Homework 10

Due: Thursday, April 15 – 10:00 EST

Problem 1: More on diagonalization [10 Points]

1. Which of the following matrices in $\mathbb{M}(3 \times 3, \mathbb{R})$ are diagonalizable?

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{1}$$

2. Diagonalize those A_i which are diagonalizable.

3. Matrices $A, B \in \mathbb{M}(n \times n, \mathbb{R})$ are similar if there is an invertible $S \in \mathbb{M}(n \times n, \mathbb{R})$ such that $A = SBS^{-1}$. Show the following:
   - If A is diagonalizable with eigenvalue matrix $\Lambda \in \mathbb{M}(n \times n, \mathbb{R})$, then A and Λ are similar.
   - If A is diagonalizable and A, B are similar, then also B is diagonalizable.

4. Which A_i in eq. (1) are similar?

5. Matrices $A, B \in \mathbb{M}(n \times n, \mathbb{R})$ are simultaneously diagonalizable if there is an invertible $S \in \mathbb{M}(n \times n, \mathbb{R})$ such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal.
   - Show that simultaneously diagonalizable matrices commute: $AB = BA$.
   - Is every pair of commuting matrices simultaneously diagonalizable?
   - **Math 513:** Show that if $AB = BA$ and all eigenvalues of A have algebraic multiplicity 1, then A, B are simultaneously diagonalizable.

Problem 2: Markov meets genetics [10 Points]

In autosomal inheritance, each individual inherits one gene from each of its parents' pairs of genes to form its own particular pair. It is believed that each of a parent's two genes are equally likely to be passed on to the offspring. The following table lists the probabilities of the possible genotypes of the offspring for all the possible combinations of the parents' genotypes.

		Parent's Genotype					
		AA-AA	AA-Aa	AA-aa	Aa-Aa	Aa-aa	aa-aa
Offspring Genotype	AA	1	1/2	0	1/4	0	0
	Aa	0	1/2	1	1/2	1/2	0
	aa	0	0	0	1/4	1/2	1

(2)

A breeder has a large population of dogs consisting of some distribution of all three genotypes AA, Aa, and aa. By design, each dog in the population mates with a dog of the same genotype and produces the same number of offsprings.

Your goal is to determine the distribution of each genotype in any given generation. For $k = 0, 1, 2, \dots$, let

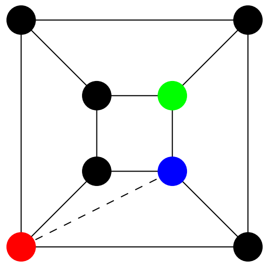
$$\begin{aligned} a_k &= \text{fraction of dogs of genotype AA in the } k\text{-th generation}, \\ b_k &= \text{fraction of dogs of genotype Aa in the } k\text{-th generation}, \\ c_k &= \text{fraction of dogs of genotype aa in the } k\text{-th generation}. \end{aligned}$$

Define the vector $\vec{x}^{(k)} = \begin{bmatrix} a_k & b_k & c_k \end{bmatrix}^T$.

1. Find a matrix M such that $\vec{x}^{(k)} = M\vec{x}^{(k-1)}$ (for $k = 1, 2, \dots$).
2. Compute M^k for $k \in \{3, 6, 9\}$. Does $\lim_{k\to\infty} M^k$ seem to approach a limit?
3. Diagonalize M.
4. For any $k \in \mathbb{N}$, compute all the entries in M^k.
5. Evaluate $\lim_{k\to\infty} M^k$ and comment on the limiting distribution $\lim_{k\to\infty} \vec{x}^{(k)}$.

Problem 3: An ant race [10 Points]

A young ant Y walks along a 3-dimensional cube from the red to the green vertex:



(3)

Y moves along the edges, namely with probability $1/3$ to each neighboring vertex. Answer the following with a Markov chain in `Python`.

1. Find the minimal number N_{min} of steps connecting the red and green vertex from your Markov chain. (No, it is not sufficient to read it off from the diagram.)
2. How likely is it that Y reaches the green vertex after N_{min} steps?
3. Y challenges an old ant O to a race along the cub. O needs to rest with probability 0.4 and moves to neighbouring vertices with probability 0.2. In favor of O, the ants agree on a race in 10 steps. Hence, what that matter is that they reach the green vertex in exactly 10 steps. Can O beat Y?
4. Y decides to explore the cube more. It make 2021 steps. Which vertex does Y reach with what probability after these 2021 steps?
5. While exploring the cube, Y found a short-cut along the dashed line in eq. (3). Y challenges O again, but uses the short-cut this time. Does O still beat Y?

Problem 4: Paths along the cube [10 Points]

A *path* is an ordered sequence of edges where any two consecutive edges share a common vertex.

1. How many length 3 paths are there between the red and green vertex in eq. (3)?
2. Without computation, justify that there are no paths of length 2020 from the red to the green vertex in eq. (3).
3. Familiarize yourself with *adjacency matrices* and write down the adjacency matrix $A \in \mathbb{M}(8 \times 8, \mathbb{R})$ for the cube in eq. (3).
4. Compute an appropriate entry of A^3 and A^{2020} in `Python` to recover your answers to 4–1 and 4–2.
5. By computing an appropriate entry of A^{2025} with `Python`, find the number of paths of length 2025 from the red to the blue vertex.