r_{a_1}, a_2) \end{matrix} & \xleftarrow{\begin{pmatrix} \rho_T(a_1, \rho_2) & \\ & \rho_T(\rho_1, a_2) \end{pmatrix}} & \begin{matrix} r_T(a_1, r_{a_2}) \\ \oplus r_T(r_{a_1}, a_2) \end{matrix} \\ \Big\downarrow \begin{pmatrix} \delta_T(\mathrm{id}_{a_1}, \rho_2) \\ \delta_T(\rho_1, \mathrm{id}_{a_2}) \end{pmatrix} & \circlearrowleft & \begin{pmatrix} \omega_T(\mathrm{id}_{a_1}, \rho_2) \\ \omega_T(\rho_1, \mathrm{id}_{a_2}) \end{pmatrix} \Big\downarrow \\ g_T(a_1, a_2) & \xleftarrow[\rho_T(a_1, a_2)]{} & r_T(a_1, a_2) \end{array} \right]$$

Algorithmic lift of $T \colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A})$

Step 3: Final algorithm

$$\hat{T}(A_1, A_2) = \left(g_T(a_1, a_2) \xleftarrow{\begin{pmatrix} \rho_T(a_1,a_2) \\ \delta_T(\mathrm{id}_{a_1},\rho_2) \\ \delta_T(\rho_1,\mathrm{id}_{a_2}) \end{pmatrix}} \begin{matrix} r_T(a_1, a_2) \\ \oplus g_T(a_1, r_{a_2}) \\ \oplus g_T(r_{a_1}, a_2) \end{matrix} \right)$$

[For $a_1, a_2 \in \mathbf{A}$: $T(a_1, a_2) = \left(g_T(a_1, a_2) \xleftarrow{\rho_T(a_1,a_2)} r_T(a_1, a_2) \right)$]

Upshot

- $T \colon \mathbf{A} \times \mathbf{A} \to \mathcal{A}(\mathbf{A}) \leftrightarrow$ **pro**tensor product
- $\widehat{T} \colon \mathcal{A}(\mathbf{A}) \times \mathcal{A}(\mathbf{A}) \to \mathcal{A}(\mathbf{A}) \leftrightarrow$ tensor product

$\Rightarrow$ Extend systematically to (pro)monoidal structures

What and how?

What? Find algorithmic relations

- tensor product $\leftrightarrow$ **pro**tensor product
- tensor unit $\leftrightarrow$ **pro**tensor unit
- ...
- internal-Hom $\widehat{\underline{\mathrm{Hom}}}$ $\leftrightarrow$ **pro**-internal Hom $\underline{\mathrm{Hom}}$

How?

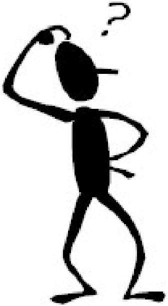
1. Consider monoidal structure on $\mathcal{A}(\mathbf{A})$
2. Restrict to $\mathbf{A}$ by universal 2-categorical property

$\Rightarrow$ **Pro**monoidal structure on $\mathbf{A}$ subject to **restricted** pentagonal identity, hexagonal identities, ...

3. Lift **pro**monoidal structure on $\mathbf{A}$ to $\mathcal{A}(\mathbf{A})$ by universal 2-categorical property

Questions so far?

- Many proper promonoidal structures ▸ Examples
- Internal hom does not always extend ▸ Example
- **A** additive, closed monoidal category
  - $\Rightarrow$ $\mathcal{A}(\mathbf{A})$ is additive, closed monoidal category
  - $\Rightarrow$ Monoidal structures on $\mathcal{A}(\mathcal{A}(\mathbf{A})^{\mathrm{op}})^{\mathrm{op}}$
- Tensor products on Freyd categories
  $\leftrightarrow$ Day convolution of f.p. functors [B. Day '70 & '72]
- Unified implementation of monoidal structures for f.p. (graded) modules and f.p. functors.
  $\rightarrow$ back to physics . . .



Strategy

Why Pati-Salam models?

