Cohomology Of Holomorphic Pullback Line Bundles On Smooth And Compact Normal Toric Varieties

Martin Bies

Feb 11, 2014

Section 1

Motivation From Physics

U(1) Charged Localised Zero Modes

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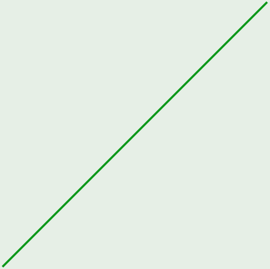
Intersecting D7-Branes



D_7 brane $\mathcal{B}_a$ with
$U(1)_a$ gauge theory $\mathcal{L}_a$

U(1) Charged Localised Zero Modes

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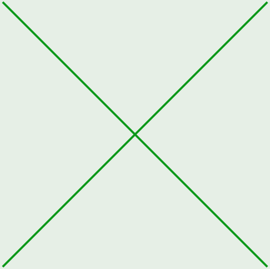

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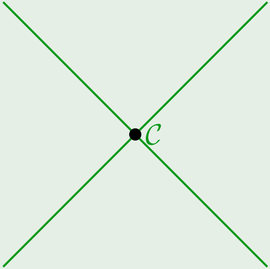

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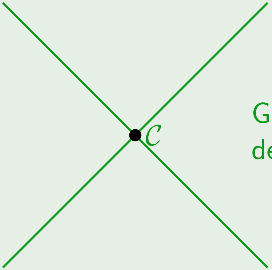

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Gauge enhancement on $\mathcal{C}$ described by $\mathcal{L}_a|_{\mathcal{C}} \otimes \mathcal{L}_b|_{\mathcal{C}}$

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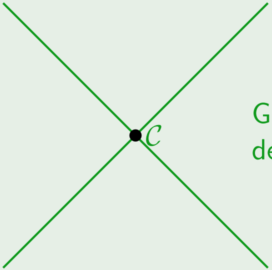

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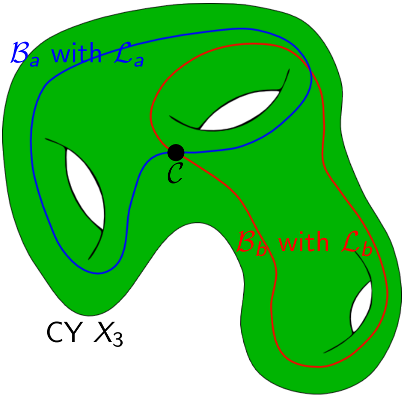
Consequence hep-th/0403166

Spectrum of massless zero modes at $\mathcal{C}$ $\Longleftrightarrow$ $H^0\left(\mathcal{C}, \mathcal{L}_a|_{\mathcal{C}} \otimes \mathcal{L}_b|_{\mathcal{C}}\right)$, $H^1\left(\mathcal{C}, \mathcal{L}_a|_{\mathcal{C}} \otimes \mathcal{L}_b|_{\mathcal{C}}\right)$

U(1) Charged Localised Zero Modes - Simplified Setup



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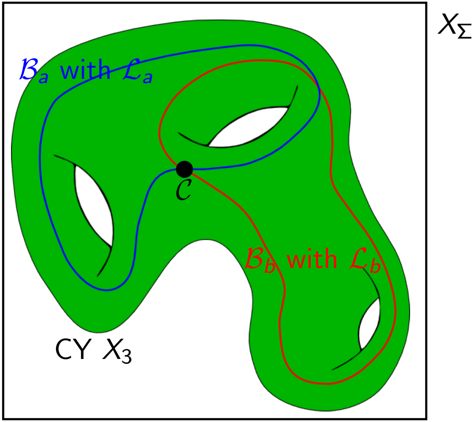

