# Dr. rer. nat. Martin Bies

## *Research Statement*

RPTU Kaiserslautern-Landau
Department of Mathematics
Gottlieb-Daimler-Straße 48 (Office 433)
67663 Kaiserslautern, Germany
December 15, 1987 (Merzig, Germany)
Single (Not Married)
+49 (0)631 205 2850
bies@mathematik.uni-kl.de
https://martinbies.github.io/

| | | |
|---|---|---|
| German | **Native** | ●●●●● |
| English | **Full Proficiency** | ●●●●● |
| French | **Modest** (*CEFR Level B1*) | ●●○○○ |

From my undergraduate days, string theory captivated me as a foremost candidate for a unified theory of nature. Central to string theory is its vast *string landscape*, with myriads of solution. Despite advances, we are far from uncovering string theory's full implications. My research aims to discern solutions consistent with observed physics, presenting exciting opportunities to unveil hitherto unknown facets of the universe. To this end, I employ geometric logic, especially in toric and algebraic geometry. My strength lies in constructive and enumerative techniques, harnessing computational tools to pinpoint optimal string theory solutions. Notably, I have added/modified 142,000+ lines of code in the open-source computer algebra system `OSCAR`.

My research trifurcates into: physics, emphasizing F-theory's geometric framework for string theory; mathematics, delving into toric and algebraic geometry as well as touching on combinatorics, graph theory, and number theory; and computer science, focusing on open-source computer algebra systems like `OSCAR` but also interests in machine learning and data science applications.

The following text is tailored for a broad readership. Experts may skip the introduction to string and F-theory.

## INTRODUCTION TO STRING THEORY

Physics has long pursued the unification of the four fundamental forces: electromagnetism, weak and strong nuclear interactions, and gravity [1–4]. While the standard model effectively unifies the first three forces [5–7], gravity's integration remains elusive by use of perturbative quantum field theories [8]. This challenge has spotlighted string theory as a potential solution [9–12].

At the heart of string theory lies the proposition that elementary particles are not point-like but instead resemble strings in shape. Intriguingly, a consistent quantum representation of these strings mandates a 10-dimensional spacetime $\mathcal{S}$ [9–12], a deviation from our conventional 4-dimensional (time plus three spatial directions) observation of the world surrounding us. The disparity is bridged through compactification, often illustrating the 10-dimensional spacetime as $\mathcal{S} = \mathcal{E} \times \mathcal{M}_6$, with $\mathcal{E}$ representing our everyday-observed 4-dimensional spacetime, and $\mathcal{M}_6$ being a 6-dimensional compact Calabi-Yau manifold. Experimental endeavors have yet to yield evidence for extra spacetime dimensions, leading to the presumption that $\mathcal{M}_6$ is exceptionally minuscule. The configuration of $\mathcal{M}_6$ profoundly affects the physics we discern in $\mathcal{E}$ according to string theory. Pinpointing an $\mathcal{M}_6$ that ensures a seamless match between string theory's predictions and experimental findings remains a pressing concern.

String theory is fundamentally defined by its action: a functional dependent on fields that denote elementary particles. From a mathematical perspective, these fields can be conceptualized as sections of vector bundles. Minimizing this action produces differential equations, the solutions of which dictate the dynamics of the elementary particles represented by these fields. As the intricacies are unpacked, an additional consistency criterion surfaces: string theory demands a particular symmetry among its elementary particles, termed supersymmetry. Despite its theoretical prominence, this symmetry has yet to be substantiated through experimentation. It's worth noting that the discovery of the Higgs boson [13, 14] took fifty years, while the detection of *gravitational waves* spanned an entire century. My optimism persists in believing that supersymmetry will eventually gain empirical validation, potentially indirectly via supersymmetry breaking mechanisms [15, 16].

String phenomenology aims to reconcile string theory with our experimentally observed physics. This involves identifying suitable geometries $\mathcal{M}_6$ and solutions to the equations of motion obtained from minimizing the action functional. Enormous efforts in this direction have been undertaken. Broadly, these research efforts diverge into two categories: matching string theory with cosmological observations and aligning it with particle accelerator results. My research focus is on the latter.

String theory has five equivalent formulations [9, 10, 17], leading to identical physical theories from varied action functionals on spaces $\mathcal{M}_6$. Early efforts focused on the $E_8 \times E_8$ heterotic string [18–25] and eventually expanded to include intersecting brane scenarios in type IIA and IIB string theory [26–33]. Said solutions to string theory solve the differential equations of motion by use of Taylor expansion. Physicist justify this strategy by saying that there is a small parameter in said Taylor series – the so-called interaction strength – which is believed to ensure convergence of said series. Put yet differently, said solutions to string theory explore the perturbative sector. The first perturbative realization of the minimally supersymmetric standard model (MSSM) – a minimal extension of the empirically-backed standard model of particle physics via supersymmetry – exists in [21, 22] with further insights in [34, 35]. However, many other perturbative models introduce unobserved exotic particles. A prevailing challenge, as noted in [23–25] and detailed in [36], is that some observed particle interactions are omitted or differ significantly from experiments like those at CERN.

Alongside ongoing studies on (perturbative) heterotic line bundle standard models, significant attention has been given to the potential of *F-theory* in probing strongly coupled IIB *string theory*. F-theory adeptly bridges the gap between geometry and physics [37–39], enriched by techniques from algebraic geometry. This framework inherently supports the required particle interactions due to its geometric consistency. By the same principle, its solutions are always globally consistent, which is not a given for perturbative string theory explorations. Pioneering studies in this realm encompass references such as [40–52]. A milestone in this domain is the discovery of the *Quadrillion F-theory standard models (QSMs)* [53], renowned for their physically appealing properties (global consistency, gauge coupling unification, and absence of chiral exotics). The QSMs stand out as the largest known class (more than $10^{15}$ constructions) of F-theory standard model solutions with these attributes.

## PAST CONTRIBUTIONS

Higgs pairs are a vital component in our understanding of particle physics, as underpinned by the Nobel Prize awarded to Higgs and Englert in 2013 [13, 14]. For instance, a *single* Higgs pair is imperative for the MSSM. To check for alignment among F-theory solutions and the MSSM, it is thus imprudent to compute the total number of Higgs pairs. This leads to the study of vector-like spectra. My engagement with this topic is not only rooted in the dire need to investigate this property of F-theory solutions, but also by the rich mathematical tapestry of this topic, which provides avenues to apply modern mathematics and computational tools in physics. My explorations encompass cohomologies of coherent sheaves on toric varieties [45, 49, 51], Freyd categories [54], and machine learning enhancements [55]. Recently, I focused on the F-theory QSMs [56–59]. While F-theory adeptly resolves consistency issues in perturbative string theory, computing the vector-like spectra, and thereby the number of Higgs pairs, is more demanding compared to the heterotic string theory. In the latter, Higgs pairs are encoded in line bundles on a Calabi-Yau 3-fold, often assumed to be a toric space for practicality, simplifying the computation of line bundle cohomologies. Conversely, F-theory centers on line bundles on smooth, irreducible curves, introducing complexities due to the curve's continuous Picard group. My QSM program – encapsulated in a recent review [60] and remaining a major focus in my future research – provides significant arithmetic approaches towards the Brill-Noether theory of root bundles on nodal curves and employ the enusing Brill-Noether numbers as upper bounds for the vector-like spectra of the F-theory QSMs.

