**Math 313/513, Spring 2022** **Martin Bies**

# Midterm 1

Tuesday, February 8: 10.15 – 11.45 EST

## Instructions

- Allowed materials: Pen and paper.
- Required materials: Penn card/ID.
- Forbidden materials: Anything not listed above.
- Fill in your information below.
- On each piece of paper, state your name and student ID.

## Student information

First name ______________________________

Last name ______________________________

Penn ID ______________________________

**Problem 1: Solutions to linear systems [10 Points]**

For the following linear systems, indicate if they can have

(I) no solution, (II) a unique solution, (III) infinitely many solutions.

No justification is required.

(a) $A\vec{x} = \vec{0}$ and $A \in \mathbb{M}(3 \times 6, \mathbb{R})$,

(b) $A\vec{x} = \vec{0}$ and $A \in \mathbb{M}(4 \times 3, \mathbb{R})$,

(c) $A\vec{x} = \vec{b}$ and $A \in \mathbb{M}(4 \times 4, \mathbb{R})$,

(d) $A\vec{x} = \vec{b}$ and $A \in \mathbb{M}(5 \times 6, \mathbb{R})$,

(e) $A\vec{x} = \vec{b}$ and $A \in \mathbb{M}(3 \times 2, \mathbb{R})$.

**Problem 2: A matrix questionary [10 Points]**

Consider a matrix $A \in \mathbb{M}(m \times n, \mathbb{R})$ with the following properties:

(a) $\mathrm{rk}(A) = 2$,

(b) $N(A) = \mathrm{Span}_{\mathbb{R}}(\vec{v})$ for some $\vec{v} \in \mathbb{R}^n$.

Justify if the following are true, false or undecided.

1. $\dim_{\mathbb{R}}(C(A)) = 2$.
2. $m = 3$.
3. For all $\vec{b} \in \mathbb{R}^m$ there exists $\vec{x} \in \mathbb{R}^n$ with $A\vec{x} = \vec{b}$.
4. **For Math 513:**
   Is $S = \{A \in \mathbb{M}(3 \times 3, \mathbb{R}) | A \text{ satisfies (a) \& (b)}\}$ a linear subspace of $\mathbb{M}(3\times 3, \mathbb{R})$?

**Problem 3: PLU-factorization [10 Points]**

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 5 & 9 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,.$$

1. Compute a PLU-factorization of $A$.
2. Determine $\mathrm{rk}(A)$.
3. Verify that $N(A) = \mathrm{Span}_{\mathbb{R}}(\vec{v}_1)$ for suitable $\vec{v}_1 \in \mathbb{R}^3$.
4. Define $R(A) := C(A^T)$. Find $\vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$ with $R(A) = \mathrm{Span}_{\mathbb{R}}(\vec{v}_2, \vec{v}_3)$.
5. **For Math 513:** Is there $\vec{\lambda} \in \mathbb{R}^3 \setminus \vec{0}$ such that $\sum_{i=1}^{3} \lambda_i \vec{v}_i = \vec{0}$?

## Problem 4: Basis [10 Points]

1. Find a basis $\mathcal{B}$ of

$$S = \mathrm{Span}_{\mathbb{R}}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}4\\0\\-3\\-1\end{bmatrix}\right) \subseteq \mathbb{R}^4 .$$

2. What is the dimension of $S$?

3. Extend $\mathcal{B}$ to a basis $\mathcal{B}'$ of $\mathbb{R}^4$.

4. Does $A \in \mathbb{M}(4 \times 4, \mathbb{R})$ with $C(A) = S$ and $\dim(N(A)) = 2$ exist?

## Problem 5: Parametric null space [10 Points]

For $a_1, a_2 \in \mathbb{R}$ we consider the linear system $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & a_2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}), \qquad \vec{b} = \begin{bmatrix}1\\1\\1\end{bmatrix} \in \mathbb{R}^3 . \tag{1}$$

Find all solutions to $A\vec{x} = \vec{b}$ as function of $a_1, a_2 \in \mathbb{R}$.

## Problem 6: A few proofs [10 Points]

1. Prove that for $A \in \mathbb{M}(m \times n, \mathbb{R})$ it holds $N(A) = \{\vec{0}\}$ iff $\mathrm{rk}(A) = n$.

2. Consider $A_1 \in \mathbb{M}(m \times n, \mathbb{R})$ and $A_2 \in \mathbb{M}(n \times l, \mathbb{R})$. Prove that

$$(A_1 \cdot A_2)^T = A_2^T \cdot A_1^T .$$

3. Let $A \in \mathbb{M}(n \times n, \mathbb{R})$ s.t. $\exists k \in \mathbb{Z}_{>0}$ with $A^k = 0$. Is $I - A$ invertible?