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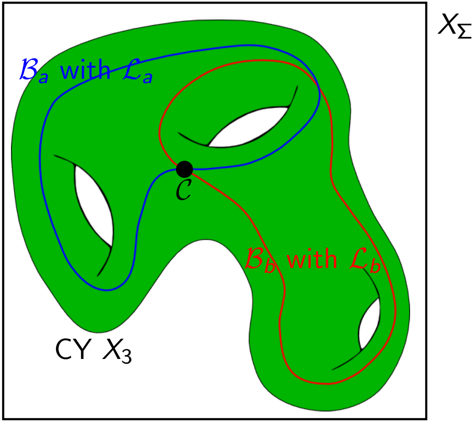

Simplifying Assumptions

- X_Σ a smooth and compact normal toric variety

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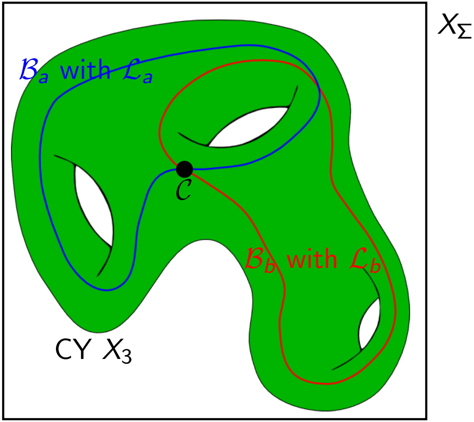

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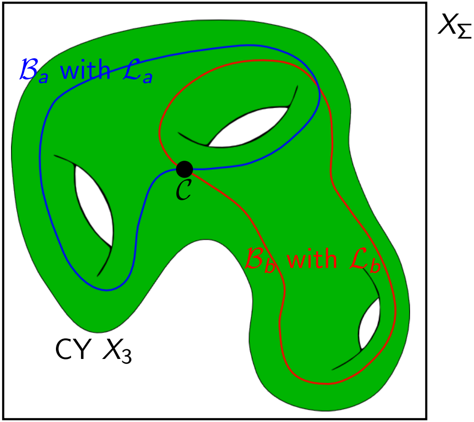

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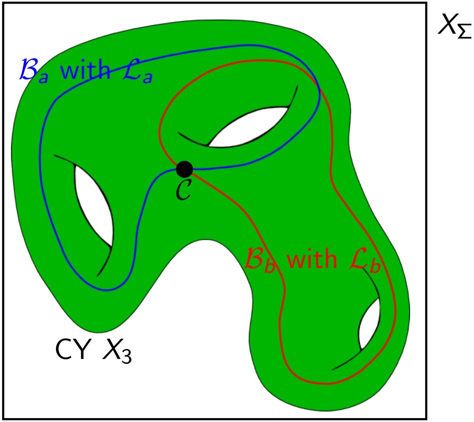

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New Question

How to compute $H^i(C, \mathcal{L}|_C)$?

What Is A Toric Variety?

Book by D. Cox, J. Little, H. Schenk 'Toric Varieties'

Toric Varieties Via Homogenisation

Every smooth and compact normal toric variety X_Σ is given by

$$X_\Sigma \cong \left(\mathbb{C}^r - Z\right) / \left(\mathbb{C}^*\right)^a$$

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Example: Complex Projective Space

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Why Smooth And Compact Normal Toric Varieties?

Let $X_\Sigma \cong (\mathbb{C}^r - Z) \, / \, (\mathbb{C}^*)^a$. Then it holds

- $\operatorname{Pic}(X_\Sigma) \cong \mathbb{Z}^a$.
- $H^i \left(X_\Sigma, \mathcal{O}_{X_\Sigma}(\mathbf{v})\right)$ are finite dimensional vector spaces.

Example: Computation Of Ambient Space Cohomologies

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cohomCalg hep-th/1003.5217, hep-th/1010.3717, math.AG/1006.0780, hep-th/1006.2392

I implemented a function into *Mathematica* which computes a basis of cohomology based on *cohomCalg*.