Navigating the geometric computations in F-theory can be arduous, which slows progress, limits the exploration of complex geometries (e.g., those requiring techniques beyond those suitable for toric varieties), and presents a steep entry barrier for newcomers. In collaboration with A. Turner from the University of Pennsylvania, I have enriched the OSCAR computer algebra system, introducing tools specifically designed for F-theory applications. A notable feature of the upcoming FTheoryTools is its capability to effortlessly extract and modify geometric constructions from existing literature. It will also incorporate advanced singularity resolution techniques, since the physics is most easiest extracted from the resolved space. Advancing these developments, e.g. by facilitating topological properties of non-toric-spaces as well as establishing `FTheoryTools` as a recognized computational tool in F-theory encapsulates my second research focus in the future.

# RESEARCH FOCUS I: FTheoryTools – TACKLING F-THEORY'S COMP. CHALLENGES

Navigating F-theory's geometric computations is challenging, slowing advancements. To streamline this, I am making major contributions to the development of `FTheoryTools` within the `OSCAR` computer algebra system [61, 62]. As of now, I have added and modified over 142,000 lines in `OSCAR`, with significant contributions to its toric geometry functionality [63]. Additionally, I sought to modernize my earlier work, the *ToricVarieties_project* [64], which is part of [65, 66] and written in the `gap` programming language [67]. In contrast, `OSCAR` uses the modern `Julia` programing language. A brief overview of F-theory's geometric intricacies ensues.

## F-Theory in a Nutshell

Type IIB *string theory* hinges on a supergravity action governed by fields which, for mathematicians, are sections of particular vector bundles. Notably, the scalar dilaton field $\phi$ (with $\phi(x) \in \mathbb{C}$ for every $x \in \mathcal{M}_6$) and the RR gauge potentials $C_0$ and $C_8$ are of primary focus. The dilaton field's significance stems from its linkage to the strength of string interactions in 10-dimensional spacetime $\mathcal{S} = \mathcal{E} \times \mathcal{M}_6$ [9–12]. Interestingly, $C_8$'s equations of motion allow only trivial solutions on a smooth $\mathcal{M}_6$. The application of an involution $\sigma$ to $\mathcal{M}_6$, i.e. $\mathcal{M}_6 \to \mathcal{B}_6 := \mathcal{M}_6/\sigma$, leads to a space $\mathcal{B}_6$ which allows non-trivial solutions to the $C_8$ equations of motion. In physics lingo, the involution introduces orientifold $O7$ planes, which are represented by the fixed points of $\sigma$ [9, 10]. Spaces akin to $\mathcal{B}_6$ serve as the foundation for type IIB orientifold theories [15,68,69]. It turns out that in such orientifold theories, the dilaton field $\phi$ can manifest singularities, implying an infinite value at specific loci and leading to strong string interactions. Such strength contradicts the essential premise of weak interactions in perturbation theory. This necessitates a transition from the perturbative type IIB supergravity description to a non-perturbative framework [70], namely *F-theory* [37]. To this end, the dilaton $\phi$ and RR gauge field $C_0$ merge into the axio-dilaton $\tau$:

$$\tau \colon \mathcal{E} \times \mathcal{B}_6 \to \mathbb{C}\ ,\ x^\mu \mapsto C_0\left(x^\mu\right) + ie^{-\phi(x^\mu)}\,.$$

Due to *Lorentz invariance*, $\tau$ is constant on $\mathcal{E}$ and a section of a holomorphic $\mathrm{SL}\left(2, \mathbb{Z}\right)$ line bundle over $\mathcal{B}_6$ [38,39]. We understand the value of $\tau$ at $x^\mu \in \mathcal{B}_6$ as the complex structure modulus of an elliptic curve. Consequently, an elliptically fibered 4-fold $\pi \colon Y_4 \mathbf{2} headrightarrow \mathcal{B}_6$ with fibre $\mathbb{C}_{1,\tau(x^\mu)}$ serves as "book-keeping" device of the axio-dilaton. At the same time, the geometry of $Y_4$ enforces consistency in that it ensures that the equation of motion for $C_8$ has a solution and leads to the encoded axio-dilaton field $\tau$.
The geometry of $Y_4$ encodes much of the physics, as detailed in [71–74]. Non-trivial physics necessitates a singular $Y_4$. In need for better alternatives, it is common to try to crepantly resolve $Y_4$ [75]. Most of my past contributions to F-theory assume that, up to $\mathbb{Q}$-factorial terminal singularities, at least one such crepant resolution $\widehat{Y}_4$ exist in the form of a sequence of blowups. Furthermore, I assume that $\widehat{Y}_4$ admits at least one smooth section.[1]

## Goals and Features of FTheoryTools

In F-theory setups, the initial geometric challenge is the crepant resolution of singular $Y_4$. A comprehensive algorithm is still elusive, especially in determining $\mathbb{Q}$-factorial terminal singularities. Typically, we apply the entire F-theory toolkit to singularities, presuming non-resolvability when standard methods fail. Hence, incorporating state-of-the-art resolution routines, for instance including the weighted blowups explored in [80], is paramount. An equally significant feature is a database to automatically utilize established literature constructions, including the set of known resolutions.
Upon resolution, the ensuing step involves examining the given geometry using topological tools, notably through the application of pushforward formulae. This technique facilitates the translation of intersection theory computations from the resolved 4-fold $\widehat{Y}_4$ to the base, often simplifying the calculations and revealing patterns. For instance, it shows that specific physical quantities solely depend on a base intersection number, as seen in F-theory QSMs which hinge on the triple intersection number of the anticanonical class of the base 3-fold. Enhancing the

[1] Certainly, F-theory has been studied in more general contexts, for instance without section [76–79].

`FTheoryTools` with intersection theory and topological intersection numbers, and venturing beyond the toric regime, presents collaborative prospects.

Incorporating prevalent F-theory methodologies into the `FTheoryTools` offers an excellent avenue for students to delve into advanced research, interact with relevant (computer) geometries, and contribute to the literature constructions database. Although these efforts are concise—fitting, they hold potential for deeper exploration. Notably, this database not only furthers the study of F-theory geometries but also initiates explorations into machine learning and data science, reflecting my interdisciplinary spirit. A case in point involves probing a theory of Brill-Noether numbers or, if unattainable, investigating a related F-theory inspired cryptosystem. These concepts form part of my secondary research focus, elaborated towards the end of this proposal, and hint at potential fruitful collaborations.

In F-theory studies, exploring beyond topology is vital. For example, the singularities of $Y_4$ determine a non-abelian (gauge) group. It is advantageous to augment this group with abelian group factors for purposes such as enforcing selection rules. These abelian factors originate from the torsion-free subgroup of the Mordell–Weil group of $\widehat{Y}_4$, which represents the group of infinite-order rational sections of the fibration, governed by elliptic curve addition. Consequently, the rank of the abelian part of the gauge algebra aligns with the Mordell–Weil rank. It is noteworthy that the torsional part of the Mordell–Weil rank relates to the gauge group's global structure in the physical theory. Similarly, exploring F-theory on elliptic fibrations with multi-sections can be pursued, the study of which leads to the Weil–Châtalet group and discrete factors (cf. [78] and references therein).