- Computation in *Quadrillion SMs* [Cvetič Halverson Lin Liu Tian '19] hard ($\leftrightarrow$ complicated matter curves)
- Models can be Higgsed to Pati-Salam model ($\leftrightarrow$ simple geometry)

$\Rightarrow$ Focus on $(SU(4) \times SU(2)^2)/\mathbb{Z}_2$-Pati-Salam models

Geometric realization

- B_3 is toric 3-fold (from Kreuzer-Skarke list 9805190)
- Matter curves: $C_{\mathbf{R}} = V(P_1, P_2)$, $\deg(P_i) = \overline{\mathcal{K}}_{B_3}$
- Matter representations: $(\mathbf{4}, \mathbf{1}, \mathbf{2})$, $(\mathbf{4}, \mathbf{2}, \mathbf{1})$, $(\mathbf{6}, \mathbf{1}, \mathbf{1})$, $(\mathbf{1}, \mathbf{2}, \mathbf{2})$

Challenges

Challenge 1: Zero mode counting of **real** reps. $(\mathbf{6},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{2},\mathbf{2})$

- No holomorphic matter surface with corresponding weights

$\Rightarrow$ Find 'normal' matter surfaces for special complex structure

Challenge 2: *fractional* pullbacks

- Spectrum of complex representation:

representation	line bundle	chiralities
$(4,1,2)$	$\mathcal{O}_{C_{(\mathbf{4},\mathbf{1},\mathbf{2})}}\left(\left(\frac{1}{2}-\frac{a}{4}\right)\overline{K}_{\mathcal{B}_3}\big\vert_{C_{\mathbf{R}}}\right)$	$-\frac{a}{4}\overline{K}_{\mathcal{B}_3}^3$
$(4,2,1)$	$\mathcal{O}_{C_{(\mathbf{4},\mathbf{2},\mathbf{1})}}\left(\left(\frac{1}{2}+\frac{a}{4}\right)\overline{K}_{\mathcal{B}_3}\big\vert_{C_{\mathbf{R}}}\right)$	$+\frac{a}{4}\overline{K}_{\mathcal{B}_3}^3$

$\Rightarrow$ $\left(\frac{1}{2}\pm\frac{a}{4}\right)\overline{K}_{\mathcal{B}_3}\big\vert_{C_{\mathbf{R}}}$ defines divisor on $C_{\mathbf{R}}$ if Freed-Witten quantization is satisfied

Simple starting point: 4 family models

Assumptions

- chirality ± 4 for reps. $(\mathbf{4}, \mathbf{1}, \mathbf{2})$ and $(\mathbf{4}, \mathbf{2}, \mathbf{1})$
- $\frac{1}{2}\overline{K}_{\mathcal{B}_3}$ is a $\mathbb{Z}$-Cartier divisor of $\mathcal{B}_3$

Total of 408 admissible setups

space	$\overline{K}^3_{\mathcal{B}_3}$	number of triangulations
X^1_j	32	1
X^2_j	32	53
$\vdots$	$\vdots$	$\vdots$
X^8_j	32	30
X^9_j	16	158

Example – space X_1^1

Make sense of the *fractional* line bundles

- *Fractional* pullback: $\mathcal{L}_{(\mathbf{4},\mathbf{1},\mathbf{2})} = \mathcal{O}_{C_{(\mathbf{4},\mathbf{1},\mathbf{2})}}\left(\frac{3}{8} \, \overline{K}_{X_1^1}\big|_{C_{(\mathbf{4},\mathbf{1},\mathbf{2})}} \right)$
- Find $\overline{K}_{X_1^1} = 4V(x_1) + 2V(x_2)$, $V(x_2)|_{C_{(\mathbf{4},\mathbf{1},\mathbf{2})}} = \emptyset$ and

$$V(x_1)|_{C_{(\mathbf{4},\mathbf{1},\mathbf{2})}} = 8V(x_1, x_3, x_4) \equiv 8r$$

$\Rightarrow \mathcal{L}_{(\mathbf{4},\mathbf{1},\mathbf{2})} = \mathcal{O}_{C_{(\mathbf{4},\mathbf{1},\mathbf{2})}}(12 \cdot r)$

Compute their cohomologies

1. Find $L_{(\mathbf{4},\mathbf{1},\mathbf{2})}, L_{(\mathbf{4},\mathbf{2},\mathbf{1})} \in S\text{-fpgrmod}$ such that $\mathcal{L}_{(\mathbf{4},\mathbf{1},\mathbf{2})} \cong \widetilde{L_{(\mathbf{4},\mathbf{1},\mathbf{2})}}$
2. Use gap-package *SheafCohomologyOnToricVarieties* to find cohomologies (computer `Plesken` – Siegen university):

$$h^i(\mathcal{L}_{(\mathbf{4},\mathbf{1},\mathbf{2})}) = (5, 9)$$

Extend the search

Strategy

- Repeat analysis for other 4-family and 3-family models
- Sometimes the spectrum follows from pullback bundles!