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Result from *Mathematica*

- $H^0\left(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(1)\right) = \{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\ ,\ \alpha_i \in \mathbb{C}\} \cong \mathbb{C}^3$
- $H^i\left(\mathbb{CP}^3, \mathcal{O}_{\mathbb{CP}^3}(1)\right) = 0$ for $i \geq 1$

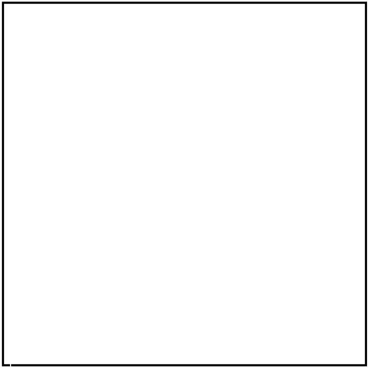
Questions so far?

Section 2

The Hypersurface Case

Task

Task



X_Σ

Ingredients

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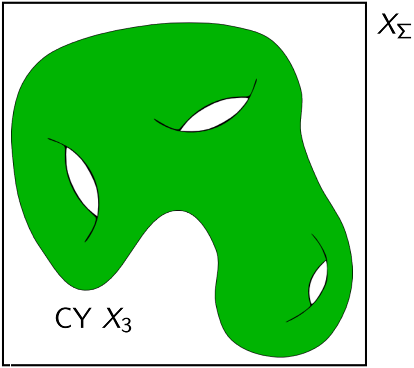

Ingredients

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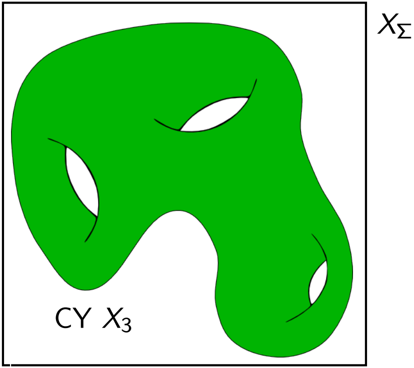
- $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$
- $\widetilde{s}_1 \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_1))$ s.t. $X_3 = \{p \in X_\Sigma \ , \ \widetilde{s}_1(p) = 0\}$

Task



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Goal

Compute $H^i\left(X_3, \mathcal{L}|_{X_3}\right)$.

Hypersurface Case

Theorem ▸ To the proof

The following sequence is sheaf exact

$$0 \to \underbrace{\mathcal{O}_{X_\Sigma}(D - S_1)}_{=\mathcal{L}'} \stackrel{\otimes \widetilde{s}_1}{\to} \underbrace{\mathcal{O}_{X_\Sigma}(D)}_{=\mathcal{L}} \stackrel{r}{\to} \mathcal{O}_{X_\Sigma}(D)|_{X_3} \to 0$$

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Consequence: There is a long exact cohomology sequence

$$\begin{array}{l} 0 \longrightarrow H^0(X_\Sigma, \mathcal{L}') \xrightarrow{\alpha^0} H^0(X_\Sigma, \mathcal{L}) \xrightarrow{\beta^0} H^0(X_3, \mathcal{L}|_{X_3}) \xrightarrow{\delta^0} \\ \quad H^1(X_\Sigma, \mathcal{L}') \xrightarrow{\alpha^1} H^1(X_\Sigma, \mathcal{L}) \longrightarrow H^1(X_3, \mathcal{L}|_{X_3}) \longrightarrow \\ \quad H^2(X_\Sigma, \mathcal{L}') \xrightarrow{\alpha^2} H^2(X_\Sigma, \mathcal{L}) \longrightarrow H^2(X_3, \mathcal{L}|_{X_3}) \end{array}$$

Hypersurface Case II

The E_1-Sheet

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
H^0\left(X_\Sigma, \mathcal{L}'\right) & H^1\left(X_\Sigma, \mathcal{L}'\right) & H^2\left(X_\Sigma, \mathcal{L}'\right) \\
\downarrow \alpha^0 & \downarrow \alpha^1 & \downarrow \alpha^2 \\
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Compute with *Mathematica*.

