Homework 7

Due: Thursday, March 18 – 10:00 EST

Problem 1: Orthogonal projections [10 Points]

1. Show that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})$ is invertible iff $ad - bc \neq 0$. Find A^{-1}.
2. In $\mathbb{R}^3$, compute the orthogonal projection P to $S = \left\{ [x, y, z]^T \in \mathbb{R}^3 \middle| x - y - 2z = 0 \right\}$.
3. Compute the orthogonal projection Q to $S^\perp$ and show that $P + Q = I$.
4. **Math 513:** Formulate the following as orthogonal projection and solve it: Find $f \in \operatorname{Span}_{\mathbb{R}} \{1, \sin(x), \cos(x)\}$ which minimizes $\int_0^{2\pi} (\sin(2x) - f(x))^2$.

Problem 2: Least square approximation [10 Points]

We compute the parabola $P(D, E, F) = \left\{ \begin{bmatrix} t \\ D + Et + Ft^2 \end{bmatrix} \mid t \in \mathbb{R} \right\}$ closest to

$$\vec{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}. \tag{1}$$

1. First consider a general $A \in \mathbb{M}(n \times m, \mathbb{R})$ and $\vec{b} \in \mathbb{R}^n$.
   - By repeating the steps in the lecture, argue that the orthogonal projection $A\vec{x}$ of $\vec{b}$ to $C(A)$ is defined by the demand that $A^T A \vec{x} = A^T \vec{b}$.
   - Consider the length of the error vector $l_e \colon \mathbb{R}^3 \to \mathbb{R}$, $\vec{x} \mapsto l_e(\vec{x}) = \left\langle A\vec{x} - \vec{b}, A\vec{x} - \vec{b} \right\rangle_{\text{Std}}$. Prove that the Jacobian matrix of l_e vanishes at $\vec{x} \in \mathbb{R}^n$ iff $A^T A \vec{x} = A^T \vec{b}$.
   - Explain the significance of the relation among $A^T A \vec{x} = A^T \vec{b}$ and l_e.
2. Now focus on the $\vec{b}_i$ in eq. (1) and the parabola $P(D, E, F)$. Find $A \in \mathbb{M}(4 \times 3, \mathbb{R})$, $\vec{b} \in \mathbb{R}^4$ such that

$$\left\{ \vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4 \right\} \subset P(D, E, F) \quad \Leftrightarrow \quad \vec{x} = \begin{bmatrix} D & E & F \end{bmatrix}^T \text{ satisfies } A\vec{x} = \vec{b}. \tag{2}$$

3. Show that no parabola contains $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$ and $\vec{b}_4$.
4. Find the best approximation parabola. You may assume that the Hessian matrix of l_e is positive definite.
5. Plot the best approximation parabola and the points $\vec{b}_i$.

Problem 3: Least square approximation in `Python` [10 Points]

1. Write a function `LineFit` which accepts $[(t_1, b_1), (t_2, b_2), \ldots, (t_n, b_n)]$ and fits a line to this data by the method of least square. You may assume that the Hessian matrix is positive definite. Plot the line and the data points (t_i, b_i).
2. Similarly, write a function `ParaFit`, which fits a parabola $C + Dt + Et^2$ to $[(t_1, b_1), (t_2, b_2), \ldots, (t_n, b_n)]$, plots the parabola and the data points.
3. Apply `LineFit` and `ParaFit` to $[(1, 2), (2, 2), (3, 5), (4, 3), (4.5, 8)]$. By looking at the plots, does the line or the parabola describe the data better?
4. Expand your functions by a criterion for the quality of the fit. Justify your criterion and use it to tell if the line or parabola fits the data better.
5. **Math 513:** Compare your line fit with the linear regression fit in *scikit-learn*.

Problem 4: Gram-Schmidt procedure [10 Points]

1. Be $\{\vec{a}, \vec{b}, \vec{c}\}$ a family of linearly independent vectors in an inner product space $(V, \langle \cdot, \cdot \rangle)$. Show that $\vec{U}, \vec{V}, \vec{W}$ are orthogonal:

$$\vec{U} = \vec{a}\,, \quad \vec{V} = \vec{b} - \frac{\langle \vec{U}, \vec{b} \rangle}{\left\langle \vec{U}, \vec{U} \right\rangle} \cdot \vec{U}\,, \quad \vec{W} = \vec{c} - \frac{\left\langle \vec{U}, \vec{c} \right\rangle}{\left\langle \vec{U}, \vec{U} \right\rangle} \cdot \vec{U} - \frac{\langle \vec{V}, \vec{c} \rangle}{\langle \vec{V}, \vec{V} \rangle} \cdot \vec{V}\,. \tag{3}$$

2. In $\mathbb{R}^4$, let $W = \mathrm{Span}_{\mathbf{R}}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ with

$$\vec{v}_1 = 2\vec{e}_1 + \vec{e}_3\,, \qquad \vec{v}_2 = -\vec{e}_1 + 3\vec{e}_2\,, \qquad \vec{v}_3 = 2\vec{e}_1 - \vec{e}_2 + 3\vec{e}_3\,, \tag{4}$$

   where $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$ is the standard basis of $\mathbb{R}^4$.

   - Find an orthonormal basis of W.
   - Find an orthonormal basis of $N(A^T)$ where $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$.
   - Use these results to find an orthonormal basis of $\mathbb{R}^4$.
   - Compute the projection of $\vec{b} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T \in \mathbb{R}^4$ to W.