Extend the search

Strategy

- Repeat analysis for other 4-family and 3-family models
- Sometimes the spectrum follows from pullback bundles!
- Find 3-family models with

$$h^i(\mathcal{L}_{(\mathbf{4},\mathbf{1},\mathbf{2})}) = (1,4)\,, \qquad h^i(\mathcal{L}_{(\mathbf{4},\mathbf{2},\mathbf{1})}) = (4,1)$$

Extend the search

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- Sometimes the spectrum follows from pullback bundles!
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Strategy

- Repeat analysis for other 4-family and 3-family models
- Sometimes the spectrum follows from pullback bundles!
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$$h^i(\mathcal{L}_{(\mathbf{4},\mathbf{1},\mathbf{2})}) = (1, 4)\,, \qquad h^i(\mathcal{L}_{(\mathbf{4},\mathbf{2},\mathbf{1})}) = (4, 1)$$

Phenomenological challenge: absence of vector-like exotics

- To Higgs the Pati-Salam model to the SM we require: **one** Higgs field in rep. $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ – **none** in $(\mathbf{4}, \mathbf{1}, \mathbf{2})$

$\Rightarrow$ Modify these models to have spectrum $(0, 3)$ and $(4, 1)$ ...

Summary

- Count vector-like spectra 1402.5144, 1706.04616, 1706.08528, 1802.08860
  - G_4-flux $\leftrightarrow A \in \mathrm{CH}^2(\hat{Y}_4)$
  - Massless matter $\leftrightarrow$ cohomologies of $\mathcal{F} \in \mathfrak{Coh}(X_\Sigma)$
- Computer model for $\mathfrak{Coh}(X_\Sigma) \leftrightarrow$ Freyd categories
  - Implementation in `CAP`-package *Freyd categories*
  - Analyse monoidal structures to improve efficiency 1909.00172
    - multilinear 2-categorical universal property
    - $\Rightarrow$ **Pro**monoidal structures $\leftrightarrow$ monoidal structures
    - Approach matches Day convolution of f.p. functors
- Applications to *Quadrillion SMs* 1903.00009
  - Simpler: Analyse Pati-Salam model via toric Higgsing
  - Challenges:
    - No holomorphic matter surface with weights of real reps.
    - *Fractional* pullbacks ($\leftrightarrow$ evaluate intersection product)
  - $\Rightarrow$ Overcome (at special complex structure): 3-family models

$$(\mathbf{4}, \mathbf{1}, \mathbf{2}) : (1, 4) \qquad (\mathbf{4}, \mathbf{2}, \mathbf{1}) : (4, 1)$$

Outlook

Phenomenological challenge: Absence of exotics

- Assumption: Pati-Salam Higgs field in $(\mathbf{4}, \mathbf{2}, \mathbf{1})$

$\Rightarrow$ Desired spectrum without exotics

$$(\mathbf{4}, \mathbf{1}, \mathbf{2}) \colon (0, 3) \, , \qquad (\mathbf{4}, \mathbf{2}, \mathbf{1}) \colon (4, 1)$$

- **But** our best models only satisfy

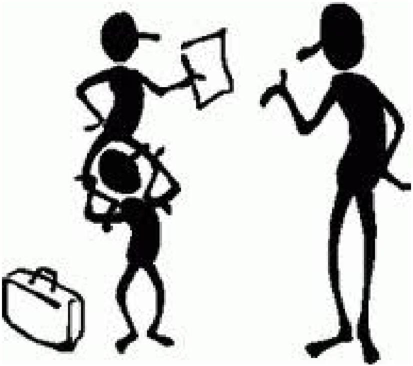
$$(\mathbf{4}, \mathbf{1}, \mathbf{2}) \colon {\color{red}(1, 4)} \, , \qquad (\mathbf{4}, \mathbf{2}, \mathbf{1}) \colon (4, 1)$$

$\Rightarrow$ Systematics of adding/removing vector-like pairs?
(Horizontal fluxes, tuning of complex structure, ...)