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\end{array}$$

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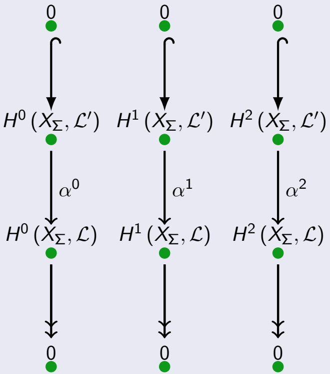
The E_2-Sheet

$$\begin{array}{ccc}
0 & 0 & 0 \\
\ker\left(\alpha^0\right) & \ker\left(\alpha^1\right) & \ker\left(\alpha^2\right) \\
V_0 & V_1 & V_2 \\
0 & 0 & 0
\end{array}$$

$$V_i := H^i\left(X_\Sigma, \mathcal{L}\right) / \mathrm{Im}\left(\alpha^i\right)$$

Hypersurface Case II

The E_1-Sheet



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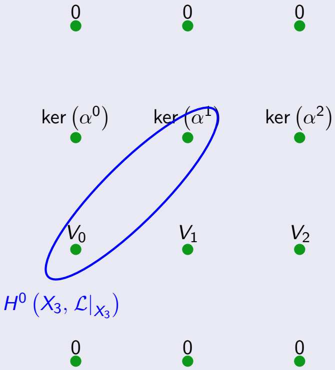

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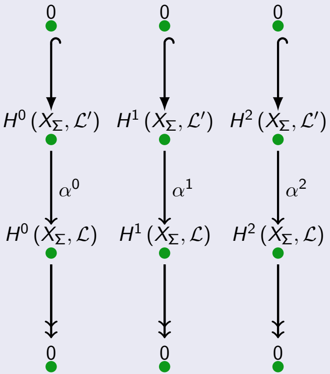

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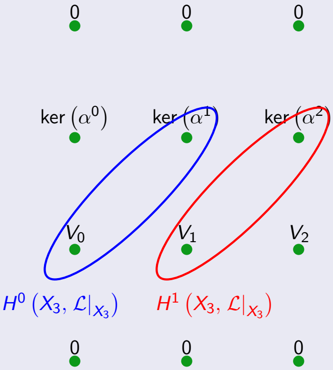

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A Computational Example

Ingredients

- Toric ambient space $X_\Sigma = \mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$
- $\widetilde{s}_1 = C_1 x_1 + C_2 x_2 + C_3 x_3 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}(1,0,0)\right)$
- $\mathcal{L} = \mathcal{O}_{X_\Sigma}(1,0,-2)$

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Result

- *cohomCalg* left an unconstraint constant A_2 in the result.
- My notebook computed this constant to be 0 for pseudo-random $C_i \in (0,1)$. So in this case

$$H^0\left(X_\Sigma, \mathcal{L}|_{X_3}\right) = 0, \qquad H^1\left(X_\Sigma, \mathcal{L}|_{X_3}\right) = 2$$

Questions?

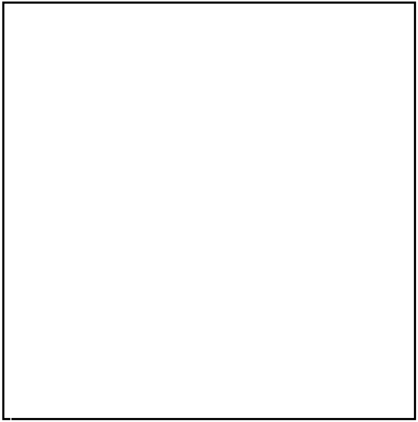
Section 3

The Codimension Two Case

The Task

Ingredients

The Task



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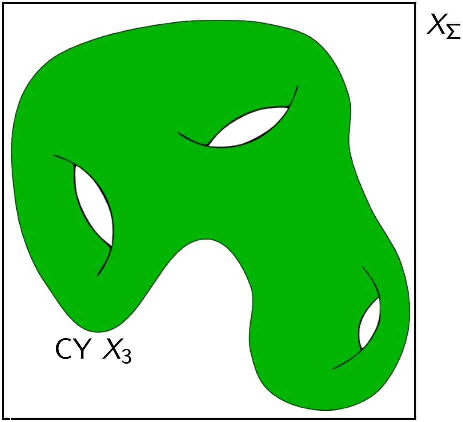

