Math 313/513, Spring 2022 **Martin Bies**

Practice questions

The following questions are designed to help you revise the course material. They are not guaranteed to cover every topic or to represent the focus of the final exam. A few selected exercises were already discussed on the weekly assignments.

1. Consider the matrix

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \in \mathbb{M}(3 \times 4, \mathbb{R}) \tag{1}$$

   a) Find a basis for the right null space $N(A)$ of A.

   b) Find a basis for the left null space $N(A^T)$ of A.

   c) Find a basis for the column space $C(A)$ of A.

   d) Find a basis for the row space $R(A)$ of A.

2. True or false? If false, give a reason.

   a) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a collection of vectors in $\mathbb{R}^5$, then the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a three dimensional linear subspace of $\mathbb{R}^5$.

   b) If $A \in \mathbb{M}(n \times n, \mathbb{R})$ has no zero singular values, then A is invertible.

   c) If $A \in \mathbb{M}(m \times n, \mathbb{R})$, then the column space of A consists of all possible vectors in $\mathbb{R}^m$ of the form $A\vec{x}$, where $x \in \mathbb{R}^n$.

   d) If $\vec{x}$ is a least-squares solution to $A\vec{x} - \vec{b}$, then $A\vec{x}$ is orthogonal to the column space $C(A)$ of A.

3. Find the line of best fit passing through the following points in $\mathbb{R}^2$:

$$(3, 1)\,, \quad (0, 0)\,, \quad (1, 4)\,, \quad (-1, -1)\,, \quad (3, 2)\,. \tag{2}$$

4. Define

$$A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}(3 \times 2, \mathbb{R})\,. \tag{3}$$

   a) Find the SVD of A.

   b) Find the rank-1 approximation to A. Why is your answer not surprising?

   c) Suppose $\vec{x} \in \mathbb{R}^2$ with length 1. What are the largest and smallest possible values of $(A\vec{x})^T(A\vec{x})$?

5. In $\mathbb{R}^4$, let W be the span of the following vectors:

$$\vec{v}_1 = 2e_1 + e_3, \qquad \vec{v}_2 = -e_1 + 3e_2, \qquad \vec{v}_3 = 2e_1 - e_2 + 3e_3 \tag{4}$$

(where e_1, e_2, e_3, e_4 is the standard basis of $\mathbb{R}^4$.

a) Find an orthonormal basis for W.

b) Find an orthonormal basis for $N(A^T)$ where $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$

c) Using your answer to (a), find the projection of $\vec{x} = [1, 0, 0, 1]$ onto W.

6. True or false? Explain your answer (in either case).

a) If A is a diagonalizable $n \times n$ matrix, then A^2 is diagonalizable.

b) If A is an $n \times n$ matrix and A^2 is diagonalizable, then A is diagonalizable.

c) If A is a matrix such that no column is a scalar multiple of any other column, then the columns of A are linearly independent.

7. Diagonalize:

$$\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R})\,. \tag{5}$$

8. Suppose A is a real-valued matrix with a two-dimensional null space and a one-dimensional column space. Suppose $\vec{b}$ belongs to the column space of A. Answer each of the following questions as true, false, or not enough information.

a) The system $A\vec{x} = \vec{b}$ is solvable.

b) A has three columns.

c) A has three rows.

d) The set of solutions to $A\vec{x} = \vec{b}$ is a plane in $\mathbb{R}^3$ (possibly shifted away from the origin).

9. Let W be the plane in $\mathbb{R}^3$ containing the origin and points $(1, 2, 3)$ and $(2, 0, 1)$. Find the closest vector in W to $y = (3, 5, 4)$ and the distance from y to W.