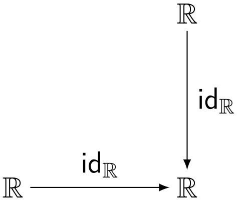
Mathematics

Tensor products on the free Abelian category

Thank you for your attention!



CAP-philosophy: Pullback from product, difference, kernel



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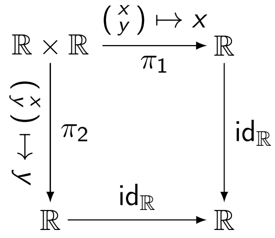
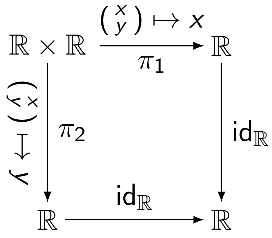


Steps

1. Product $\mathbb{R} \times \mathbb{R}$

CAP-philosophy: Pullback from product, difference, kernel

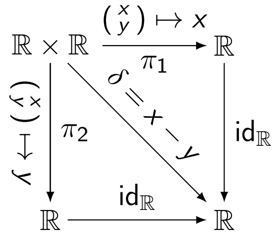


Steps

1. Product $\mathbb{R} \times \mathbb{R}$

$\Rightarrow$ $\mathrm{id}_{\mathbb{R}} \circ \pi_1 \neq \mathrm{id}_{\mathbb{R}} \circ \pi_2$

CAP-philosophy: Pullback from product, difference, kernel



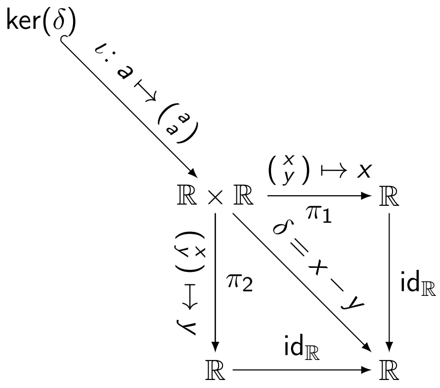
Steps

1. Product $\mathbb{R} \times \mathbb{R}$

$\Rightarrow$ $\mathrm{id}_{\mathbb{R}} \circ \pi_1 \neq \mathrm{id}_{\mathbb{R}} \circ \pi_2$

2. Consider difference $\delta = \mathrm{id}_{\mathbb{R}} \circ \pi_1 - \mathrm{id}_{\mathbb{R}} \circ \pi_2$

CAP-philosophy: Pullback from product, difference, kernel



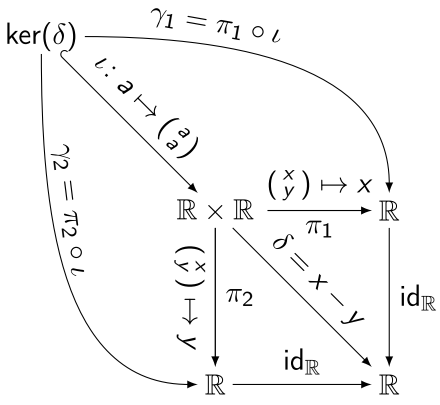
Steps

1. Product $\mathbb{R} \times \mathbb{R}$

$\Rightarrow$ $\mathrm{id}_{\mathbb{R}} \circ \pi_1 \neq \mathrm{id}_{\mathbb{R}} \circ \pi_2$

2. Consider difference
   $\delta = \mathrm{id}_{\mathbb{R}} \circ \pi_1 - \mathrm{id}_{\mathbb{R}} \circ \pi_2$
3. Kernel embedding
   $\iota \colon \ker(\delta) \cong \mathbb{R} \hookrightarrow \mathbb{R} \times \mathbb{R}$