My secondary research aim explores the alignment of F-theory solutions with empirical observations in particle physics. Central to this endeavor is the $G_4$-flux, an element of $H^{2,2}(\widehat{Y}_4, \mathbb{Z})$, which encodes the count of matter particle families in a specified compactification. Initially, the chiral spectrum, founded on topological computations as outlined in [44, 81–88], provides insight. Advancing beyond, I aim to ascertain whether an F-theory solution encompasses zero, one, or multiple instances of the notable Higgs boson - a pivotal step in connecting F-theory solutions with observed particle physics. Presently, `FTheoryTools` has limited capacity for such advanced inquiries. However, support for the renowned `cohomCalg`-algorithm [89–93], along with the vanishing sets from `cohomCalg` in [51], lays a promising groundwork in `OSCAR`. Further explorations include Deligne cohomology and root bundles and necessitate foundational investigations, marking the core of my second research focus.

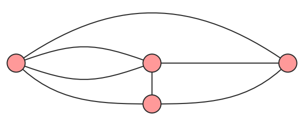
I am excited by the numerous applications of `FTheoryTools`, from uncovering crepant resolutions to enabling previously unmanageable geometries, and streamlining standard processes for novices and experts.

# RESEARCH FOCUS 2: FROM F-THEORY QSMS TO BRILL-NOETHER NUMBERS AND BACK

## Brill-Noether Numbers – A Noval Introduction

The *F-theory QSMs* [53] provide $10^{15}$ solutions apt for the *standard model* of particle physics. My investigation into their vector-like spectra – a crucial ingredient to compare these solutions to experimental findings from particle accelerators – directed me to root bundles on nodal curves. Before we explain how this topic arises from the physics, I wish to provide a noval introduction to Brill-Noether numbers.

Root bundles generalize spin bundles. We recall that there are $2^{2g}$ spin bundles on a smooth, irreducible genus $g$ curve, each of which corresponds to a divisor class $D$ with $2 \cdot D = K_C$, where $K_C$ is the canonical bundle. For root bundles, we consider divisors $D$ with $r \cdot D = E$, $r \in \mathbb{Z}_{\geq 2}$, and $E$ not necessarily the canonical bundle, yielding $r^{2g}$ roots, if existent. Nodal curves $C^\bullet$ bring two nuances: they typically have multiple irreducible components due to nodal singularities, and their root bundles can be enumerated as limit roots [94] (see also [95–97]). Here is the dual graph (vertices are irreducible components and edges nodal singularities) of a nodal curve for which all irreducible components are $\mathbb{P}^1$s:



(1)

Our objective is to enumerate limit roots $P^\bullet$ that satisfy $12P^\bullet = 12K_{C^\bullet}$ and discern their global section count. As we shall motivate below, this count of global sections filters F-theory solutions potentially aligned with experimental findings. For the given case, we have $12^8$ limit roots. Our techniques reveal that $12^4$ roots have $h^0(C^\bullet, P^\bullet) = 4$, whereas the rest possess precisely three global sections:

| Roots Count | $h^0 = 3$ | $h^0 \geq 3$ | $h^0 = 4$ | $h^0 \geq 4$ |
|---|---|---|---|---|
| $12^8$ | $12^4 \cdot (12^4 - 1)$ | $0$ | $12^4$ | $0$ |

(2)

While a unique determination of the number of global sections is elusive for some roots $P^\bullet$, an optimal lower bound can be computed, as would be reflected in every 2nd column of the prior table for a more complicated example. This gives rise to the partition:

$$12^8 = 12^4 \cdot \left(12^4 - 1\right) + 0 + 12^4 + 0\,. \tag{3}$$

The order of the summands is crucial, and summands can reappear. Our technology recently culminated in [59], computing in a certain sense an optimal partition. This optimal partition is likely to bear a deeper meaning and definitely carries a striking resemblance to the Brill-Noether theory for line bundles on smooth, irreducible curves [98]. This leads me to dub the summands *Brill-Noether numbers*. It is worth recalling the historic works [99,100], showing that roughly half of the spin bundles on a genus $g$ curve possess an even numbers of global sections. Beyond this, knowledge is limited. The following subsections will explain the significance of Brill-Noether-numbers for F-theory QSMs, our current understanding, and tasks for future studies.

## Spin Bundle: A Critical Piece in F-theory's Puzzle

Vector-like spectra are essential in analyzing string theory solutions and elucidating the number of Higgs pairs in a theory [13,14] – a salient component highlighted by the Nobel Prize awarded to Higgs and Englert. For instance, a single Higgs pair is imperative for the MSSM. My engagement with this topic, also driven by its rich mathematical tapestry, was initiated in [45]. The relevant matter fields, arising from strings between D7-branes in the perturbative type II string theory, localized in F-theory on *matter curves* $C_{\mathbf{R}} \subset \mathcal{B}_6$ [101–103]. The gauge groups in F-theory, including the group representations of the matter fields, are determined by $Y_4$'s geometry [71,72]. The gauge fields correspond to the Deligne cohomology $H^4_D\left(Y_4, \mathbb{Z}\left(2\right)\right)$ [104–110]. The results in [45] imply that these gauge fields induce a line bundle $\mathcal{L}_{\mathbf{R}}$ on $C_{\mathbf{R}}$. Classical results guide our computation of massless matter [111, 112]:

- $\mathcal{N} = 1$ chiral multiplets: $H^0\left(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}}\right)$.
- $\mathcal{N} = 1$ anti-chiral multiplets: $H^1\left(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}}\right)$.[2]

On a curve of genus $g$, the selection of the appropriate spin bundle $\sqrt{K_{C_{\mathbf{R}}}}$ from $2^{2g}$ possibilities significantly influences the number of sections of $L_{\mathbf{R}} = \mathcal{L}_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}}$ and the Freed-Witten anomaly cancelation [113]. As highlighted in [102], the anomaly cancellation requires spin$^c$-structures on gauge surfaces $S \subset B_3$ in F-theory GUTs. Identifying the correct spin bundles is essential but complex. While traditionally underserved, recent advances underscore the need to address this question. I aspire to explore this area in subsequent research endeavors.

## Brill-Noether Numbers: Upper Bound to Vector-Like Spectra of F-theory QSMs

The pioneering work on F-theory QSMs [53] concentrated on the $G_4$-flux, which governs the chiral index [40–42, 108, 114–119] but leaves the *F-theory* gauge field $A$ undetermined. Amidst the intricacies of the spin bundle (cf. section 4.2), we deduced a pivotal constraint: the line bundle $L_{\mathbf{R}}$ is a specific root bundle $P_{\mathbf{R}}$ [56]. Prior investigations centered on the quark-doublet curve $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ of genus $g = \frac{\overline{K}_{\mathcal{B}_6}{}^3 + 2}{2}$, where $\overline{K}_{\mathcal{B}_6}{}^3$ denotes the triple intersection of the F-theory base's anticanonical class $\overline{K}_{\mathcal{B}_6}{}^3 \in \{6, 10, 18, 30\}$. The root bundle constraint is

$$P_{(\mathbf{3},\mathbf{2})_{1/6}}^{\otimes 2\overline{K}_{\mathcal{B}_6}{}^3} = K_{(\mathbf{3},\mathbf{2})_{1/6}}^{\otimes\left(6+\overline{K}_{\mathcal{B}_6}{}^3\right)}\,. \tag{4}$$

[2] A *supersymmetric field theory* only contains chiral fields. We count chiral superfields in the charge conjugate representation $\overline{\mathbf{R}}$. The wording "anti-chiral" is inspired from low energy physics.

