Math 313/513, Spring 2021 **Martin Bies**

Homework 3

Due: Thursday, February 11 – 10:00 am EST

Problem 1: PLU factorization [10 Points]

1. Consider n invertible matrices $A_i \in \mathbb{M}(n \times n, \mathbb{R})$. Prove that
$$\left(\prod_{i=1}^{n} A_i \right)^{-1} = \prod_{i=0}^{n-1} A_{n-i}^{-1} \,. \tag{1}$$

2. Let $i, j \in \mathbb{Z}_{>0}$ with $i > j$. Consider the elementary matrix $E_{ij}(k)$, whose non-trivial entries are 1's along the diagonal and k in row i column j. Find $E_{ij}^{-1}(k)$.

3. **For Math 513:**
Prove that if a lower triangular matrix has an inverse, it is lower triangular.

4. Find a lower triangular matrix L such that LA is upper trangular:
$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 4 & 6 & 1 & 0 \\ -2 & 2 & 0 & 4 \\ 1 & 3 & 5 & 7 \end{bmatrix} . \tag{2}$$

5. Find the PLU factorization of A. What are the pivots of A? Find $\mathrm{rk}(A)$.

6. Explain how the PLU factorization efficiently solves $A\vec{x} = \vec{b}$ for varying $\vec{b} \in \mathbb{R}^4$.

Problem 2: Exotic vector spaces [10 Points]

1. Consider $M := \mathbb{M}(2 \times 2, \mathbb{R})$ and
$$+_M \colon M \times M \to M \,,\; (A, B) \mapsto \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} , \tag{3}$$
$$\cdot_M \colon \mathbb{R} \times M \to M \,,\; (c, A) \mapsto \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} \\ c \cdot a_{21} & c \cdot a_{22} \end{bmatrix} . \tag{4}$$
Show that $(M, +_M, \cdot_M)$ is a vector space over $\mathbb{R}$.

2. $P := \mathrm{Pol}_n$ is the set of polynomials in the variable x with degree at most n. Find operations $+_P$ and $\cdot_P$ such that $(\mathrm{Pol}_n, +_P, \cdot_P)$ is a vector space over $\mathbb{R}$.

3. **Math 513**:
Argue that $(\mathrm{Pol}_n, +_P, \cdot_P) \cong (\mathbb{R}^m, +, \cdot)$ for a suitable $m \in \mathbb{Z}_{\geq 0}$. $\mathbb{R}^m$ is considered with its standard vector space operations.

Problem 3: Nullspace [10 Points]

1. Compute the row echelon form (REF):

$$A_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 7 & 2 \\ 0 & 1 & 5 & 2 \\ 1 & 4 & 3 & 2 \end{bmatrix}. \tag{5}$$

2. For each free column, find a non-trivial solution to $A_i\vec{x} = \vec{0}$.
3. Compute the row *reduced* echelon form (RREF).
4. For each free column, read off a non-trivial solution to $A_i\vec{x} = \vec{0}$ from the RREF.
5. Find all solutions to $A_1\vec{x} = \vec{0}$ and $A_2\vec{x} = \vec{0}$, respectively. Justify your answer.

Problem 4: Elementary row operations in `Python` [10 Points]

In this exercise, $\mathcal{A}$ denotes a `numpy`-array. We perform computations with the matrices

$$B = \begin{bmatrix} 0 & 2 & 3 & 5 \\ 5 & 6 & 7 & 13 \\ 9 & 10 & 11 & 21 \\ 13 & 14 & 15 & 29 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 4 & -2 & 1 \\ 4 & 6 & 1 & 0 \\ 0 & 1 & 0 & 5 \\ 1 & 3 & 5 & 7 \end{bmatrix}. \tag{6}$$

1. Write a function *add_rows* with the following properties:
   - Input: $\mathcal{A}$, k, i, j.
   - Output: Numpy-array resulting from adding k times row j to row i.

   For $i = j$, rescale row i by $k + 1$.
2. Write a function *scale_row* with the following properties:
   - Input: $\mathcal{A}$, k, i.
   - Output: Numpy-array resulting from k times row i.
3. Write a function *switch_rows* with the following properties:
   - Input: $\mathcal{A}$, i, j.
   - Output: Numpy-array resulting from switching rows i and j.
4. Use these functions to compute the row reduced echelon form of B.
5. Write a function *elementary_matrix* with the following properties:
   - Input: k, i, j
   - Output: 4×4 numpy-array matching $E_{ij}(k)$ as defined in exercise 1.
6. Use the above functions to compute the LU-factorization of C step-by-step by Gauss elimination. Print your matrices L, U and verify that $LU = C$.