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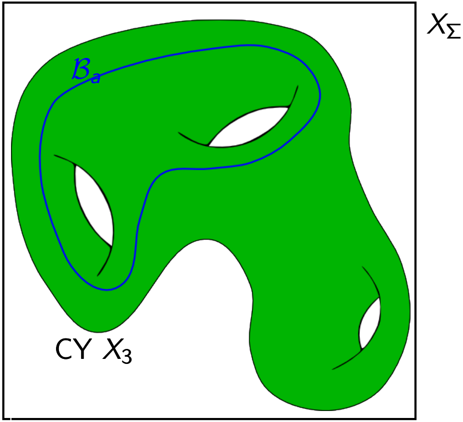

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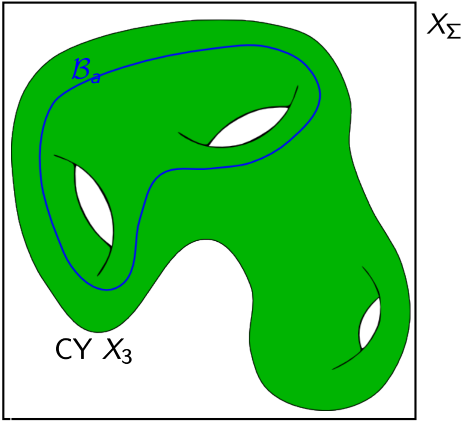

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Codimension 2 Case

Theorem

The following sequence is sheaf exact

$$0 \to \mathcal{L}' \xrightarrow{\otimes \begin{pmatrix} \widetilde{s}_2 \\ -\widetilde{s}_1 \end{pmatrix}} \mathcal{V}_1 \xrightarrow{\otimes(\widetilde{s}_1, \widetilde{s}_2)} \mathcal{L} \xrightarrow{r} \mathcal{L}|_{\mathcal{B}_a} \to 0$$

where

- $\mathcal{L}' = \mathcal{O}\left(D - S_1 - S_2\right)$
- $\mathcal{V}_1 = \mathcal{O}\left(D - S_1\right) \oplus \mathcal{O}\left(D - S_2\right)$

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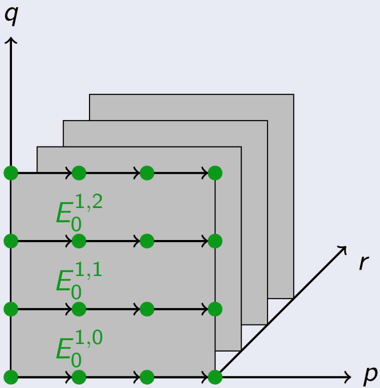
Watch out!

There is **no** associated long exact sequence in cohomology.

Spectral Sequence Construction

Book by D. Cox, J. Little, H. Schenk 'toric varieties', 'Aspects of (2,0) string compactifications' by B. Green, J. Distler

Rough Picture



- $E_0^{p,q}$ are Abelian groups.
- Derive E_0 cohomologies.
- Write those into sheet E_1.
- Derive E_1 cohomologies.
- ...