CAP-philosophy: Pullback from product, difference, kernel



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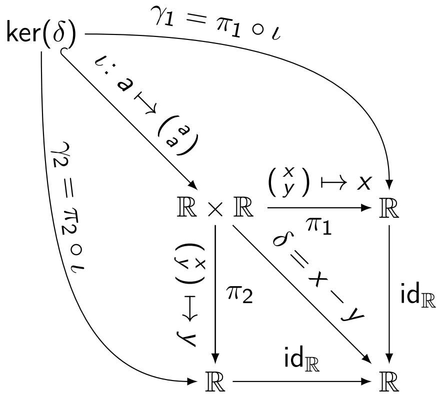
$\Rightarrow$ $\mathrm{id}_{\mathbb{R}} \circ \pi_1 \neq \mathrm{id}_{\mathbb{R}} \circ \pi_2$

2. Consider difference
   $\delta = \mathrm{id}_{\mathbb{R}} \circ \pi_1 - \mathrm{id}_{\mathbb{R}} \circ \pi_2$
3. Kernel embedding
   $\iota \colon \ker(\delta) \cong \mathbb{R} \hookrightarrow \mathbb{R} \times \mathbb{R}$
4. Define $\gamma_i := \pi_i \circ \iota$

CAP-philosophy: Pullback from product, difference, kernel



Steps

1. Product $\mathbb{R} \times \mathbb{R}$

$\Rightarrow$ $\mathrm{id}_{\mathbb{R}} \circ \pi_1 \neq \mathrm{id}_{\mathbb{R}} \circ \pi_2$

2. Consider difference $\delta = \mathrm{id}_{\mathbb{R}} \circ \pi_1 - \mathrm{id}_{\mathbb{R}} \circ \pi_2$
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$\Rightarrow$ $\mathrm{id}_{\mathbb{R}} \circ \gamma_1 = \mathrm{id}_{\mathbb{R}} \circ \gamma_2$ and $(\ker(\delta), \gamma_1, \gamma_2)$ satisfy universal property

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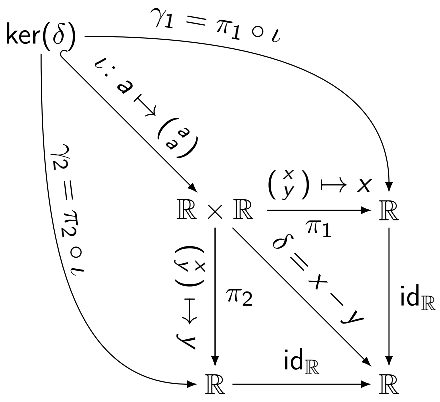

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Many such derived algorithms available in CAP

https://github.com/homalg-project/CAP_project ◂ Back

Step 1: From $\mathrm{Hom}((\mathbf{A}_i)_{i \in 1,2}, \mathbf{B})$ to $\mathscr{H}om^{\mathrm{r}}((\mathcal{A}(\mathbf{A}_i))_{i \in 1,2}, \mathbf{B})$

1. Start with bilinear functor $F \colon \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{B}$
2. Consider objects $A_i = (a_i \xleftarrow{\rho_{a_i}} r_{a_i})$, $B_i = (b_i \xleftarrow{\rho_{b_i}} r_{b_i})$ and morphisms $A_i \xleftarrow{\{\alpha_i, \omega_i\}} B_i$
3. Define $\hat{F} \colon \mathcal{A}(\mathbf{A}_1) \times \mathcal{A}(\mathbf{A}_2) \to \mathbf{B}$ by exactness of the diagram

$$
\begin{array}{ccccccc}
0 & \longleftarrow & \widehat{F}(A_1, A_2) & \longleftarrow & F(a_1, a_2) & \xleftarrow{\begin{pmatrix} F(\mathrm{id}_{a_1}, \rho_{a_2}) \\ F(\rho_{a_1}, \mathrm{id}_{a_2}) \end{pmatrix}} & F(a_1, r_{a_2}) \oplus F(r_{a_1}, a_2) \\
 & & \Big\downarrow {\scriptstyle \widehat{F}(\{\alpha_1\}, \{\alpha_2\})} & \circlearrowleft & \Big\downarrow {\scriptstyle F(\alpha_1, \alpha_2)} & \circlearrowleft & \Big\downarrow {\scriptstyle \begin{pmatrix} F(\alpha_1, \omega_{\alpha_2}) & \\ & F(\omega_{\alpha_1}, \alpha_2) \end{pmatrix}} \\
0 & \longleftarrow & \widehat{F}(B_1, B_2) & \longleftarrow & F(b_1, b_2) & \xleftarrow{\begin{pmatrix} F(\mathrm{id}_{b_1}, \rho_{b_2}) \\ F(\rho_{b_1}, \mathrm{id}_{b_2}) \end{pmatrix}} & F(b_1, r_{b_2}) \oplus F(r_{b_1}, b_2)
\end{array}
$$

