Math 313/513, Spring 2022 **Martin Bies**

Homework 8

Due: Thursday, April 7 – 10:00 EST

Problem 1T: General properties of Eigenvalues [20 Points]

1. Consider $A \in \mathbb{M}(n \times n, \mathbb{R})$ and prove the following:
   - Be $k \in \mathbb{Z}_{\geq 0}$ and λ eigenvalue of A. Then λ^k is eigenvalue of A^k.
   - If A is invertible, then λ is eigenvalue of A iff λ^{-1} is eigenvalue of A^{-1}.
2. Take $n = 3$. For each of the following, name one matrix $A \in \mathbb{M}(n \times n, \mathbb{R})$:
   - A has n distinct eigenvalues.
   - A has less than n distinct eigenvalues.
   - At least one eigenvalue of A is not real.
3. **Math 513:** Repeat for arbitrary but fixed $n \in \mathbb{Z}$ with $n \geq 4$.

Problem 2T: An Eigenbasis [10 Points]

In this exercise we compute the *Eigenbasis* of a projection $\varphi_P \colon \mathbb{R}^3 \to \mathbb{R}^3$ with

$$P_{\mathcal{B}_{\text{std}}\mathcal{B}_{\text{std}}} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}), \quad \mathcal{B}_{\text{std}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \quad (1)$$

1. Compute the eigenvalues and eigenvectors of $P_{\mathcal{B}_{\text{std}}\mathcal{B}_{\text{std}}}$.
2. Verify that the eigenvectors of $P_{\mathcal{B}_{\text{std}}\mathcal{B}_{\text{std}}}$ furnish a basis of $\mathbb{R}^3$.
   Remark: This basis is the so-called *eigenbasis* $\mathcal{B}_{\text{eig}}$.
3. Find the mapping matrix $P_{\mathcal{B}_{\text{eig}}\mathcal{B}_{\text{eig}}}$ with respect to the eigenbasis.
   Hint: The eigenvalues of the vectors in $\mathcal{B}_{\text{eig}}$ are sufficient to identify this matrix.

Problem 3T: Eigenvalues, traces and determinants [10 Points]

In this exercise, we compare the eigenvalues, the trace and the determinant.

1. For $A \in \mathbb{M}(n \times n, \mathbb{R})$ one can show that the eigenvalues λ_i satisfy

   $$\operatorname{tr}(A) = \sum_{i=1}^{N} \lambda_i, \qquad \det(A) = \prod_{i=1}^{N} \lambda_i. \quad (2)$$

   Verify these results for $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 2 & 3 \\ 5 & 6 & 3 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})$. (Hint: $\lambda_1 = -2$.)
2. Show that all eigenvalues of $A \in \mathbb{M}(2 \times 2, \mathbb{R})$ are real iff $\operatorname{tr}(A)^2 - 4 \cdot \det(A) \geq 0$.
   Hint: Use eq. (2) and express the eigenvalues in terms of $\det(A)$ and $\operatorname{tr}(A)$.
3. **Bonus (for both 313 and 513)**: Prove eq. (2) for arbitrary $A \in \mathbb{M}(n \times n, \mathbb{R})$.