Homework 10 – Theory

Due: Tuesday, April 26 – 23:59 EST

Problem 1T: Matrix exponentials [20 Points]

Consider $X, Y \in \mathbb{M}(n \times n, \mathbb{R})$. We define $e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!}$ and set X^0 to be the $n \times n$ identity matrix. In addition we define the so-called *commutator* $[X, Y] := XY - YX$.

1. Suppose X is diagonalizable. Find an expression for e^X in terms of a base change change matrix $S \in \mathbb{M}(n \times n, \mathbb{R})$ and the eigenvalues of X.

2. Compute e^X for $X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})$.

3. There are remarkable results concerning matrix exponentials:
   - $\left(e^X\right)^T = e^{\left(X^T\right)}$.
   - $\left(e^X\right)^{-1} = e^{-X}$ (even when X is not invertible).
   - If $XY = YX$, then $e^X e^Y = e^{X+Y}$. (In general $e^X e^Y \neq e^Y e^X$.)
   - If $[X, [X, Y]] = [Y, [X, Y]] = 0$, then $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$.

   Prove two of these four results.
   Hint: The last result is a special instance of the so-called *Campbell-Baker-Hausdorff* formula and the toughest challenge. For a proof, consider $f(\lambda) = e^{\lambda X} e^{\lambda Y} e^{-\lambda(X+Y)}$ and establish $f'(\lambda) = \lambda [X, Y] \cdot f(\lambda)$.

4. **Bonus:** Find $X \in \mathbb{M}(2 \times 2, \mathbb{R})$ with $e^X = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.