Step 2: Consequences and definition of restriction

Properties of $F \mapsto \widehat{F}$

- $\widehat{\mathrm{id}_F} = \mathrm{id}_{\widehat{F}}$ for all bilinear functors $F : \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{B}$
- $\widehat{\nu \circ \mu} = \widehat{\nu} \circ \widehat{\mu}$ for all composable natural transformation ν, μ

$\Rightarrow$ Have a well-defined functor

$$\mathrm{Hom}((\mathbf{A}_i)_{i \in 1,2}, \mathbf{B}) \longrightarrow \mathscr{H}om^{\mathrm{r}}((\mathcal{A}(\mathbf{A}_i))_{i \in 1,2}, \mathbf{B}) : F \mapsto \widehat{F}$$

Definition of restriction

Let $\mathrm{emb} : \prod_{i \in 1,2} \mathbf{A}_i \hookrightarrow \prod_{i \in 1,2} \mathcal{A}(\mathbf{A}_i)$ denote componentwise embedding. Then consider

$$\mathscr{H}om^{\mathrm{r}}((\mathcal{A}(\mathbf{A}_i))_{i \in 1,2}, \mathbf{B}) \longrightarrow \mathrm{Hom}((\mathbf{A}_i)_{i \in 1,2}, \mathbf{B})$$
$$G \mapsto G|_{\mathbf{A}_1 \times \mathbf{A}_2} := G \circ \mathrm{emb}$$

Step 3: Argue for natural isomorphisms

Back to strategy

- For $(a_1, a_2) \in \mathbf{A}_1 \times \mathbf{A}_2$ obtain natural isomorphism

$$\widehat{F}(\mathrm{emb}(a_1, a_2)) \simeq \mathrm{cok}\left(F(a_1, a_2) \xleftarrow{\begin{pmatrix} F(\mathrm{id}_{a_1}, 0) \\ F(0, \mathrm{id}_{a_2}) \end{pmatrix}} \begin{matrix} F(a_1, 0) \\ \oplus F(0, a_2) \end{matrix} \right)$$

$$\simeq \mathrm{cok}\left(F(a_1, a_2) \longleftarrow 0 \right) \simeq F(a_1, a_2)$$

- For $(A_1, A_2) \in \mathcal{A}(\mathbf{A}_1) \times \mathcal{A}(\mathbf{A}_2)$ obtain natural isomorphism

$$G(A_1, A_2) \simeq \mathrm{cok}\left(G(\mathrm{emb}(a_1, a_2)) \longleftarrow \begin{matrix} G(\mathrm{emb}(a_1, b_2)) \\ \oplus G(\mathrm{emb}(b_1, a_2)) \end{matrix} \right)$$

$$\simeq \mathrm{cok}\left(G|_{\mathbf{A}_1 \times \mathbf{A}_2}(a_1, a_2) \longleftarrow \begin{matrix} G|_{\mathbf{A}_1 \times \mathbf{A}_2}(a_1, b_2) \\ \oplus \; G|_{\mathbf{A}_1 \times \mathbf{A}_2}(b_1, a_2) \end{matrix} \right)$$

$$\simeq \widehat{G|_{\mathbf{A}_1 \times \mathbf{A}_2}}(A_1, A_2)$$

Necessity of promonoidal structures

Back

What?

There are promonoidal structures which are not monoidal.