A limited number of these solutions likely arise from an apt choice of the spin bundle, yet its correct identification remains elusive. Instead of pinpointing this, our prior research took a statistical approach: We enumerated all solutions to eq. (4), counted all roots with exactly three global sections, and determined probable F-theory geometries devoid of absence of exotic vector-like quark-doublets. Still, enumerating all roots and distinguishing their global sections is formidable on smooth, irreducible curves. Fortunately, the Brill-Noether numbers offer a practical upper bound. Our methodology ensues:

1. Deforming $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ into $C^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$, which is shared across various geometries [53] due to their origin from toric K3-surfaces desingularizations [120–122], see [57] for a detailed explanation. This key observation facilitated a computer scan of the majority of the $10^{15}$ F-theory QSM geometries [53, 123].
2. Employing techniques from [94] (cf. [95–97]), we list all *limit root* $P^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$ with our software [124].
3. Computing the global sections of each *limit root* $P^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$ with the techniques developed in [56–58], which recently culminated in an optimal approach [59]. The interested reader may which to consult [60] for a summary of this program. This step leads to the Brill-Noether numbers introduced above.
4. Lastly, we bridge the number of global sections between all limit roots and the roots on the smooth curve $C_{(\mathbf{3},\mathbf{2})_{1/6}}$. The number of global sections may decrease if $h^0(C^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}, P^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}) > \chi(P_{(\mathbf{3},\mathbf{2})_{1/6}})$. Hence, the Brill-Noether numbers serve as upper bound to the desired statistics on $C_{(\mathbf{3},\mathbf{2})_{1/6}}$.

## Queston 1: Towards a Theory of Brill-Noether Numbers

Even though the computation of the Brill-Noether numbers are resource-intensive, we currently use them as upper bound to the F-theory QSMs' vector-like spectra. A deeper understanding of the link between a nodal curve and these numbers is desired. One may posit if the Brill-Noether numbers can be inferred from the nodal curve's dual graph and the root bundle constraint. Employing both *machine learning tools* and *analytic/algebraic insights* can be beneficial in this pursuit, mimicking efforts in [55]. If the systematics are revealed, it opens doors for analogous analyses on the intricate Higgs curve of the F-theory QSMs. To illustrate the Higgs curve's complexity, consider a base $\mathcal{B}_6$ with $\overline{K}_{\mathcal{B}_6}{}^3 = 6$. Then, $g(C_{(\mathbf{3},\mathbf{2})_{1/6}}) = 4$, while $g(C_{(\mathbf{1},\mathbf{2})_{-1/2}}) = 28$. The complexity not only manifests in a much larger number of limit roots to be enumerated, but also in a much more complicated dual graph.

## Question 2: An F-Theory-Inspired Cryptosystem?

Recently, the idea of a cryptosystem based on Brill-Noether numbers was sparked: For a given integer partition, can we identify a graph and a root bundle constraint, such that the ensuing Brill-Noether numbers match exactly with the initial partition? The inverse of this question is feasible with our existing methods, but this is already computationally taxing. Attempting the direct approach seems, at the very least, daunting. This disparity raises the possibility of unveiling a new cryptosystem: A promising avenue for future studies.

## Question 3: Jumps Meet Yukawa Interactions

Computing the Brill-Noether numbers for the nodal Higgs curve is imperative. Yet, linking these numbers to the vector-like spectra on the smooth, irreducible Higgs curve introduces challenges. Specifically, a drop in the number of global sections might occur if $h^0(C^{\bullet}_{(\mathbf{1},\mathbf{2})_{1/6}}, P^{\bullet}_{(\mathbf{1},\mathbf{2})_{1/6}}) > \chi(P^{\bullet}_{(\mathbf{1},\mathbf{2})_{1/6}})$. Realizing a Higgs pair [13, 14] indeed necessitates such a non-minimal number of sections:

$$h^0\left(C_{(\mathbf{1},\mathbf{2})_{-1/2}}, P_{(\mathbf{1},\mathbf{2})_{-1/2}}\right) = h^1\left(C_{(\mathbf{1},\mathbf{2})_{-1/2}}, P_{(\mathbf{1},\mathbf{2})_{-1/2}}\right) = 1\,. \tag{5}$$

Changes in $\widehat{Y}_4$'s complex structure can alter $P_{\mathbf{R}}$ and $C_{\mathbf{R}}$, potentially causing $h^0$ and $h^1$ jumps, as explained by Brill-Noether theory [98, 125]. To refine our analysis [56] towards a single Higgs pair, understanding the cohomology differences between the limit root line bundles on $C^{\bullet}_{(\mathbf{1},\mathbf{2})_{-1/2}}$ and $C_{(\mathbf{1},\mathbf{2})_{-1/2}}$ is crucial. Physically, interactions are anticipated at the nodes of $C^{\bullet}_{(\mathbf{1},\mathbf{2})_{-1/2}}$, leading to a mass matrix $M$, whose rank is expected to be difference in vector-like spectra between $C^{\bullet}_{(\mathbf{1},\mathbf{2})_{-1/2}}$ and $C_{(\mathbf{1},\mathbf{2})_{-1/2}}$. The challenge lies in computing $M$ [126, 127] and aligning the predicted jump to mathematical concepts like Brill-Noether jumps [98] or *limit linear series* [128–130].

# REFERENCES


[1] Sheldon L. Glashow, *Partial-symmetries of weak interactions*, *Nuclear Physics* **22**, no. 4 (1961) 579–588, 1961, https://doi.org/10.1016/0029-5582(61)90469-2.

[2] Weinberg, Steven, *A Model of Leptons*, *Phys. Rev. Lett.* **19** (1967) 1264–1266, Nov 1967, https://doi.org/10.1103/PhysRevLett.19.1264.

[3] Abdus Salam, *Weak and electromagnetic interactions*, *Selected Papers of Abdus Salam* pp. 244–254 (1994), https://doi.org/10.1142/9789812795915_0034.

[4] Einstein, Albert, *Die Feldgleichungen der Gravitation*, pp. 88–92, Albert Einstein: Akademie-Vorträge, John Wiley & Sons, Ltd (2005), https://doi.org/10.1002/3527608958.ch5.

[5] Halzen, F. and Martin, A.D. and John Wiley & Sons, *Quarks and Leptones: An Introductory Course in Modern Particle Physics*, Wiley, 1984, ISBN 9780471887416, https://books.google.de/books?id=zwDvAAAAMAAJ.

[6] Griffiths, D., *Introduction to Elementary Particles*, Physics textbook, Wiley, 2008, ISBN 9783527618477, https://books.google.de/books?id=Wb9DYrjcoKAC.

[7] Schwartz, M.D., *Quantum Field Theory and the Standard Model*, Quantum Field Theory and the Standard Model, Cambridge University Press, 2014, ISBN 9781107034730, https://books.google.de/books?id=HbdEAgAAQBAJ.

[8] A. Shomer, *A Pedagogical explanation for the non-renormalizability of gravity* (2007), [0709.3555].

[9] Polchinski, J., *String Theory: Volume 1, An Introduction To The Bosonic String*, Cambridge monographs on mathematical physics, Cambridge University Press, 1998, ISBN 9781139457408, http://books.google.de/books?id=jbM3t_usmX0C.

[10] Polchinski, J., *String Theory: Volume 2, Superstring Theory and Beyond*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2005, ISBN 9780521672283, http://books.google.de/books?id=tJtOMAEACAAJ.

[11] Green, M.B. and Schwarz, J.H. and Witten, E., *Superstring Theory: Volume 1, Introduction*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1988, ISBN 9780521357524, http://books.google.co.in/books?id=ItVsHqjJo4gC.