10. Give an example of a 2×2 diagonalizable matrix that is not invertible.

11. Give an example of a (2×2 or 3×3) matrix that is neither diagonalizable nor invertible.

12. What is the determinant of the following matrix?

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \tag{6}$$

(Hint: quick and short)

13. Briefly answer the following questions, giving a short explanation

a) What does it mean for a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ to be linear?

b) Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are vectors in $\mathbb{R}^3$ such that none is a scalar multiple of the either of the others. Must these vectors be linearly independent?

c) If A is a 3×2 matrix with rank 2, is every vector $\vec{b}$ in $\mathbb{R}^3$ in the span of the columns of A?

d) If A is a diagonalizable $n \times n$ matrix, what can you say about sum of the dimensions of all the eigenspaces?

e) Do the tips of all vectors along the line $y = 2x + 5$ form a linear subspace of $\mathbb{R}^2$?

14. For each of the three elementary row operations (adding a multiple of one row to another, scaling by a nonzero constant, and swapping), state which ones do affect ...

a) whether a matrix is invertible?

b) the eigenvalues of a matrix?

c) whether a matrix is diagonalizable?

d) the determinant of a matrix?

e) the rank of a matrix?

f) the null space of a matrix?

g) the column space of a matrix?

15. Prove that if A and B are invertible matrices, then $(AB)^{-1}$ equals to $B^{-1}A^{-1}$. Also, prove that $(A^T)^{-1} = (A^{-1})^T$.

16. Suppose you are given a 3×3 matrix A. You performed the following row operations:
    - added the first row to the second row,
    - swapped rows 2 and 3,
    - scaled row 1 by the number 2,

    to arrive at the matrix

$$E = \begin{bmatrix} 2 & 1 & 5 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) . \tag{7}$$

    What is the determinant of A? Explain your answer.

17. The following linear system is inconsistent. Find all least-squares solutions.

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ -x_1 + 2x_2 &= 0 \\ 2x_1 - 4x_2 &= 1 \end{aligned}$$

18. Consider the matrix A below, and answer the following questions.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -2 \\ -2 & 3 & 4 \\ 1 & 1 & 3 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R}) . \tag{8}$$

    a) Find a basis for the column space of A.
    b) Find an orthonormal basis for the column space of A.
    c) How many nonzero singular values does A have? Why?

19. Consider the matrix A below and answer the following questions.

$$A = \begin{bmatrix} 3 & 1 \\ -1 & -3 \end{bmatrix} \in \mathbb{M}(2 \times 2, \mathbb{R}) . \tag{9}$$

    a) Find a singular value decomposition of A.
    b) Find the rank-one approximation of A.

20. Find an LU factorization of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})$.

21. Find the inverse of the following matrices (if possible)

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix} \tag{10}$$

For additional practice, compute their determinants as well.

22. Which of the following are vector spaces? Explain your answers:
    a) The set of $n \times n$ matrices with determinant 1
    b) The set of $n \times n$ matrices with trace 0
    c) The set of $n \times n$ upper-triangular matrices
    d) The set of $n \times n$ matrices or rank $\leq r$ for some number r
    e) Vectors in $\mathbb{R}^n$ orthogonal to some vector $\vec{v}$.
    f) Vectors in $\mathbb{R}^2$ whose tips lie in the first quadrant
    g) Vectors in $\mathbb{R}^2$ whose tips lie in the first quadrant or third quadrant.

23. Prove directly that if A is $n \times n$ invertible matrix, then the equation $A\vec{x} = b$ always has a unique solution.

24. Let A be a symmetric matrix with eigenvalues 2 and 3 and respective eigenvectors v and w. Prove that v and w must be orthogonal.

25. Find the steady-state vector of the Markov matrix:

$$\begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \in \mathbb{M}(3 \times 3, \mathbb{R})\,. \tag{11}$$

26. Find an SVD of:

$$\begin{bmatrix} 7 & 0 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} \in \mathbb{M}(3 \times 2, \mathbb{R})\,. \tag{12}$$

27. Find an SVD of:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \in \mathbb{M}(3 \times 2, \mathbb{R})\,. \tag{13}$$

28. Find an SVD of:

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \in \mathbb{M}(2 \times 3, \mathbb{R})\,. \tag{14}$$