Math 313/513, Spring 2021 **Martin Bies**

Homework 9

Due: Thursday, April 8 – 10:00 EST

Problem 1: General properties of Eigenvalues [10 Points]

1. Consider $A \in \mathbb{M}(n \times n, \mathbb{R})$, $\vec{x} \in \mathbb{R}^n \backslash \vec{0}$ and $\lambda \in \mathbb{C}$. Show the following equivalence:

$$A\vec{x} = \lambda\vec{x} \qquad \Leftrightarrow \qquad \det(A - \lambda I) = 0\,. \tag{1}$$

2. Show that $A \in \mathbb{M}(n \times n, \mathbb{R})$ can have at most n distinct eigenvalues.
3. Name $A \in \mathbb{M}(n \times n, \mathbb{R})$ with strictly less than n distinct eigenvalues.
4. Name $A \in \mathbb{M}(n \times n, \mathbb{R})$ for which all eigenvalues are complex numbers.
5. Be $k \in \mathbb{Z}_{\geq 0}$. Show the following implication:

$$\lambda \text{ eigenvalue of } A \qquad \Rightarrow \qquad \lambda^k \text{ eigenvalue of } A^k\,. \tag{2}$$

6. Be A invertible. Show the following equivalence:

$$\lambda \text{ is an eigenvalue of } A \qquad \Leftrightarrow \qquad \lambda^{-1} \text{ is an eigenvalue of } A^{-1}\,. \tag{3}$$

Problem 2: Towards Eigenbasis [10 Points]

In this exercise, we study the *Eigenbasis* a projection matrix $P \in \mathbb{M}(n \times n, \mathbb{R})$.

1. Show that all eigenvalues λ of $P \in \mathbb{M}(n \times n, \mathbb{R})$ satisfy $\lambda \in \{0, 1\}$.
2. Show that $\mathbb{R}^n$ admits a basis $\mathcal{B}_{\text{eig}}$ of eigenvectors of P.
3. Describe the mapping matrix of P in this so-called *Eigenbasis* $\mathcal{B}_{\text{eig}}$ of P.
4. Compute the eigenvalues and eigenspaces of the projection matrix

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{4}$$

5. Find the eigenbasis $\mathcal{B}_{\text{eig}}$ of P in eq. (4) and its mapping matrix in $\mathcal{B}_{\text{eig}}$.

Problem 3: Eigenvalues, traces and determinants [10 Points]

Be $A \in \mathbb{M}(n \times n, \mathbb{R})$. We denote its eigenvalues as $\{\lambda_i\}_{1 \leq i \leq N}$. The trace $\mathrm{tr}(A)$ of the square matrix A is defined as the sum of its diagonal entries.

1. Compute the eigenvalues of the following two matrices:

$$A_1 = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 8 \\ 3 & 8 & 1 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) \,, \qquad A_2 = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 8 \\ 0 & 0 & 4 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) \,. \tag{5}$$

2. Compare the sum of the eigenvalues of A_1 to $\mathrm{tr}(A_1)$. Repeat for A_2.

3. Compare the product of the eigenvalues of A_1 to $\det(A_1)$. Repeat for A_2.

4. For a general matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$, show that

$$\mathrm{tr}(A) = \sum_{i=1}^{N} \lambda_i \,, \qquad \det(A) = \prod_{i=1}^{N} \lambda_i \,. \tag{6}$$

Problem 4: The type of local extremum [10 Points]

In this exercise we study local extrema of maps

$$f \colon \mathbb{R}^n \to \mathbb{R} \,, \; \vec{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \mapsto f(x_1, x_2, \dots, x_n) \,. \tag{7}$$

At a local extremum $\vec{a}$ of f, the Jacobian $J(f)(\vec{a}) \in \mathbb{M}(n \times 1, \mathbb{R})$ necessarily vanishes:

$$0 \equiv J(f)(\vec{a}) = \begin{bmatrix} \left(\frac{\partial f}{\partial x_1}\right)(\vec{a}) & \dots & \left(\frac{\partial f}{\partial x_n}\right)(\vec{a}) \end{bmatrix}^T \,. \tag{8}$$

The type of local extremum is identified by studying the Hessian matrix of f at $\vec{a}$:

$$H(f)(\vec{a}) = \begin{bmatrix} \left(\frac{\partial^2 f}{\partial x_1 \partial x_1}\right)(\vec{a}) & \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)(\vec{a}) & \dots & \left(\frac{\partial^2 f}{\partial x_1 \partial x_n}\right)(\vec{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\partial^2 f}{\partial x_n \partial x_1}\right)(\vec{a}) & \left(\frac{\partial^2 f}{\partial x_n \partial x_2}\right)(\vec{a}) & \dots & \left(\frac{\partial^2 f}{\partial x_n \partial x_n}\right)(\vec{a}) \end{bmatrix} \in \mathbb{M}(n \times n, \mathbb{R}) \,. \tag{9}$$

Namely, it can be shown that the following holds true:

$$\begin{aligned} \vec{a} \text{ is local maximum} \quad &\Leftrightarrow \quad H(f)(\vec{a}) \text{ negative definite} \,, \\ \vec{a} \text{ is local minimum} \quad &\Leftrightarrow \quad H(f)(\vec{a}) \text{ positive definite} \,, \\ \vec{a} \text{ is saddle point} \quad &\Leftrightarrow \quad H(f)(\vec{a}) \text{ indefinite} \,. \end{aligned} \tag{10}$$

There can be local extrema, which are none of the above types.

We will eventually prove that a *symmetric* matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$ (i.e. $A = A^T$) has only real eigenvalues. By definition, it then holds:

$$\begin{aligned} A \text{ is positive definite} \quad &\Leftrightarrow \quad \text{all eigenvalues of } A \text{ are positive}\,, \\ A \text{ is negative definite} \quad &\Leftrightarrow \quad \text{all eigenvalues of } A \text{ are negative}\,, \\ A \text{ is indefinite} \quad &\Leftrightarrow \quad A \text{ has positive and negative eigenvalues}\,. \end{aligned} \tag{11}$$

Use this information to complete the following tasks:

1. Write a `Python` function `PositiveDefinite`:
   - Input: $A \in \mathbb{M}(n \times n, \mathbb{R})$
   - Output:
     - Check if $A = A^T$. If not, raise an error and exit.
     - Otherwise, return *True* if A is positive definite and *false* otherwise.
2. Similarly, write a `Python` function `NegativeDefinite` and `Indefinite`.
3. Use analytic arguments to find all local extrema of

   $$V\colon \mathbb{R}^2 \to \mathbb{R}\,,\ (x, y) \mapsto \left(1 - x^2 - y^2\right)^2\,. \tag{12}$$

   Aside: This is the potential V of the famous *Higgs boson*.
4. Use the above `Python` functions to study the type of at least 3 local extrema.
   Bonus: Study the type of *all* local extrema *analytically*.
5. Make a plot of V in `Python` for $(x, y) \in [-1, 1] \times [-1, 1]$. Compare this plot with the type of local extrema analyzed in the previous part of this exercise.