The Codimension 2 Strategy

The E_1-Sheet

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\big\downarrow & \big\downarrow & \big\downarrow \\
H^0(X_\Sigma, \mathcal{L}') & H^1(X_\Sigma, \mathcal{L}') & H^2(X_\Sigma, \mathcal{L}') \\
\big\downarrow \alpha^0 & \big\downarrow \alpha^1 & \big\downarrow \alpha^2 \\
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\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}$$

The E_2-Sheet

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0 & 0 & 0 \\
\ker(\alpha^0) & \ker(\alpha^1) & \ker(\alpha^2) \\
V_0 & V_1 & V_2 \\
W_0 & W_1 & W_2 \\
0 & 0 & 0
\end{array}$$

Definitions

$V_i := \ker(\beta^i) / \mathrm{im}(\alpha^i)$ $W_i := H^i(X_\Sigma, \mathcal{L}) / \mathrm{im}(\beta^i)$

The Codimension 2 Strategy

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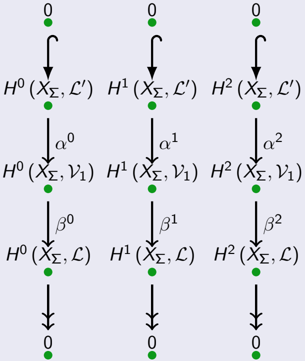
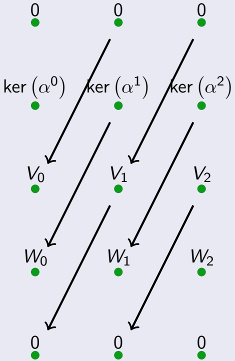


The E_2-Sheet



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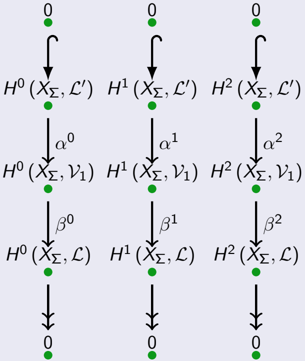
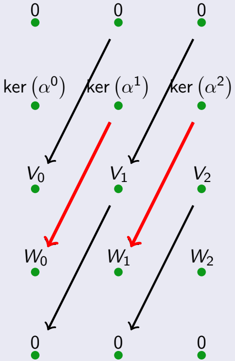


The E_2-Sheet

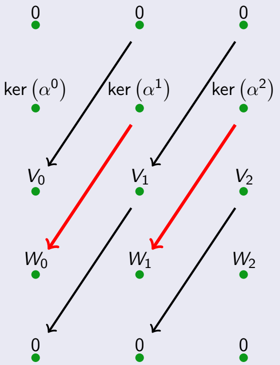


Definitions

$V_i := \ker(\beta^i) / \mathrm{im}(\alpha^i)$ $W_i := H^i(X_\Sigma, \mathcal{L}) / \mathrm{im}(\beta^i)$

The Codimension 2 Strategy

The E_2-Sheet



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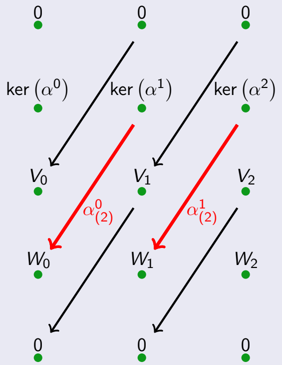

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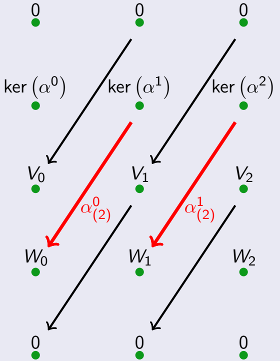
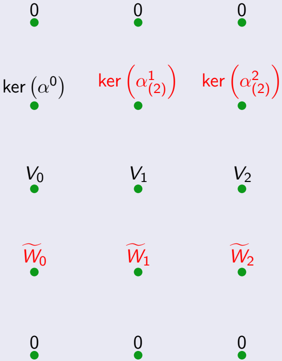


