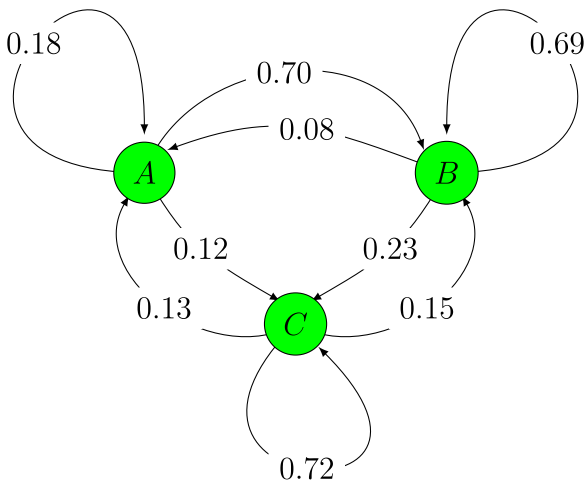
Math 313/513, Spring 2022 **Martin Bies**

Homework 8 – Coding

Due: Thursday, April 7 – 10:00 am EST

Problem 1C: A Markov process [20 Points]

We consider three towns A, B, C with 1 million inhabitants each. Every month, 70% of the inhabitants of A move to B. Similar movements exist among all pairs of the cities. The monthly rates are as follows:



For completeness, we have marked those people that do not move by an arrow from their city back to itself.

1. Find $M \in \mathbb{M}(3 \times 3, \mathbb{R})$ and $\vec{x} \in \mathbb{R}^3$ s.t. the components of $M\vec{x}$ match the number of inhabitants after one month.
   Hint: Fractional citizens are not meaningful. Still, round to 3 decimal places.
2. After n months, the number of people in A, B, C are given by the components of $\vec{x}^{(n)} := M^n \vec{x}$. Compute $\vec{x}^{(n)}$ for $n \in I = \{0, 1, \ldots, 10\}$.
3. Draw the three components of $\vec{x}_n$ against $n \in I$.
4. For each $n \in I$, verify that the total number of people
   $$\Sigma^{(n)} := x_1^{(n)} + x_2^{(n)} + x_3^{(n)}, \tag{1}$$
   is constant. To this end, draw $\Sigma^{(n)}$ against $n \in I$.
5. Is the existence of $\vec{x} \in \mathbb{R}^3$ with $x_i \geq 0$ and $M\vec{x} = \vec{0}$ consistent with part 4?
6. **Math 513**: Find $\vec{x} \in \mathbb{R}^3$ such that $M\vec{x} = \vec{x}$. Interpret the components of $\vec{x}$.
   Hint: Compare with your plot in part 2.
7. **Bonus (for 313 and 513)**: Find transition rates s.t. $\lim\limits_{n\to\infty}(M^n\vec{x})$ does not exist. Plot the components of $\vec{x}_n$ and the sum $\Sigma^{(n)}$ for $n \in I$.