[12] Green, M.B. and Schwarz, J.H. and Witten, E., *Superstring Theory: Volume 2, Loop Amplitudes, Anomalies and Phenomenology*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1987, ISBN 9780521357531, http://books.google.de/books?id=Z-uz4svcl0QC.

[13] Englert, F. and Brout, R., *Broken Symmetry and the Mass of Gauge Vector Mesons*, *Phys. Rev. Lett.* **13** (1964) 321–323, Aug 1964, https://doi.org/10.1103/PhysRevLett.13.321.

[14] Higgs, Peter W., *Broken Symmetries and the Masses of Gauge Bosons*, *Phys. Rev. Lett.* **13** (1964) 508–509, Oct 1964, https://doi.org//10.1103/PhysRevLett.13.508.

[15] Blumenhagen, Ralph and Körs, Boris and Lüst, Dieter and Stieberger, Stephan, *Four-dimensional string compactifications with D-branes, orientifolds and fluxes*, *Physics Reports* **445**, no. 1-6 (2007) 1–193, Jul 2007, https://doi.org/10.1016/j.physrep.2007.04.003.

[16] Ibáñez, L.E. and Uranga, A.M., *String Theory and Particle Physics: An Introduction to String Phenomenology*, Cambridge University Press, 2012, ISBN 9780521517522, http://books.google.de/books?id=vAUUu6DpVkUC.

[17] Blumenhagen, R. and Lüst, D. and Theisen, S., *Basic Concepts of String Theory*, Theoretical and Mathematical Physics, Springer, 2012, ISBN 9783642294969, `http://books.google.de/books?id=-3PNFQn6AzcC`.

[18] P. Candelas and Gary T. Horowitz and Andrew Strominger and Edward Witten, *Vacuum configurations for superstrings*, *Nuclear Physics B* **258** (1985) 46–74, 1985, `https://doi.org/10.1016/0550-3213(85)90602-9`.

[19] Brian R. Greene and Kelley H. Kirklin and Paul J. Miron and Graham G. Ross, *A superstring-inspired standard model*, *Physics Letters B* **180**, no. 1 (1986) 69–76, 1986, `https://doi.org/10.1016/0370-2693(86)90137-1`.

[20] Braun, Volker and He, Yang-Hui and Ovrut, Burt A. and Pantev, Tony, *A heterotic standard model*, *Physics Letters B* **618**, no. 1-4 (2005) 252–258, Jul 2005, `https://doi.org/10.1016/j.physletb.2005.05.007`.

[21] Bouchard, Vincent and Donagi, Ron, *An SU(5) heterotic standard model*, *Physics Letters B* **633**, no. 6 (2006) 783–791, Feb 2006, `https://doi.org/10.1016/j.physletb.2005.12.042`.

[22] Bouchard, Vincent and Cvetič, Mirjam and Donagi, Ron, *Tri-linear couplings in an heterotic minimal supersymmetric Standard Model*, *Nuclear Physics B* **745**, no. 1-2 (2006) 62–83, Jun 2006, `https://doi.org/10.1016/j.nuclphysb.2006.03.032`.

[23] L. B. Anderson, J. Gray, Y.-H. He and A. Lukas, *Exploring positive monad bundles and a new heterotic standard model*, *Journal of High Energy Physics* **2010**, no. 2, Feb 2010, `https://doi.org/10.1007/JHEP02(2010)054`.

[24] *Two hundred heterotic standard models on smooth Calabi-Yau threefolds*, *Physical Review D* **84**, no. 10, Nov 2011, `https://doi.org/10.1103/PhysRevD.84.106005`.

[25] L. B. Anderson, J. Gray, A. Lukas and E. Palti, *Heterotic line bundle standard models*, *Journal of High Energy Physics* **2012**, no. 6, Jun 2012, `https://doi.org/10.1007/JHEP06(2012)113`.

[26] M. Berkooz, M. R. Douglas and R. G. Leigh, *Branes intersecting at angles*, *Nuclear Physics B* **480**, no. 1-2 (1996) 265–278, Nov 1996, `https://doi.org/10.1016/S0550-3213(96)00452-X`.

[27] G. Aldazabal, S. Franco, L. E. Ibáñez, R. Rabadan and A. M. Uranga, *D=4 chiral string compactifications from intersecting branes*, *Journal of Mathematical Physics* **42**, no. 7 (2001) 3103–3126, Jul 2001, `https://doi.org/10.1063/1.1376157`.

[28] G. Aldazabal, S. Franco, L. E. Ibáñez, R. Rabadan and A. M. Uranga, *Intersecting brane worlds*, *Journal of High Energy Physics* **2001**, no. 02 (2001) 047–047, Feb 2001, `https://doi.org/10.1088/1126-6708/2001/02/047`.

[29] L. E. Ibáñez, F. Marchesano and R. Rabadan, *Getting just the standard model at intersecting branes*, *Journal of High Energy Physics* **2001**, no. 11 (2001) 002–002, Nov 2001, `https://doi.org/10.1088/1126-6708/2001/11/002`.

[30] R. Blumenhagen, B. Kors, D. Lüst and T. Ott, *The Standard Model from stable intersecting brane world orbifolds*, *Nuclear Physics B* **616**, no. 1-2 (2001) 3–33, Nov 2001, `https://doi.org/10.1016/S0550-3213(01)00423-0`.

[31] M. Cvetič, G. Shiu and A. M. Uranga, *Three-Family Supersymmetric Standardlike Models from Intersecting Brane Worlds*, *Physical Review Letters* **87**, no. 20, Oct 2001, `https://doi.org/10.1103/PhysRevLett.87.201801`.

[32] M. Cvetič, G. Shiu and A. M. Uranga, *Chiral four-dimensional N=1 supersymmetric type IIA orientifolds from intersecting D6-branes*, *Nuclear Physics B* **615**, no. 1-3 (2001) 3–32, Nov 2001, https://doi.org/10.1016/S0550-3213(01)00427-8.

[33] R. Blumenhagen, M. Cvetič, P. Langacker and G. Shiu, *Toward realistic intersecting D-brane models*, *Annual Review of Nuclear and Particle Science* **55**, no. 1 (2005) 71–139, Dec 2005, https://doi.org/10.1146/annurev.nucl.55.090704.151541.

[34] Gómez, Tomás L. and Lukic, Sergio and Sols, Ignacio, *Constraining the Kähler Moduli in the Heterotic Standard Model*, *Communications in Mathematical Physics* **276**, no. 1 (2007) 1–21, Sep 2007, https://doi.org/10.1007/s00220-007-0338-8.

[35] V. Bouchard and R. Donagi, *On heterotic model constraints*, *Journal of High Energy Physics* **2008**, no. 08 (2008) 060–060, Aug 2008, https://doi.org/10.1088/1126-6708/2008/08/060.

[36] C. Brodie, A. Constantin, J. Gray, A. Lukas and F. Ruehle, *Recent Developments in Line Bundle Cohomology and Applications to String Phenomenology*, in *Nankai Symposium on Mathematical Dialogues*: In celebration of S.S.Chern's 110th anniversary, 12 2021, [2112.12107].

[37] Vafa, Cumrun, *Evidence for F-theory*, *Nuclear Physics B* **469**, no. 3 (1996) 403–415, Jun 1996, https://doi.org/10.1016/0550-3213(96)00172-1.

