Homework 8

Due: Thursday, April 1 – 10:00 EST

Problem 1: Hesse normal form [10 Points]

1. Be $\vec{a}, \vec{b} \in \mathbb{R}^3 \setminus \vec{0}$ two linearly independent vectors. For $\vec{x}_0 \in \mathbb{R}^3$ consider

$$S(\vec{x}_0) = \left\{ \mu\vec{a} + \nu\vec{b} + \vec{x}_0 \,|\, \mu, \nu \in \mathbb{R} \right\} \subseteq \mathbb{R}^3 \,. \tag{1}$$

   Show that there exist $\vec{n} \in \mathbb{R}^3$ and $d \in \mathbb{R}$ such that $\vec{n}^T\vec{n} = 1$ and

$$\vec{x} \in S(\vec{x}_0) \qquad \Leftrightarrow \qquad \langle \vec{n}, \vec{x} \rangle_{\text{std}} - d = 0 \,. \tag{2}$$

2. Give a geometric interpretation of $\vec{n} \in \mathbb{R}^3$ and $|d| \in \mathbb{R}$.
3. Are $\vec{n} \in \mathbb{R}^3$ and $d \in \mathbb{R}$ unique? If not, name conditions under which they are.
4. Under what condition is $S(\vec{x}_0)$ a linear subspace of $\mathbb{R}^3$?
5. Be $\vec{v} \in \mathbb{R}^3$ arbitary but fixed. Compute the orthogonal projection of $\vec{v}$ to $S(\vec{x}_0)$.

Problem 2: Determinants and applications [10 Points]

1. Use Cramer's rule to solve $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) \,, \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} . \tag{3}$$

2. Repeat for A as above but $\vec{b} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T$.
3. Show that the Vandermonde determinant satisfies ($a_i \in \mathbb{R}$)

$$\det \left(\begin{bmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{bmatrix} \right) = \prod_{1 \leq i < j \leq n}^{n} (a_j - a_i) \,. \tag{4}$$

4. You are given points $\{(x_i, y_i) \in \mathbb{R}^2 | 1 \leq i \leq n \text{ and } x_i \neq x_j \text{ whenever } i \neq j\}$. We are looking for a polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \,, \tag{5}$$

   with $P(x_i) = y_i$ for all $1 \leq i \leq n$. Express this condition as matrix equation.
5. Under what condition does such a polynomial exist?

Problem 3: A first encounter with diagonalization [10 Points]

In this problem, we find a basis in which a linear transformation is diagonal.

1. Compute the polynomial $\mathrm{ch}_A(\lambda) = \det(A - \lambda I) \in \mathbb{R}[\lambda]$ for

$$A = \begin{bmatrix} -2 & -2 & -2 \\ -2 & 1 & -5 \\ -2 & -5 & 1 \end{bmatrix} . \tag{6}$$

2. Find the three zeros $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{R}$ of this polynomial.

3. Find linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$ with

$$A\vec{v}_1 = \lambda_1 \cdot \vec{v}_1 \,, \qquad A\vec{v}_2 = \lambda_2 \cdot \vec{v}_2 \,, \qquad A\vec{v}_3 = \lambda_3 \cdot \vec{v}_3 \,. \tag{7}$$

4. Find the base change matrix $T_{\mathcal{B}_2\mathcal{B}_1}$ where $\mathcal{B}_2 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\mathcal{B}_1 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

5. For the linear transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $A_{\mathcal{B}_2\mathcal{B}_2} = A$, find $A_{\mathcal{B}_1\mathcal{B}_1}$ by use of $T_{\mathcal{B}_2\mathcal{B}_1}$. You should find that $A_{\mathcal{B}_1\mathcal{B}_1}$ is diagonal.

Problem 4: Basic diagonalization in `Python` [10 Points]

1. Use `numpy` to write a `Python` function `BasicDiag` which realizes the following algorithm:
   - Input: $A \in \mathbb{M}(n \times n, \mathbb{R})$,
   - Output: $A_{\mathcal{B}_1\mathcal{B}_1}$, $T_{\mathcal{B}_1\mathcal{B}_2}$.

   The matrix $A_{\mathcal{B}_1\mathcal{B}_1}$ is to be computed by the following algorithm:

   a) Check that the input matrix A is a square matrix.

   b) The zeros of $\mathrm{ch}_A(\lambda) = \det(A - \lambda I) \in \mathbb{R}[\lambda]$ are known as *eigenvalues* of A. For deep mathematical reasons, they are considered as complex numbers. Use the build in functions in `numpy` to compute the eigenvalues of A.

   c) Proceed if there are exactly n *distinct* and *real* eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Otherwise, raise a warning.

   d) For each λ_i compute a so-called *eigenvector* $\vec{v}_i \in \mathbb{R}^n$, that is $A\vec{v}_i = \lambda_i \cdot \vec{v}_i$.

   e) Proceed if $\mathcal{B}_1 = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$. Otherwise, raise an error.

   f) Let $\mathcal{B}_2 = \{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis of $\mathbb{R}^n$. Construct the base change matrix $T_{\mathcal{B}_2\mathcal{B}_1}$ and compute $A_{\mathcal{B}_1\mathcal{B}_1} = T_{\mathcal{B}_1\mathcal{B}_2} A_{\mathcal{B}_2\mathcal{B}_2} T_{\mathcal{B}_2\mathcal{B}_1}$.

2. Apply `BasicDiag` to $A = I_3$. "Too few eigenvalues" should be triggered.

3. Apply `BasicDiag` to eq. (6). You should find a result *equivalent* to yours in 3-5.

4. Apply `BasicDiag` to eq. (3) and rederive your answer to problem 2-2.