Math 313/513, Spring 2021 **Martin Bies**

Midterm 2

Due: Thursday, March 25 – 10:00 am EST

Problem 1: QR-decomposition [10 Points]

1. Be $A \in \mathbb{M}(n \times n, \mathbb{R})$ invertible. Show that it admits a QR-decomposition $A = QR$ with $Q, R \in \mathbb{M}(n \times n, \mathbb{R})$, Q orthogonal and R upper triangular.
2. Explain why QR-decompositions are useful.
3. Find all lower triangular $A \in \mathbb{M}(n \times n, \mathbb{R})$ which are orthogonal.
4. Compute a QR-decomposition for $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})$.
5. **Math 513:** Is the QR-decomposition unique? What conditions make it unique?

Problem 2: Determinants [10 Points]

1. Derive the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$.
2. You were given $A \in \mathbb{M}(3 \times 3, \mathbb{R})$ and performed the following row operations:
   - Added the first row to the second row.
   - Swapped rows 2 and 3.
   - Scaled row 1 by the number 2.

   You arrived at $E = \begin{bmatrix} 2 & 1 & 5 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. What is $\det(A)$? Explain your answer.
3. Is $\{A \in \mathbb{M}(n \times n, \mathbb{R}) \,|\, \det(A) = 0\}$ a vector space over $\mathbb{R}$?
4. Is $\{A \in \mathbb{M}(n \times n, \mathbb{R}) \,|\, \det(A) = 1\}$ a vector space over $\mathbb{R}$?
5. Let $A, B \in \mathbb{M}(n \times n, \mathbb{R})$ with $AB = 0$. Give a proof or counterexample for each of the following:
   - $BA = 0$,
   - $A = 0$ or $B = 0$,
   - If $\det(A) = -3$, then $B = 0$,
   - If B is invertible, then $A = 0$.

Problem 3: Fourier series in `Python` [10 Points]

1. Write a function r:
   - Input: $x \in [0, 2\pi]$
   - Output: 1 if $0 \leq x \leq \pi$ and 0 otherwise.
2. Write a function `FourierTransform`:
   - Input: A function f (such as r from part 1) and $d \in \mathbb{Z}_{\geq 0}$.
   - Output: Scatter plot of a_0, a_k, b_k $(1 \leq k \leq d)$ in different colours.
3. Apply `FourierTransform` to r for $d = 10$. Qualitatively, describe the plot.
4. Write a function `FourierSeries`:
   - Input: A function f (such as r from part 1) and $d \in \mathbb{Z}_{\geq 0}$.
   - Output:
     - A `Python` function $F \colon [0, 2\pi] \to \mathbb{R}\,,\ x \mapsto F(x)$ with

$$F(x) = a_0 + \sum_{k=1}^{d} a_k \cdot \cos(kx) + \sum_{k=1}^{d} b_k \cdot \sin(kx)\,. \tag{1}$$

     - Plot of $f(x)$ and $F(x)$ for $0 \leq x \leq 2\pi$ – at least 500 values.
5. Apply `FourierSeries` to r for $d = 3, 10, 50$ and describe the plots.
6. For $d = 3, 10$, apply `FourierTransform` and `FourierSeries` to

$$G \colon [0, 2\pi] \to \mathbb{R}\,,\ x \mapsto e^{-(x-\pi)^2}\,. \tag{2}$$

7. `FourierSeries` converges much quicker for G than for r. Explain this.

Problem 4: True or false? Give a proof or counterexample.

1. Let $n > 2$, then $\{A \in \mathbb{M}(n \times n, \mathbb{R}) | \mathrm{rk}(A) \leq 2\}$ is a vector space over $\mathbb{R}$.
2. Let $A, B \in \mathbb{M}(n \times n, \mathbb{R})$. Then AB is invertible if and only if A, B are invertible.
3. Let $A \in \mathbb{M}(n \times (n+1), \mathbb{R})$, $B \in \mathbb{M}((n+1) \times n, \mathbb{R})$. Then AB is not invertible.
4. Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ s.t. $\exists k \in \mathbb{Z}_{>0}$ with $A^k = 0$, then $I - A$ is invertible.
5. **Math 513:** Let $A \in \mathbb{M}(n \times n, \mathbb{R})$. If $AB = BA$ for all invertible matrices $B \in \mathbb{M}(n \times n, \mathbb{R})$, then $A = c \cdot I$ for a scalar $c \in \mathbb{R}$.