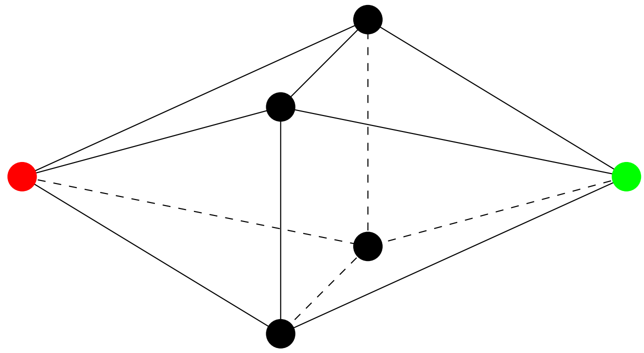
Homework 10 – Coding

Due: Tuesday, April 26 – 23:59 EST

Problem 1C: Walking on a diamond [20 Points]

A young ant Y walks along a diamond. With each step, it moves from one vertex to a neighbouring vertex along one of the edges of the diamond. It picks the edges that it passes along at random and with equal probability.

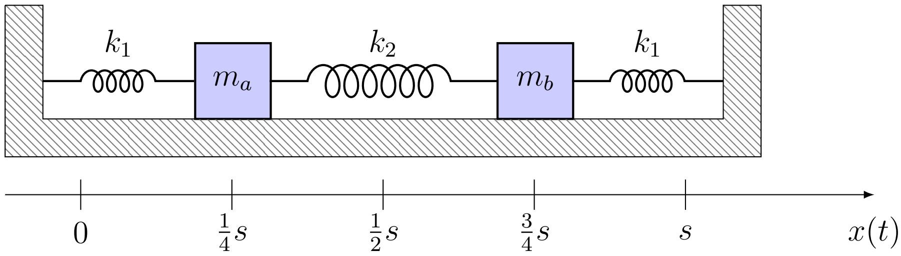


(1)

1. Y explores the diamond in 2022 steps. Which vertex does Y reach with what probability after these 2022 steps? (Hint: Markov chain.)
2. Y challenges an old ant O to a race. O needs to rest with probability 0.2 and moves to neighbouring vertices with equal probability. In favor of O, the ants agree on a race in 5 steps. They start in the red vertex R. The goal is to reach the green vertex G in exactly 5 steps:
   - How likely is it at Y arrives at G after exactly 5 steps?
   - How likely is it at O arrives at R after exactly 5 steps?
   - Who is more likely to win this race?
3. In ant school, Y learns about *paths* – ordered sequences of edges where any two consecutive edges share a common vertex.
   - Count the paths of length 3 between R and G in eq. (1).
   - Explain that for any $n \in \mathbb{Z}_{\geq 2}$, there is a path of length n from R to G.
4. Y is fascinated by paths. It decides to count paths between the green and red vertex. We help it by computing their number.
   - Familiarize yourself with adjacency matrices.
   - Find the adjacency matrix $A \in \mathbb{M}(6 \times 6, \mathbb{Z})$ for the diamond in eq. (1).
   - Compute an appropriate entry of A^3 to find the number of paths of length 3 between R and G.
   - How many paths of length 30 exist between R and G?
   - **Bonus (for 313 and 513)**: How many paths of length 2023 are there between R and G? (Hint: The answer is not zero.)

Problem 2C: A coupled spring-mass-system [20 Points]

We will continue on the coupled spring-mass system discussed in section 6.4.3 of the lecture notes. The goal is to see how rich the dynamics of this simple system is:



1. Write a `Python` function:
   - Input: Initial values x_a, v_a, x_b, v_b, positive spring constants k_1, k_2, positive masses m_a, m_b, the positive box length s and times $t_{\min}, t_{\max} \in \mathbb{R}$.
   - Processing 1: Compute $\vec{y}(t)$ and $\vec{x}(t)$ as discussed in the lecture.
   - Processing 2: Construct the set $T := \{t_{\min}, t_{\min} + 0.1, t_{\min} + 0.2, \ldots, t_{\max}]$.
   - Output 1: Plot $\frac{s}{4} + y_a(t)$ and $\frac{3 \cdot s}{4} + y_b(t)$ in one diagram for $t \in T$.
   - Output 2: Plot $\frac{s}{4} + x_a(t)$ and $\frac{3 \cdot s}{4} + x_b(t)$ in a second diagram for $t \in T$.