The E_3-Sheet



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The Codimension 2 Strategy

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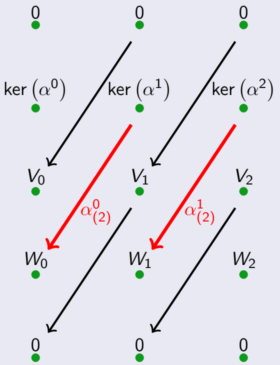

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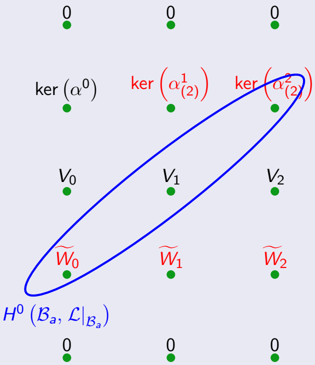

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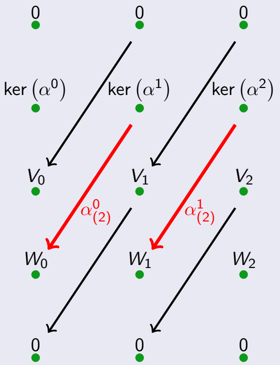

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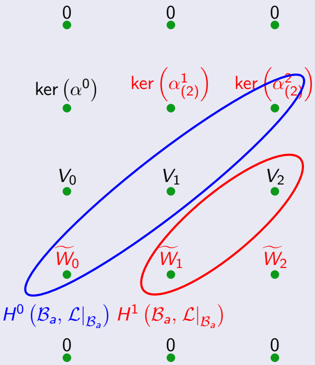

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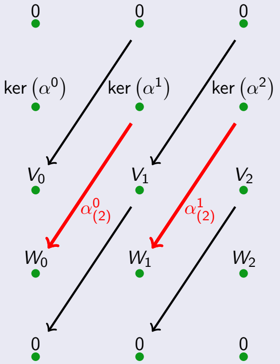

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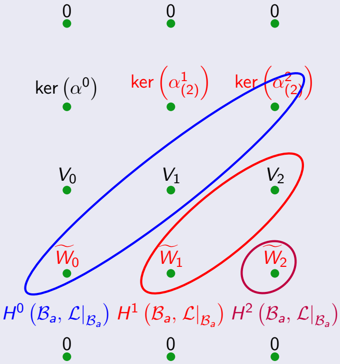

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Example With Knight's Move I

Ingredients

- Toric ambient space $X_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$
- $S_1 = (1, 0, 1)$ and $S_2 = (0, 0, 1)$
- $\mathcal{L} = \mathcal{O}_{X_\Sigma}(1, 1, 0)$

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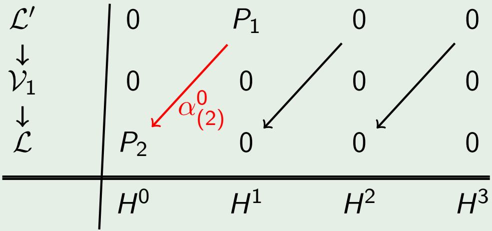

Example With Knight's Move II

Spaces And Polynomials

- $\widetilde{s}_1 = C_4 x_1 x_5 + C_2 x_2 x_5 + C_3 x_1 x_6 + C_1 x_2 x_6$
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$\alpha^0_{(2)} \colon P_1 \to P_2$ is given by

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Note

$\alpha^0_{(2)}$ respects the symmetries in P_1 and P_2.

Open Questions

Fact hep-th/0808.3621, book by T. Huebsch 'Calabi-Yau Manifolds: A Bestiary for Physicists'

- The cohomologies $H^i\left(\mathbb{CP}^n, \mathcal{L}\right)$ are labeled by representations of $U(1) \times U(n)$.

$\Rightarrow$ The cohomologies $H^i\left(\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{L}\right)$ have (anti)-symmetrisation properties.

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Question ▸ To the definition

Is every smooth and compact normal toric variety X_Σ a generalised Flag variety?

