Math 313/513, Spring 2021 **Martin Bies**

Homework 5 – Theory

Due: Thursday, February 24 – 10:00 am EST

Problem 1T: Base change [10 Points]

In this exercise, you will find that in some basis it is easy to understand a linear transformation. We exemplify this for the linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\mathcal{B}_1 = \left\{ \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad A_{\mathcal{B}_1\mathcal{B}_1} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{1}$$

To this end, consider the $\mathcal{B}_2 = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$:

1. Find the transformation matrix $T_{\mathcal{B}_2\mathcal{B}_1}$ for the base change from $\mathcal{B}_1$ to $\mathcal{B}_2$.
2. Compute $T_{\mathcal{B}_1\mathcal{B}_2}$, $A_{\mathcal{B}_1\mathcal{B}_2} = A_{\mathcal{B}_1\mathcal{B}_1} T_{\mathcal{B}_1\mathcal{B}_2}$ and $A_{\mathcal{B}_2\mathcal{B}_2} = T_{\mathcal{B}_2\mathcal{B}_1} A_{\mathcal{B}_1\mathcal{B}_1} T_{\mathcal{B}_1\mathcal{B}_2}$.
3. Use $A_{\mathcal{B}_1\mathcal{B}_2}$ to tell if φ is a rotation, reflection, projection, ...

Problem 2T: Orthogonal vector spaces and decomposition [20 Points]

1. For the standard inner product $\langle \cdot, \cdot \rangle_{\text{Std}}$ in $\mathbb{R}^n$, verify all axioms of inner products.
2. Name another inner product $\langle \cdot, \cdot \rangle_2$ in $\mathbb{R}^n$ and verify that it satisfies all axioms.
3. Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R})$.
   - Verify that for any $\vec{x} \in N(A)$ and any $\vec{y} \in R(A)$ it holds $\langle \vec{x}, \vec{y} \rangle_{\text{Std}} = 0$.
   - For every $\vec{x} \in \mathbb{R}^3$, find $\vec{x}_N \in N(A)$ and $\vec{x}_R \in R(A)$ with $\vec{x} = \vec{x}_N + \vec{x}_R$.

Problem 3T: Inner products in exotic vector spaces [10 Points]

1. Prove that the following is an inner product in Pol_n:

$$\langle \cdot, \cdot \rangle_1 : \text{Pol}_n \times \text{Pol}_n \to \mathbb{R}, \, (p, q) \mapsto \int_0^1 p(x) \cdot q(x) dx. \tag{2}$$

2. Compute the length of $x, x^2, x^3 \in \text{Pol}_4$ with $\langle \cdot, \cdot \rangle_1$.
3. Find a basis $\mathcal{B} = \{P_1, P_2, P_3\}$ of Pol_2 such that for all $i, j \in \{1, 2, 3\}$ and $i \neq j$:

$$\langle P_i, P_j \rangle_1 = 0, \qquad \langle P_i, P_i \rangle_1 = 1. \tag{3}$$