   Test your function for

   $$k_1 = k_2 = 1\,, \qquad m_a = m_b = 1 \qquad x_a = x_b = 1\,, \qquad v_a = v_b = 1\,, \tag{2}$$

   $t_{\min} = 0$, $t_{\max} = 100$ and $s = 16$. Does the plot fit with your expectation?

2. Let us now study the limit $m_b \to \infty$. To this end, set $m_b = 10$ and describe how the plot changes. Qualitatively, explain the changed behavior.

3. Without using your function, explain what behavior to expect for $m_b \to 0$.

4. Let us plot the system for asymmetric initial conditions. To this end, we consider

   $$k_1 = k_2 = 1\,, \qquad m_a = m_b = 1 \qquad x_a = v_a = 1\,, \qquad x_b = v_b = 2\,, \tag{3}$$

   and $t_{\min} = 0$, $t_{\max} = 100$, $s = 16$. Qualitatively, describe how the diagrams change relative to the symmetric case in 4-2. Does this fit your expectation?

5. Finally, let us consider the plots for a choice with different spring constants:

   $$k_1 = 10, \qquad k_2 = 1\,, \qquad m_a = m_b = 1 \qquad x_a = v_a = 1\,, \qquad x_b = v_b = 2\,, \tag{4}$$

   and $t_{\min} = 0$, $t_{\max} = 100$, $s = 24$. Qualitatively, describe the plots.

Problem 3C: Gradient descent [20 Points]

We will code a gradient descent method to find the global minima of

$$f\colon \mathbb{R}^2 \to \mathbb{R}\,,\ (x, y) \mapsto 2 - 2x^2 + x^4 + xy - 2y^2 + y^4\,. \tag{5}$$

1. Write a function `move(p,r)`:
   - Input: A point $p = (x, y) \in \mathbb{R}^2$ and $r \in \mathbb{R}_{>0}$.
   - Processing:
     Let C be a circle with radius r about p. Construct a family $F \subseteq C$ of 360 equally-spaced points. For each point $q = (x_q, y_q) \in F$ compute $\Delta_q = f(x_q, y_q) - f(x, y)$.
   - Output:
     - If for all $q \in F$ it holds $\Delta_q \geq 0$, return p.
     - Otherwise return the (or, if at least two exist, one) point $q \in F$ for which Δ_q is smallest among all points in F.
2. Write a function `chain(p,r)`:
   - Input: A point $p_0 = (x, y) \in \mathbb{R}^2$ and $r \in \mathbb{R}_{>0}$.
   - Output: Family $F = \{p_0, p_1, p_2, \dots, p_N\}$ where $p_{i+1} = \texttt{move}(p_i,\ \mathrm{r})$ and p_N is defined by $p_N = \texttt{move}(p_N,\ \mathrm{r})$.
3. Write a function `plot_descent(p,r)`:
   - Input: A point $p_0 = (x, y) \in \mathbb{R}^2$ and $r \in \mathbb{R}_{>0}$.
   - Output:
     - Contour plot of f for $(x, y) \in R = [-1.5, 1.5] \times [-1.5, 1.5]$ including the maxima, minima and saddle points identified on assignment 9 and all points of in `chain(p,r)`. Indicate maxima in red, saddle points in green, minima in blue and the points in `chain(p,r)` in orange color.
     - Print the length of `chain(p,r)` and the value of f at its last point.
4. Let $S = \{(0.1, 0.1), (-0.2, -0.4), (-0.1, -0.2)\}$. Execute `plot_descent(p,r)` for $r = 0.2$ and each point in S. Qualitatively describe the chains.
5. There are two local minima $m_1, m_2 \in R = [-1.5, 1.5] \times [-1.5, 1.5]$ with $f(m_i) = -1.125$. We consider those the global minima of f in R. Let us estimate the chances to find those points m_1, m_2 with our gradient descent algorithm.
   - Construct a family T of 50×50 uniformly distributed points in R.
   - For each $q \in T$ and $r = 0.2$ execute `chain(p,r)`. If f at the final point of the chain is between -1.15 and -1.10 we consider a global minimum found. Use T to estimate the chances to find m_1, m_2. Comment on your result. How could we improve the chances?