Homework 2 – Theory

Due: Thursday, January 27 – 10:00 am EST

Problem 1T: Solving linear systems [20 Points]

1. Solve $A\vec{x} = \vec{b}$ with the method of elimination and back substitution:

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 1 & 3 & 4 & 5 \\ -1 & 4 & 5 & 6 \\ -2 & 4 & 3 & 7 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}. \tag{1}$$

2. Find B^{-1} and C^{-1} (if they exist) by Gauss-Jordan elimination:

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} -4 & 2 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}. \tag{2}$$

3. Verify or falsify if the following vectors are contained in a plane in $\mathbb{R}^3$:

$$\vec{u} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \qquad \vec{w} = \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix}. \tag{3}$$

4. Consider the matrix

$$D = \begin{bmatrix} 5 & 3 & -2 \\ -4 & 2 & 6 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}. \tag{4}$$

   - Use the result of part 3 to verify or falsify if $D\vec{x} = \vec{0}$ has a solution $\vec{x} \neq \vec{0}$.
   - Suppose that $D\vec{x} = \vec{d}$ has a solution. Explain if it has other solutions.

Problem 2T: "Order" of elimination with back substitution [20 Points]

Consider $A \in \mathbb{M}(n \times n, \mathbb{R})$ and $\vec{b} \in \mathbb{R}^n$. We apply the method of elimination with back substitution (MEB) to $[A|\vec{b}]$. It is assumed that no permutations of the rows of A are required and that $A\vec{x} = \vec{b}$ is non-singular. Hence, we perform the transformation

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \to \left[\begin{array}{cccc|c} u_{11} & u_{12} & \cdots & u_{1n} & c_1 \\ & u_{22} & \cdots & u_{2n} & c_2 \\ & & \ddots & \vdots & \\ & & & u_{nn} & c_n \end{array}\right], \tag{5}$$

followed by back substitution:

- The first elimination step is to turn the second row into the form

$$\left[\begin{array}{ccccc|c} 0 & a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12} & \dots & a_{2n} - \frac{a_{21}}{a_{11}} \cdot a_{1n} & b_2 - \frac{a_{21}}{a_{11}} \cdot b_1 \end{array}\right]. \tag{6}$$

  For this we compute $\frac{a_{21}}{a_{11}}$ and then use this result to perform the remaining additions/multiplications. Consequently, this first step requires at most n additions and $n+1$ multiplications. Note that we count $\frac{a_{21}}{a_{11}} = a_{21} \cdot a_{11}^{-1}$ as multiplication.

- The first step of back substitution $x_n = \frac{c_n}{u_{nn}}$ requires at most one multiplication.

We now generalize these findings:

1. Consider $A \in \mathbb{M}(2 \times 2, \mathbb{R})$, $\vec{b} \in \mathbb{R}^2$:
   - Show that MEB requires at most 6 multiplications and 3 additions.
   - Find A, $\vec{b}$ for which MEB requires stricly less than 9 operations.
2. Repeat for $A \in \mathbb{M}(3 \times 3, \mathbb{R})$, $\vec{b} \in \mathbb{R}^3$.
3. **For Math 513:**
   Show that for $A \in \mathbb{M}(n \times n, \mathbb{R})$, $\vec{b} \in \mathbb{R}^n$, MEB requires (at most) $\frac{n \cdot (n^2+3n-1)}{3}$ multiplications and $\frac{(n-1) \cdot n \cdot (2n+5)}{6}$ additions. Hint: Use induction by n.
4. Use the result of part 3 to estimate the order of MEB:
   - Find the total number of operations (additions and multiplications).
   - Classify MEB with the computer science $\mathcal{O}$-notation.