F-TheorY Tools: String theory applications of OSCAR

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Motivation

Questions to challenge string theory:

- Does one of these solutions describe our universe?
  (↔ Holy grail of string phenomenology)

- Can we make predictions beyond current experiments?
  (↔ Cool new technology?)

- Can we understand the physical universe better?
  (↔ 19 experimentally determined parameters for particle physics – Really?!?)
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My work: Vector-like spectra

- Relatively recent with M. Cvetič, R. Donagi, L. Lin, M. Liu, M. Ong, F. Ruehle
  (2007.00009, 2102.10115, 2104.08297, 2205.00008, 2303.08144, and work in progress)

- Many algorithmic aspects
  (toric varieties, intersection theory, categories, sheaf cohomologies, root bundles on nodal curves, . . . )
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Most recent additions:
- Toric geometry in OSCAR with L. Kastner. (2303.08110)
- FTheoryTools in OSCAR with A. P. Turner, M. Zach, A. Frühbis-Krüger. (work in progress)
Goal: For computations in algebra, geometry, and number theory.
OSCAR: Open Source Computer Algebra Research system

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  - Toric geometry: Polyhedral geometry, algebraic geometry, combinatorics + ...
  - FTheoryTools: Toric and algebraic geometry, intersection theory, Lie groups, ...
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- **More information e.g. in latest *Computeralgebra-Rundbrief***:
  - M. Horn: OSCAR: An introduction
  - M. Bies, L. Kastner: Toric geometry in OSCAR. (2303.08110)
Toric Geometry in OSCAR

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- Check it out!
Questions so far?
String theory = General relativity + Standard Model?
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4-dim. world $\mathcal{W}$ 'small' 6 real-dim. manifold $\mathcal{B}_3$

Challenge: Find $\mathcal{B}_3$ s.t. ST reproduces 4d physics.
Different types of String theory

- type I
- type IIB
- type I
- heterotic $SO(32)$
- heterotic $E_8 \times E_8$
- 11d SUGRA
- M-Theory

- $g_s, \alpha' \ll 1 \ (\alpha' = \frac{l_s^2}{4\pi})$
- $\alpha' \ll 1$ but $g_s$ strongly coupled

String and F-Theory in a nutshell
More on crepant resolutions in F-theory
Singular elliptic fibrations for F-theory

- Particle physics deals with “fields”.
- Fields are, loosely speaking, functions. (More precisely: Sections of vector bundles.)
- F-theory: Axio-dilaton field $\tau$ is super important:
  - $\tau : B_3 \rightarrow \mathbb{C}, \ p \mapsto C_0(p) + \frac{i}{g_s(p)}$.
  - Physics invariant under $SL(2, \mathbb{Z})$ transformation of $\tau$:
    $$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1.$$  

$\Rightarrow$ Axio-dilation $\tau$ is complex structure modulus of elliptic curve $\mathbb{C}_{1,\tau}$.

- $\tau(p) \rightarrow i\infty$ when $g_s(p) \rightarrow 0$ (Physics lingo: Vicinity of D7-branes.)
  $\Rightarrow \mathbb{C}_{1,\tau}$ becomes singular.
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**Consequence** \[ \text{[Vafa '96], [Morrison Vafa '96]} \]

Singular elliptic fibration as book-keeping device/consistency check of F-theory.
Cartoon: Singular elliptic fibrations for F-theory

<table>
<thead>
<tr>
<th>IIB-SUGRA</th>
<th>Geometry</th>
</tr>
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<tbody>
<tr>
<td>union of loci of D7-branes</td>
<td>Singular locus $\Delta$ of elliptic fibration $C_{1,\tau} \hookrightarrow Y_4 \xrightarrow{\pi} B_3$</td>
</tr>
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</table>

$\pi$ base $B_3$

$\pi$ fibre $C_{1,\tau}$

$\pi$ total space $Y_4$
Singularities meet F-theory

1. Strategy 1: Do not resolve the singularities
   - Hard to extract the physics, but some attempts do exist.
   - [Anderson Heckman Katz '13], [Collinucci Savelli '14], [Collinucci Giacomelli Savelli Valandro '16]

2. Strategy 2: Resolve the singularities
   - For (simple) physics interpretation, must resolve crepantly.
   - Employ (weighted) blowup sequence.
   - [Arena Jefferson Obinna '23]

Challenges to find a crepant resolution:
- $Q$-factorial terminal singularities cannot be resolved crepantly.
- Currently, no algorithm for crepant (weighted blowup) resolution.
- Sometimes find non-flat fibrations: Physics not clear.
  - [Lawrie Schafer-Nameki '12], [Apruzzi Heckman Morrison Tizzano '18], ...