Example in R-fpmod (R commutative ring)

- Every $M \in R\text{-fpmod}$ gives rise to a right-exact bilinear functor

$$T \colon R\text{-fpmod} \times R\text{-fpmod} \to R\text{-fpmod}\,,\ (A, B) \mapsto A \otimes_R M \otimes_R B$$

$\Rightarrow$ R-fpmod becomes semimonoidal category & T tensor product

- Restriction to Rows_R gives prosemimonoidal structure
- Protensor product of two objects in Rows_R lies outside of Rows_R whenever M is not a row module

Internal Homs do not always extend

Back

- Consider $R = \mathbb{Q}[x_i, z | i \in \mathbb{N}]$ and $\mathbf{A} = \mathrm{Rows}_R$ with ordinary tensor product

$\Rightarrow$ Induced tensor product on R-fpmod is the ordinary tensor product

- We argue that it has no right-adjoint:
  - $\mathrm{Hom}_R(R/\langle z \rangle, R) \cong \langle \{x_i | i \in \mathbb{N}\} \rangle$ – not finitely presented
  - Assume there was f.p. $\underline{\mathrm{Hom}}_R$ on R-fpmod. Then:
    - $\underline{\mathrm{Hom}}_R(R/\langle z \rangle, R) = \mathrm{cok}\left(R^{1 \times a} \xleftarrow{M} R^{1 \times b} \right)$
    - Tensor-Hom-adjunction implies

$$\begin{aligned} \mathrm{Hom}_R(R/\langle z \rangle, R) &\cong \mathrm{Hom}_R(1, \underline{\mathrm{Hom}}_R(R/\langle r \rangle, R)) \\ &\cong \mathrm{cok}\left(R^{1 \times a} \xleftarrow{M} R^{1 \times b} \right) \end{aligned}$$

$\Rightarrow$ Contradiction: $\mathrm{Hom}_R(R/\langle z \rangle, R) \cong \langle \{x_i | i \in \mathbb{N}\} \rangle$ is **not** f.p.

Koszul resolution I

Koszul resolution for $\mathcal{O}_\Sigma(D_S)$ with $D_S = \frac{1}{2}\overline{K}_{\mathcal{B}_3}\big|_\Sigma$

- Set $\mathcal{L} = \mathcal{O}_\Sigma\left(\frac{1}{2}\overline{K}_{\mathcal{B}_3}\right)$.
- Matter curve is complete intersection $\Sigma = \{P_1 = P_2 = 0\}$ ($\deg(P_i) = \overline{K}_{\mathcal{B}_3}$)

$\Rightarrow$ Have Koszul resolution $0 \to \mathcal{V}_2 \xrightarrow{M_2} \mathcal{V}_1 \xrightarrow{M_1} \mathcal{L} \to \mathcal{L}|_\Sigma \to 0$ with

$$\mathcal{V}_2 = \mathcal{O}_{\mathcal{B}_3}\left(-\frac{3}{2}\overline{K}_{\mathcal{B}_3}\right), \qquad \mathcal{V}_1 = \mathcal{O}_{\mathcal{B}_3}\left(-\frac{1}{2}\overline{K}_{\mathcal{B}_3}\right)^{\oplus 2}.$$

Strategy

1. Use *cohomCalg* (Blumenhagen et all 2010) and compute cohomologies of $\mathcal{L}$, $\mathcal{V}_1$, $\mathcal{V}_2$
2. Try to deduce cohomologies of $\mathcal{L}$

Koszul resolution II

For $\mathcal{B}_3 = X_1^1$ compute cohomologies of $\mathcal{L}$, $\mathcal{V}_1$, $\mathcal{V}_2$

Non-trivial cohomologies are $h^3(X_1^1, \mathcal{V}_2) = 6$ and $h^0(X_1^1, \mathcal{L}) = 6$

Deduction of cohomologies of $\mathcal{L}$

- Introduce auxilliary sheaf I to split Kozsul resolution

$$0 \to \mathcal{V}_2 \to \mathcal{V}_1 \to I \to 0\,, \qquad 0 \to I \to \mathcal{L} \to \mathcal{L}|_\Sigma \to 0\,.$$

- Use the two induced long exact sequences in cohomologies

$$0 \to h^0(\mathcal{V}_2) \to h^0(\mathcal{V}_1) \to h^0(I) \to h^1(\mathcal{V}_2) \to h^1(\mathcal{V}_1) \to \ldots$$

$$0 \to h^0(I) \to h^0(\mathcal{L}) \to h^0(\mathcal{L}|_\Sigma) \to h^1(I) \to h^1(\mathcal{L}) \to \ldots$$

$\Rightarrow$ $h^0(I) = h^1(I) = h^3(I) = 0$ and $h^2(I) = 6$

$\Rightarrow$ $h^0(\mathcal{L}|_\Sigma) = h^1(\mathcal{L}|_\Sigma) = 6$