Math 313/513, Spring 2022 **Martin Bies**

Midterm 2

Tuesday, March 1: 10.15 – 11.45 EST

Instructions

- Allowed materials: Pen and paper.
- Required materials: Penn card/ID.
- Forbidden materials: Anything not listed above.
- Fill in your information below.
- On each piece of paper, state your name and student ID.

Student information

First name ______________________________

Last name ______________________________

Penn ID ______________________________

Result

Exercise	1	2	3	4	5	6	Σ
Points							

Problem 1: True or false? No justification required. [10 Points]

1. Consider a map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$. Then $\varphi = \varphi_A$ for a suitable $A \in \mathbb{M}(m \times n, \mathbb{R})$. Recall: $\varphi_A\colon \mathbb{R}^n \to \mathbb{R}^m\,,\ \vec{x} \mapsto A\vec{x}$.

2. Let $\mathcal{A}$ the standard basis of $\mathbb{R}^n$ and $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ another basis of $\mathbb{R}^n$. Then

$$T_{\mathcal{AB}} = \begin{bmatrix} \vec{b}_1 & \ldots & \vec{b}_n \end{bmatrix} \in \mathbb{M}(n \times n, \mathbb{R})\,. \tag{1}$$

3. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a projection onto a 1-dimensional linear subspace of $\mathbb{R}^2$. Then there exists a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the mapping matrix $A_{\mathcal{BB}}$ of φ is

$$A_{\mathcal{BB}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{2}$$

4. **Math 513:** Let $A \in \mathbb{M}(m \times n, \mathbb{R})$. Then $A^T A \in \mathbb{M}(n \times n, \mathbb{R})$ is invertible.

Problem 2: Orthogonal projection [10 Points]

Consider $\mathbb{R}^3$ wth the standard inner product and $S = \left\{ [x, y, z]^T \in \mathbb{R}^3 \,\middle|\, 2x + y + z = 0 \right\}$.

1. Compute the orthogonal projection $\varphi_P\colon \mathbb{R}^3 \to S$.

2. Find a basis of $S^\perp$. Use it to compute the orthogonal projection $\varphi_Q\colon \mathbb{R}^3 \to S^\perp$.

3. Verify that $P + Q = I$.

Problem 3: Orthogonal vectors [10 Points]

Consider linearly independent $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n \backslash \{\vec{0}\}$. Show that with regard to the standard inner product in $\mathbb{R}^n$, the following $\vec{U}, \vec{V}, \vec{W} \in \mathbb{R}^n$ are pairwise orthogonal:

$$\vec{U} = \vec{a}\,, \quad \vec{V} = \vec{b} - \frac{\langle \vec{U}, \vec{b} \rangle}{\langle \vec{U}, \vec{U} \rangle} \cdot \vec{U}\,, \quad \vec{W} = \vec{c} - \frac{\langle \vec{U}, \vec{c} \rangle}{\langle \vec{U}, \vec{U} \rangle} \cdot \vec{U} - \frac{\langle \vec{V}, \vec{c} \rangle}{\langle \vec{V}, \vec{V} \rangle} \cdot \vec{V}\,. \tag{3}$$

Problem 4: Orthogonal basis [10 Points]

Consider $\mathbb{R}^4$ with standard inner product and $S = \mathrm{Span}_{\mathbb{R}}(\vec{a}, \vec{b}, \vec{c})$ with

$$\vec{a} = 2\vec{e}_1 + \vec{e}_3\,, \qquad \vec{b} = -\vec{e}_1\,, \qquad \vec{c} = 2\vec{e}_1 - \vec{e}_2 + 3\vec{e}_3\,, \tag{4}$$

where $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$ is the standard basis of $\mathbb{R}^4$.

1. A basis $\{\vec{U}, \vec{V}, \vec{W}\}$ of S is orthogonal iff $\vec{U}$, $\vec{V}$, $\vec{W}$ are pairwise orthogonal. Find an orthogonal basis of S. Hint: Use problem 3.

2. Find an orthogonal basis of $S^\perp$.

3. **Math 513:** Use these results to construct an orthogonal basis of $\mathbb{R}^4$.

Problem 5: Least square approximation [10 Points]

Consider the following three points in $\mathbb{R}^2$:

$$\vec{b}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{5}$$

We seek $C, D \in \mathbb{R}$ such that the following line approximates these points:

$$L(C,D) = \left\{ \begin{bmatrix} t \\ C + Dt \end{bmatrix} \mid t \in \mathbb{R} \right\} \subseteq \mathbb{R}^2. \tag{6}$$

1. Find $A \in \mathbb{M}(3 \times 2, \mathbb{R})$ and $\vec{b} \in \mathbb{R}^3$ such that

$$\left\{ \vec{b}_1, \vec{b}_2, \vec{b}_3 \right\} \subset L(C,D) \quad \Leftrightarrow \quad A \cdot \begin{bmatrix} C \\ D \end{bmatrix} = \vec{b}. \tag{7}$$

2. Find the best approximation line.

   Hint: You may use $\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5/6 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$.

3. **Math 513:** Quantify how good this line approximates $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$.

Problem 6: Projection vs. least square [10 Points]

Consider $A \in \mathbb{M}(m \times n, \mathbb{R})$, $\vec{b} \in \mathbb{R}^m$ and $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T \in \mathbb{R}^n$.

1. Consider $l_e(\vec{x}) = \langle A\vec{x} - \vec{b}, A\vec{x} - \vec{b} \rangle_{\text{Std}}$. Verify that

$$l_e(\vec{x}) = \vec{x}^T \left(A^T A \right) \vec{x} - 2\vec{x}^T A^T \vec{b} + \vec{b}^T \vec{b}. \tag{8}$$

2. Use this to conclude that

$$\left(\frac{\partial l_e}{\partial x_k} \right) (\vec{x}) = 2 \cdot \left(A^T A \vec{x} - A^T \vec{b} \right)_k. \tag{9}$$

3. Consider the Jacobian matrix

$$J_{l_e}(\vec{x}) = \begin{bmatrix} \left(\frac{\partial l_e}{\partial x_1} \right) (\vec{x}) \\ \left(\frac{\partial l_e}{\partial x_2} \right) (\vec{x}) \\ \vdots \\ \left(\frac{\partial l_e}{\partial x_n} \right) (\vec{x}) \end{bmatrix}. \tag{10}$$

   Argue that $J_{l_e}(\vec{x}) = \vec{0}$ iff $A^T A \vec{x} = A^T \vec{b}$.