[38] Morrison, David R. and Vafa, Cumrun, *Compactifications of F-theory on Calabi-Yau threefolds (II)*, *Nuclear Physics B* **476**, no. 3 (1996) 437–469, Sep 1996, https://doi.org/10.1016/0550-3213(96)00369-0.

[39] Morrison, David R. and Vafa, Cumrun, *Compactifications of F-theory on Calabi-Yau threefolds. (I)*, *Nuclear Physics B* **473**, no. 1-2 (1996) 74–92, Aug 1996, https://doi.org/10.1016/0550-3213(96)00242-8.

[40] T. W. Grimm and H. Hayashi, *F-theory fluxes, chirality and Chern-Simons theories*, *Journal of High Energy Physics* **2012**, no. 3, Mar 2012, https://doi.org/10.1007/JHEP03(2012)027.

[41] Krause, Sven and Mayrhofer, Christoph and Weigand, Timo, *$G_4$-flux, chiral matter and singularity resolution in F-theory compactifications*, *Nuclear Physics B* **858**, no. 1 (2012) 1–47, May 2012, https://doi.org/10.1016/j.nuclphysb.2011.12.013.

[42] S. Krause, C. Mayrhofer and T. Weigand, *Gauge Fluxes in F-theory and Type IIB Orientifolds*, *Journal of High Energy Physics* **2012**, no. 8, Aug 2012, https://doi.org/10.1007/JHEP08(2012)119.

[43] V. Braun, T. W. Grimm and J. Keitel, *Geometric Engineering in Toric F-Theory and GUTs with U(1) Gauge Factors*, *Journal of High Energy Physics* **2013**, no. 12, Dec 2013, https://doi.org/10.1007/JHEP12(2013)069.

[44] M. Cvetič, A. Grassi, D. Klevers and H. Piragua, *Chiral four-dimensional F-theory compactifications with SU(5) and multiple U(1)-factors*, *Journal of High Energy Physics* **2014**, no. 4, Apr 2014, https://doi.org/10.1007/JHEP04(2014)010.

[45] M. Bies, C. Mayrhofer, C. Pehle and T. Weigand, *Chow groups, Deligne cohomology and massless matter in F-theory* 2 (2014), [1402.5144].

[46] M. Cvetič, D. Klevers, D. K. M. Peña, P.-K. Oehlmann and J. Reuter, *Three-family particle physics models from global F-theory compactifications*, *Journal of High Energy Physics* **2015**, no. 8, Aug 2015, https://doi.org/10.1007/JHEP08(2015)087.

[47] L. Lin, C. Mayrhofer, O. Till and T. Weigand, *Fluxes in F-theory Compactifications on Genus-One Fibrations*, *Journal of High Energy Physics* **2016**, no. 1, Jan 2016, https://doi.org/10.1007/JHEP01(2016)098.

[48] L. Lin and T. Weigand, *G4-flux and standard model vacua in F-theory*, *Nuclear Physics B* **913** (2016) 209–247, Dec 2016, https://doi.org/10.1016/j.nuclphysb.2016.09.008.

[49] Bies, Martin and Mayrhofer, Christoph and Weigand, Timo, *Gauge backgrounds and zero-mode counting in F-theory*, *Journal of High Energy Physics* **2017**, no. 11, Nov 2017, https://doi.org/10.1007/JHEP11(2017)081.

[50] M. Bies, C. Mayrhofer and T. Weigand, *Algebraic Cycles and Local Anomalies in F-Theory*, *Journal of High Energy Physics* **2017**, no. 11, Nov 2017, https://doi.org/10.1007/JHEP11(2017)100.

[51] Bies, Martin, *Cohomologies of coherent sheaves and massless spectra in F-theory*, Ph.D. thesis, Heidelberg U., 2 2018, https://doi.org/10.11588/heidok.00024045.

[52] M. Cvetič, L. Lin, M. Liu and P.-K. Oehlmann, *An F-theory Realization of the Chiral MSSM with $\mathbb{Z}_2$-Parity*, *Journal of High Energy Physics* **2018**, no. 9, Sep 2018, https://doi.org/10.1007/JHEP09(2018)089.

[53] M. Cvetič, J. Halverson, L. Lin, M. Liu and J. Tian, *Quadrillion F-Theory Compactifications with the Exact Chiral Spectrum of the Standard Model*, *Physical Review Letters* **123**, no. 10, Sep 2019, https://doi.org/10.1103/PhysRevLett.123.101601.

[54] Bies, Martin and Posur, Sebastian, *Tensor products of finitely presented functors*, *Journal of Algebra and Its Applications* p. 2250186, Jul 2021, https://doi.org/10.1142/s0219498822501869.

[55] M. Bies, M. Cvetič, R. Donagi, L. Lin, M. Liu and F. Ruehle, *Machine Learning and Algebraic Approaches towards Complete Matter Spectra in 4d F-theory*, *Journal of High Energy Physics* **2021**, no. 1, Jan 2021, https://doi.org/10.1007/JHEP01(2021)196.

[56] Bies, Martin and Cvetič, Mirjam and Donagi, Ron and Liu, Muyang and Ong, Marielle, *Root bundles and towards exact matter spectra of F-theory MSSMs*, *Journal of High Energy Physics* **2021**, no. 9, Sep 2021, https://doi.org/10.1007/JHEP09(2021)076.

[57] M. Bies, M. Cvetič and M. Liu, *Statistics of limit root bundles relevant for exact matter spectra of F-theory MSSMs*, *Physical Review D* **104**, no. 6, Sep 2021, https://doi.org/10.1103/PhysRevD.104.L061903.

[58] M. Bies, M. Cvetič, R. Donagi and M. Ong, *Brill-Noether-general limit root bundles: absence of vector-like exotics in F-theory Standard Models*, *J*ournal Of High Energy Physics **11**, Apr 2022, https://doi.org/10.1007/JHEP11(2022)004.

[59] Bies, Martin and Cvetič, Mirjam and Donagi, Ron and Ong, Marielle, *Improved statistics for F-theory standard models* (2023), [2307.02535].

[60] M. Bies, *Root bundles: Applications to F-theory Standard Models*, (2023), [2303.08144].

[61] The OSCAR Team, *OSCAR – Open Source Computer Algebra Research system, Version 0.14.0-DEV*, https://www.oscar-system.org, 2023.

[62] Decker, Wolfram and Eder, Christian and Fieker, Claus and Horn, Max and Joswig, Michael, editor, *The OSCAR book*, 2024.

[63] M. Bies and L. Kastner, *Toric Geometry in OSCAR* 3 2023, [2303.08110].

[64] The Toric Varieties project authors, *The* ToricVarieties *project*, (https://github.com/homalg-project/ToricVarieties_project), 2019–2023.

[65] The homalg project authors, *The* `homalg` *project – Algorithmic Homological Algebra*, (`https://homalg-project.github.io/`), 2003–2023.

[66] Gutsche, Sebastian and Skartsæterhagen, Øystein and Posur, Sebastian, *The* `CAP` *project – Categories, Algorithms, Programming*, (`http://homalg-project.github.io/CAP_project`), 2013–2023.

[67] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.12.2*, (`https://www.gap-system.org`), 2023.

[68] F. G. Marchesano Buznego, *Intersecting D-brane models*, Ph.D. thesis, May 2003, [hep-th/0307252].

[69] Cremades, D. and Ibáñez, L.E. and Marchesano, F., *Intersecting Brane Models of Particle Physics and the Higgs Mechanism*, *Journal of High Energy Physics* **2002**, no. 07 (2002) 022–022, Jul 2002, `https://doi.org/10.1088/1126-6708/2002/07/022`.