One goal of **FTheoryTools**: Automate strategy 2.
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Goals for FTheoryTools

1. Find and implement algorithm for **crepant** resolution:
   - Many details known in F-theory literature.
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2. Generalize/implement techniques:
   - A lot of toric functionality in OSCAR [Bies Kastner '23]
   - Many interesting techniques known [Jefferson Taylor Turner '21], [Jefferson Turner '22], ...
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3. Many models studied in large detail in F-theory literature:
   - Resolutions, topological data, ... known.
   - Study same model with different techniques.
   ⇒ LiteratureModels
Questions so far?
Cartoon: Singular elliptic fibration for F-theory

IIB-SUGRA

union of loci of D7-branes in IIB-compactification

Geometry

Singular locus $\Delta$ of elliptic fibration $\mathbb{C}_{1,\tau} \hookrightarrow Y_4 \xrightarrow{\pi} B_3$
Cartoon of blow-up resolution
Massless matter

[Katz Vafa '96], [Witten '96], [Grassi,Morrison '00 & '11], [Morrison,Taylor '11], [Grassi, Halverson, Shaneson '13],
[Cvetič, Klevers, Piragua, Taylor '15], [Anderson, Gray, Raghuram, Taylor '15], [Klevers, Taylor '16], [Klevers, Morrison, Raghuram, Taylor '17], ...
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Gauge group $G$

Massless matter:

Formal linear sum of $\mathbb{P}^1$ fibrations ($\leftrightarrow$ weight $\beta^a(R)$ of irrep. $R$ of $G$)

matter curve ($\leftrightarrow$ intersections of 7-branes)

Singular locus $\Delta$
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\[ \text{Singular locus } \Delta \]

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Consider \( \mathbb{P}^{2,3,1} \) with coordinates \([x : y : z]\).

(In analogy to "ordinary" projective space: \((\lambda^2 x, \lambda^3 y, \lambda z) \sim (x, y, z)\) and \(x = y = z = 0\) is forbidden.)

Let \( B_3 \) be a complete, Kaehler 3-fold s.t. there exist

\[
0 \neq a_i \in H^0 \left( B_3, K_{B_3}^\otimes i \right), \quad i \in \{1, 2, 3, 4, 6\}.
\]

Define the Tate polynomial ("long Weierstrass equation"):

\[
P_T = y^2 + a_1 xyz + a_3 yz^3 - x^3 - a_2 x^2 z^2 - a_4 xz^4 - a_6 z^6.
\]

Fix \( p \in B_3 \). Then \( V(P_T(p)) \subset \mathbb{P}^{2,3,1} \) is a torus surface.

\(\Rightarrow\) Elliptic fibration \( \pi : Y_4 \rightarrow B_3 \) with section \([x : y : z] = [1 : 1 : 0]\).

("Global": \( P_T \) defines the model for every \( p \in B_3 \).)
Global Tate model to Weierstrass model  

- Consider global Tate model defined by $a_i \in H^0\left(B_3, \mathcal{K}_{B_3}^\times\right)$ and

$$P_T = y^2 + a_1 xyz + a_3 yz^3 - x^3 - a_2 x^2 z^2 - a_4 xz^4 - a_6 z^6.$$ 

- We define a few quantities:

$$b_2 = 4 a_2 + a_1^2, \quad b_4 = 2 a_4 + a_1 a_3, \quad b_6 = 4 a_6 + a_3^2, \quad f = -\frac{1}{48} \left(b_2^2 - 24 b_4\right), \quad g = \frac{1}{864} \left(b_3^2 - 36 b_2 b_4 + 216 b_6\right).$$

$\Rightarrow$ (Short) Weierstrass model defined by $f$, $g$ and

$$P_W = y^2 - x^3 - fxz^4 - gz^6.$$ 

The singular loci of the Tate/Weierstrass model are

$$V(\Delta) = V(4f^3 + 27g^2) \subset B_3.$$
An $SU(5) \times U(1)$ F-theory global Tate model

Fine tune F-theory global Tate model

Wish to have particular singularity over hypersurface $V(w) \subset B_3$.

One particular model [Krause Mayrhofer Weigand '11]

- Assume that $B_3$ allows us to factor the sections $a_i$:

$$a_1 = a_1, \ a_2 = a_{2,1}w, \ a_3 = a_{3,2}w^2, \ a_4 = a_{4,3}w^3, \ a_6 \equiv 0.$$

$$\Rightarrow \Delta = 4f^3 + 27g^2 = w^5 \cdot P,$$ with complicated polynomial $P$.

- Singularities:
  - $\text{ord}_{V(w)}(f, g, \Delta) = (0, 0, 5):$ $I_5$-singularity $\leftrightarrow SU(5)$
  - $\text{ord}_{V(P)}(f, g, \Delta) = (0, 0, 1):$ $I_1$-singularity $\leftrightarrow$ "Not relevant"

$U(1)$ from Mordell-Weil group of elliptic fibration . . .

(More information: Kodaira classification, Tate table, Weierstrass table)
Resolution for $SU(5) \times U(1)$ F-theory global Tate model

- Blowup sequence worked out in literature [Krause Mayrhofer Weigand '11]:

\[
\begin{align*}
(x, y, w) &\rightarrow (xe_1, ye_1, we_1), \\
(y, e_1) &\rightarrow (ye_4, e_1 e_4), \\
(x, e_4) &\rightarrow (xe_2, e_4 e_2), \\
(y, e_2) &\rightarrow (ye_3, e_2 e_3), \\
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- Demonstrate with experimental stage of FTheoryTools: https://docs.oscar-system.org/dev/Experimental/FTheoryTools/tate/
Outlook and status of FTheoryTools

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⇒ Opportunity: Testing ground for new techniques and finding new physics.
Thank you for your attention!