Math 313/513, Spring 2022 **Martin Bies**

Homework 4 – Coding

Due: Thursday, February 17 – 10:00 am EST

Problem 1C: Rank-nullity theorem [20 Points]

1. Write a `Python` function `lin_independent` and justify why it operates correctly:
   - Input: k vectors in $\mathbb{R}^n$.
   - Output: `True` if the vectors are linearly independent and `false` otherwise.
2. Write a `Python` function `basis_check` and justify why it operates correctly:
   - Input: k vectors in $\mathbb{R}^n$.
   - Output: `True` if the vectors are a basis of $\mathbb{R}^n$ and `false` otherwise.
3. Write a `Python` function `CA_dimension` and justify why it operates correctly:
   - Input: k vectors $\vec{a}_1, \ldots, \vec{a}_k \in \mathbb{R}^n$.
   - Output: $\dim_{\mathbb{R}} S$ where $S = \mathrm{Span}_{\mathbb{R}}\left(\vec{a}_1, \ldots, \vec{a}_k\right)$.
4. Generate a family $\mathcal{A}$ of 1000 random matrices in $\mathbb{M}(5 \times 5, \mathbb{R})$ with integer entries $0, 1$. Compute $\dim_{\mathbb{R}}(C(A))$ of all matrices in $\mathcal{A}$. What dimensions occur?
5. The rank-nullity theorem states that for any $A \in \mathbb{M}(m \times n, \mathbb{R})$ it holds

   $$\dim_{\mathbb{R}}(N(A)) + \dim_{\mathbb{R}}(C(A)) = n\,. \qquad (1)$$

   Write a `Python` function:

   Input: $A \in \mathbb{M}(m \times n, \mathbb{R})$. Output: $\dim_{\mathbb{R}}(N(A)) + \dim_{\mathbb{R}}(C(A)) - n$.

   Use this function to test the rank-nullity theorem for all matrices in $\mathcal{A}$.
6. For $A \in \mathbb{M}(m \times n, \mathbb{R})$, formulate a hypothesis how $\dim_{\mathbb{R}}(N(A^T))$ and $\dim_{\mathbb{R}}(C(A^T))$ are related. Verify your hypothesis for all matrices in $\mathcal{A}$.