[70] Brian R. Greene and Alfred Shapere and Cumrun Vafa and Shing-Tung Yau, *Stringy cosmic strings and noncompact Calabi-Yau manifolds*, *Nuclear Physics B* **337**, no. 1 (1990) 1–36, 1990, `https://doi.org/10.1016/0550-3213(90)90248-C`.

[71] Denef, Frederik, *Les Houches Lectures on Constructing String Vacua*, *Les Houches* **87** (2008) 483–610, 2008, [0803.1194].

[72] Weigand, Timo, *Lectures on F-theory compactifications and model building*, *Classical and Quantum Gravity* **27**, no. 21 (2010) 214004, Oct 2010, `https://doi.org/10.1088/0264-9381/27/21/214004`.

[73] M. Cvetič and L. Lin, *TASI Lectures on Abelian and Discrete Symmetries in F-theory*, *P*oS **TASI2017** (2018) 020, 2018, `https://doi.org/10.22323/1.305.0020`.

[74] T. Weigand, *F-theory*, *P*oS **TASI2017** (2018) 016, 2018, `https://doi.org/10.22323/1.305.0016`.

[75] Griffiths, P. and Harris, J., *Principles of Algebraic Geometry*, Wiley Classics Library, Wiley, 2011, ISBN 9781118030776, `http://books.google.de/books?id=Sny48qKdW40C`.

[76] Witten, Edward, *Non-perturbative superpotentials in string theory*, *Nuclear Physics B* **474**, no. 2 (1996) 343–360, Aug 1996, `https://doi.org/10.1016/0550-3213(96)00283-0`.

[77] Braun, Volker and Morrison, David R., *F-theory on genus-one fibrations*, *Journal of High Energy Physics* **2014**, no. 8, Aug 2014, `https://doi.org/10.1007/JHEP08(2014)132`.

[78] Morrison, David R. and Taylor, Washington, *Sections, multisections, and U(1) fields in F-theory*, *Journal of Singularities* **15** (2016) 126–149, Oct 2016, `https://doi.org/10.5427/jsing.2016.15g`.

[79] Anderson, Lara B. and García-Etxebarria, Iñaki and Grimm, Thomas W. and Keitel, Jan, *Physics of F-theory compactifications without section*, *Journal of High Energy Physics* **2014**, no. 12, Dec 2014, `https://doi.org/10.1007/JHEP12(2014)156`.

[80] V. Arena, P. Jefferson and S. Obinna, *Intersection Theory on Weighted Blowups of F-theory Vacua* Apr 2023, [2305.00297].

[81] T. W. Grimm and H. Hayashi, *F-theory fluxes, chirality and Chern-Simons theories*, *J*ournal of High Energy Physics **2012**, no. 3, Mar 2012, `https://doi.org/10.1007/JHEP03(2012)027`.

[82] S. Krause, C. Mayrhofer and T. Weigand, *Gauge Fluxes in F-theory and Type IIB Orientifolds*, *J*ournal of High Energy Physics **2012**, no. 8, Aug 2012, `https://doi.org/10.1007/JHEP08(2012)119`.

[83] V. Braun, T. W. Grimm and J. Keitel, *Geometric Engineering in Toric F-Theory and GUTs with U(1) Gauge Factors*, *J*ournal Of High Energy Physics **12** (2013) 069, 2013, `https://doi.org/10.1007/JHEP12(2013)069`.

[84] M. Cvetič, D. Klevers, D. K. M. Peña, P.-K. Oehlmann and J. Reuter, *Three-Family Particle Physics Models from Global F-theory Compactifications*, *J*ournalOf High Energy Physics **08** (2015) 087, 2015, https://doi.org/10.1007/JHEP08(2015)087.

[85] L. Lin, C. Mayrhofer, O. Till and T. Weigand, *Fluxes in F-theory Compactifications on Genus-One Fibrations*, *Journal of High Energy Physics* **01** (2016) 098, 2016, https://doi.org/10.1007/JHEP01(2016)098.

[86] L. Lin and T. Weigand, *G4-flux and standard model vacua in F-theory*, *N*uclear Physics B **9**13 (2016) 209–247, Dec 2016, https://doi.org/10.1016/j.nuclphysb.2016.09.008.

[87] P. Jefferson, W. Taylor and A. P. Turner, *Chiral matter multiplicities and resolution-independent structure in 4D F-theory models* 8 2021, [2108.07810].

[88] P. Jefferson and A. P. Turner, *Generating functions for intersection products of divisors in resolved F-theory models*, *N*ucl. Phys. B **991** (2023) 116177, 2023, https://doi.org/10.1016/j.nuclphysb.2023.116177.

[89] Blumenhagen, Ralph and Jurke, Benjamin and Rahn, Thorsten and Roschy, Helmut, *Cohomology of line bundles: A computational algorithm*, *Journal of Mathematical Physics* **51**, no. 10 (2010) 103525, Oct 2010, https://doi.org/10.1063/1.3501132.

[90] *cohomCalg package*, Download link, 2010, http://wwwth.mppmu.mpg.de/members/blumenha/cohomcalg/, (High-performance line bundle cohomology computation based on [89]).

[91] Jow, Shin-Yao, *Cohomology of toric line bundles via simplicial Alexander duality*, *Journal of Mathematical Physics* **52**, no. 3 (2011) 033506, Mar 2011, https://doi.org/10.1063/1.3562523.

[92] Roschy, Helmut and Rahn, Thorsten, *Cohomology of line bundles: Proof of the algorithm*, *Journal of Mathematical Physics* **51**, no. 10 (2010) 103520, Oct 2010, https://doi.org/10.1063/1.3501135.

[93] Blumenhagen, Ralph and Jurke, Benjamin and Rahn, Thorsten and Roschy, Helmut, *Cohomology of line bundles: Applications*, *Journal of Mathematical Physics* **53**, no. 1 (2012) 012302, Jan 2012, https://doi.org/10.1063/1.3677646.

[94] Lucia Caporaso and Cinzia Casagrande and Maurizio Cornalba, *Moduli of Roots of Line Bundles on Curves*, *Transactions of the American Mathematical Society* **359**, no. 8 (2007) 3733–3768, 2007, https://doi.org/10.1090/S0002-9947-07-04087-1.

[95] T. J. Jarvis, *Geometry of the moduli of higher spin curves*, *International Journal of Mathematics* **11** (1998) 637–663, Sep 1998, https://doi.org/10.1142/S0129167X00000325.

[96] Jarvis, Tyler J., *The Picard group of the moduli of higher spin curves*, *The New York Journal of Mathematics* **7** (2001) 23–47, 2001, https://nyjm.albany.edu/j/2001/7-3.html.

[97] Sergey Natanzon and Anna Pratoussevitch, *Higher Spin Klein Surfaces*, *Moscow Mathematical Journal* **16** (2016) 95–124, 2016, http://www.mathjournals.org/mmj/2016-016-001/2016-016-001-004.html.

[98] Brill, Alexander and Noether, Max, *Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie*, *Mathematische Annalen* **7**, no. 2 (1874) 269–310, 1874, https://doi.org/10.1007/BF02104804.

[99] M. F. Atiyah, *Riemann surfaces and spin structures*, in *Annales Scientifiques de L'Ecole Normale Superieure*, volume 4, pp. 47–62, Société mathématique de France, 1971, http://www.numdam.org/item/10.24033/asens.1205.pdf.