Section 4

Summary And Future Work

Summary

Take-Away-Message

- Computing the spectrum of massless zero modes requires spectral sequence technology.
- The existing Koszul extension of *cohomCalg* leaves unconstraint constants in these computations.

$\Rightarrow$ My notebook computes the E_1-sheet and thereby fixes many, but not all, of these constants.

Future Works

Open Tasks

- Prove of disprove that every smooth and compact normal toric variety X_Σ is a generalised Flag variety.

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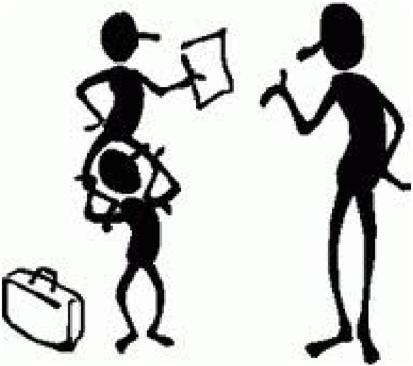
Open Tasks

- Prove of disprove that every smooth and compact normal toric variety X_Σ is a generalised Flag variety.
- Extend the functionality of the notebook beyond hypersurfaces.
- Improve the performance of the notebook.
- Apply the notebook to model building.

Thank you for your attention! Questions?



Proof I

Claim

Let X a smooth and compact normal toric variety given by

$$X_\Sigma \cong (\mathbb{C}^r - Z) / (\mathbb{C}^*)^a$$

We pick $\widetilde{s}_1 \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_1))$ non-trivial and define

$$X_3 = \{p \in X_\Sigma \ , \ \widetilde{s}_1(p) = 0\}$$

Then for any divisor class $D \in \mathrm{Cl}(X_\Sigma)$ the following sequence is sheaf exact

$$0 \to \mathcal{O}_{X_\Sigma}(D - S_1) \stackrel{\otimes \widetilde{s}_1}{\to} \mathcal{O}_{X_\Sigma}(D) \stackrel{r}{\to} \mathcal{O}_{X_\Sigma}(D)|_{X_3} \to 0$$

Proof II

◂ Back to orginal material.

Proof

- Sheaf exactness is a local property. So let $p \in X_\Sigma$ a point. Then we have to show that the following sequence is exact

$$0 \to \mathcal{O}_{X_\Sigma,p} \overset{[\widetilde{s}_1]_p}{\to} \mathcal{O}_{X_\Sigma,p} \overset{r}{\to} \mathcal{O}_{X_\Sigma,p} / \left([\widetilde{s}_1]_p \right) \to 0$$

- Note that $\mathcal{O}_{X_\Sigma,p}$ is the local power series ring. This ring is an integral domain. Therefore $[\widetilde{s}_1]_p$ is not a zero- divisor, so that the map $\mathcal{O}_{X_\Sigma,p} \overset{[\widetilde{s}_1]_p}{\to} \mathcal{O}_{X_\Sigma,p}$ is injective.
- The map $\mathcal{O}_{X_\Sigma,p} \overset{r}{\to} \mathcal{O}_{X_\Sigma,p} / \left([\widetilde{s}_1]_p \right)$ is surjective.

Generalised Flag Varieties

To the conjecture

Definition

A simply connected, compact, complex, homogeneous G-space is termed a generalised Flag variety.

Generalised Flag Varieties ◂ To the conjecture

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Example

It holds

$$\mathbb{CP}^n \cong U(n+1)/(U(1) \times U(n))$$

Generalised Flag Varieties

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Definition

A simply connected, compact, complex, homogeneous G-space is termed a generalised Flag variety.

Example

It holds

$$\mathbb{CP}^n \cong U(n+1) / (U(1) \times U(n))$$

Consequence

The cohomology groups $H^i\left(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k)\right)$ are labeled by representations of $U(1) \times U(n)$ for all $k \in \mathbb{Z}$.