[100] M. David, *Theta characteristics of an algebraic curve*, *Ann. Sci. Ecole Norm. Sup* **4**, no. 4 (1971) 181–192, 1971, http://www.numdam.org/article/ASENS_1971_4_4_2_181_0.pdf.

[101] S. H. Katz and C. Vafa, *Matter from geometry*, *Nuclear Physics B* **497**, no. 1-2 (1997) 146–154, Jul 1997, https://doi.org/10.1016/S0550-3213(97)00280-0.

[102] C. Beasley, J. J. Heckman and C. Vafa, *GUTs and exceptional branes in F-theory — I*, *Journal of High Energy Physics* **2009**, no. 01 (2009) 058–058, Jan 2009, https://doi.org/10.1088/1126-6708/2009/01/058.

[103] Donagi, Ron and Wijnholt, Martijn, *Gluing Branes — I*, *Journal of High Energy Physics* **2013**, no. 05 (2013) 068–068, May 2013, https://doi.org/10.1007/JHEP05(2013)068.

[104] R. Donagi, *Heterotic / F theory duality: ICMP lecture*, in *12th International Congress of Mathematical Physics (ICMP 97)*, pp. 206–213, 2 1998, [hep-th/9802093].

[105] Curio, Gottfried and Donagi, Ron, *Moduli in N = 1 heterotic/F-theory duality*, *Nuclear Physics B* **518**, no. 3 (1998) 603–631, May 1998, https://doi.org/10.1016/S0550-3213(98)00185-0.

[106] Diaconescu, Emanuel and Freed, Daniel S. and Moore, Gregory, *The M-theory 3-form and E8 gauge theory*, p. 44–88, London Mathematical Society Lecture Note Series, Cambridge University Press, 2007, https://doi.org/10.1017/CBO9780511721489.005.

[107] Freed, Daniel S. and Moore, Gregory W., *Setting the Quantum Integrand of M-Theory*, *Communications in Mathematical Physics* **263**, no. 1 (2006) 89–132, Jan 2006, https://doi.org/10.1007/s00220-005-1482-7.

[108] Intriligator, Kenneth and Jockers, Hans and Mayr, Peter and Morrison, David R. and Plesser, M. Ronen, *Conifold transitions in M-theory on Calabi-Yau fourfolds with background fluxes*, *Advances in Theoretical and Mathematical Physics* **17**, no. 3 (2013) 601–699, 2013, https://doi.org/10.4310/ATMP.2013.v17.n3.a2.

[109] Eisenbud, David and Harris, Joe, *3264 and All That: A Second Course in Algebraic Geometry*, Cambridge University Press, 2016, https://doi.org/10.1017/CBO9781139062046.

[110] Fulton, W., *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics, Springer Berlin Heidelberg, 2013, ISBN 9783662024218, https://books.google.de/books?id=gCXsCAAAQBAJ.

[111] E. Witten, *Topological sigma models*, *Communications in Mathematical Physics* **118**, no. 3 (1988) 411–449, 1988, https://doi.org/10.1007/BF01466725.

[112] Witten, Edward, *Mirror manifolds and topological field theory*, *AMS/IP Stud. Adv. Math.* **9** (1991) 121–160, 1991, [hep-th/9112056].

[113] Freed, Daniel S. and Witten, Edward, *Anomalies in string theory with D-branes*, *A*sian J. Math. **3**, 1999, https://doi.org/10.4310/AJM.1999.v3.n4.a6.

[114] Donagi, Ron and Wijnholt, Martijn, *Model building with F-theory*, *Advances in Theoretical and Mathematical Physics* **15** (2011) 1237 – 1317, 10 2011, https://doi.org/10.4310/ATMP.2011.v15.n5.a2.

[115] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, *New Aspects of Heterotic-F Theory Duality*, *Nuclear Physics B* **806**, no. 1-2 (2009) 224–299, Jan 2009, https://doi.org/10.1016/j.nuclphysb.2008.07.031.

[116] Donagi, Ron and Wijnholt, Martijn, *Higgs Bundles and UV Completion in F-Theory*, *Commun. Math. Phys.* **326** (2014) 287–327, 2014, https://doi.org/10.1007/s00220-013-1878-8.

[117] Heckman, Jonathan J., *Particle Physics Implications of F-Theory*, *Annual Review of Nuclear and Particle Science* **60**, no. 1 (2010) 237–265, Nov 2010, https://doi.org/10.1146/annurev.nucl.012809.104532.

[118] Marsano, Joseph and Schäfer-Nameki, Sakura, *Yukawas, G-flux, and spectral covers from resolved Calabi-Yau's*, *Journal of High Energy Physics* **2011**, no. 11, Nov 2011, https://doi.org/10.1007/JHEP11(2011)098.

[119] Braun, A.P. and Collinucci, A. and Valandro, R., *G-flux in F-theory and algebraic cycles*, *Nuclear Physics B* **856**, no. 2 (2012) 129–179, Mar 2012, https://doi.org/10.1016/j.nuclphysb.2011.10.034.

[120] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, *J. Alg. Geom.* **3** (1994) 493–545, 1994, [alg-geom/9310003].

[121] Cox, D.A. and Katz, S., *Mirror Symmetry and Algebraic Geometry*, Mathematical surveys and monographs, American Mathematical Society, 1999, ISBN 9780821821275, https://doi.org/https://doi.org/10.1090/surv/068.

[122] M. Kreuzer and H. Skarke, *Classification of reflexive polyhedra in three-dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 853–871, 1998, https://doi.org/10.4310/ATMP.1998.v2.n4.a5.

[123] J. Halverson and J. Tian, *Cost of seven-brane gauge symmetry in a quadrillion F-theory compactifications*, *Physical Review D* **95**, no. 2, Jan 2017, https://doi.org/10.1103/PhysRevD.95.026005.

[124] M. Bies, *RootCounter*, https://github.com/Julia-meets-String-Theory/RootCounter, 2023.

[125] D. Eisenbud, M. Green and J. Harris, *Cayley-Bacharach theorems and conjectures*, *B*ulletin of the American Mathematical Society **33**, no. 03 (1996) 295–325, Jul 1996, https://doi.org/10.1090/s0273-0979-96-00666-0.

[126] Cecotti, Sergio and Cheng, Miranda C. N. and Heckman, Jonathan J. and Vafa, Cumrun, *Yukawa Couplings in F-theory and Non-Commutative Geometry*, *Surveys in differential geometry* **15** (2009) 37–98, 2009, [0910.0477].

[127] Cvetič, Mirjam and Lin, Ling and Liu, Muyang and Zhang, Hao Y. and Zoccarato, Gianluca, *Yukawa Hierarchies in Global F-theory Models*, *Journal of High Energy Physics* **2020**, no. 1, Jan 2020, https://doi.org/10.1007/JHEP01(2020)037.

[128] Eisenbud, D. and Harris, J., *Limit linear series: Basic theory*, *Inventiones mathematicae* **85** (1986) 337–372, 1986, https://doi.org/10.1007/BF01389094.

[129] Osserman, Brian, *A limit linear series moduli scheme*, *Annales de l'institut Fourier* **56**, no. 4 (2006) 1165–1205, 2006, https://doi.org/10.5802/AIF.2209.

[130] Gavril Farkas, *Theta characteristics and their moduli*, *Milan Journal of Mathematics* **80** (2012) 1–24, 2012, https://doi.org/10.1007/s00032-012-0178-7.